Group Theory Methods in Physics
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Lecture – 42
U(n) and SU(n) groups

Other question you can ask is O 3, how many parameters are there? SO 3 how many parameters are there? Whether you are working with O 3 or SO 3? The number of continuous parameters is only 3, but if I have working with U 2 how many number of if you do not put the determinant to be plus 1 ok. This is the question to you. If you do not put the determinant to be plus 1 and if you put the determinant to be plus 1 is the number of parameters same or different.

So, let us take the simple example of a U 1 group, what is U 1 group? It is just 1 cross 1.

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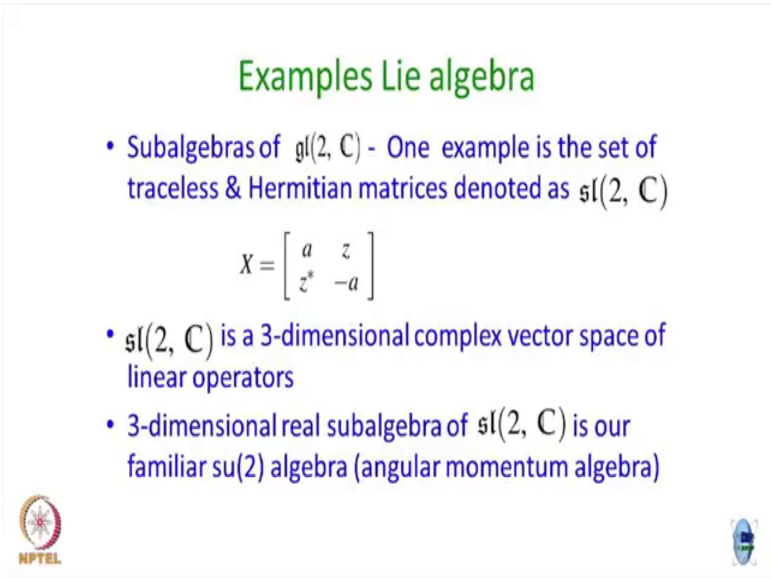

So, let me take U 1 capital U to denote it is a group.

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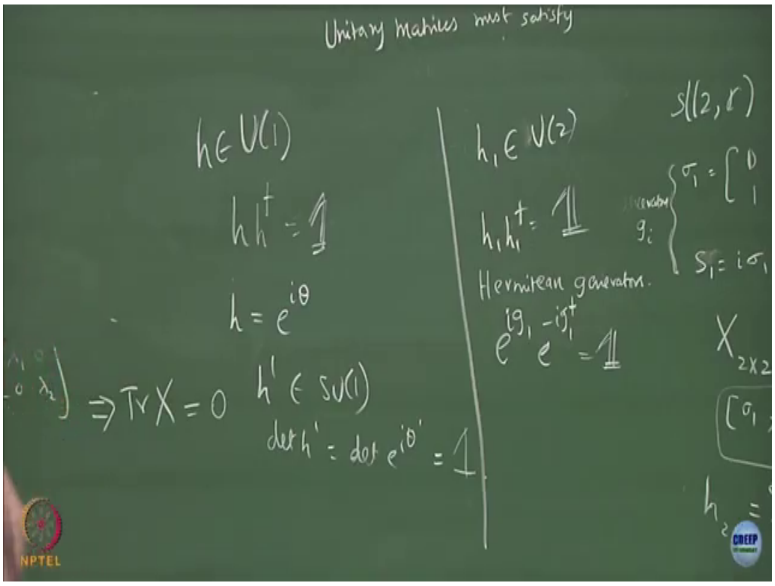

U 1 is a 1 cross 1 matrices right and you have a condition. So, let us take an element g which belongs to or let me take it as a h h which belongs to U 1 what is the meaning of it? h h dagger has to be 1 that is unitary, it is nothing to do with determinant right. This does not imply determinant is 1 you all agree? U 1 I have not put SU 1. I have just put U 1. U 1 is a 1 cross 1 matrix which is like a trivial matrix to you with a complex entry with satisfies this condition.

What happens once you put in this condition? You reduce it by one more parameter. Essentially you will have that the element will be just e to the i 1 parameter which is theta clear. So, it is a one parameter group continuous group and once I put determinant equal to 1, what happens? Suppose, I want to look at element h prime which belongs to SU 1 then what will happen? I have to put determinant of h prime which is determinant of e to the i theta prime to be 1.

When is theta arbitrary? It has a periodicity let us put it between 0 and 2 to pi, theta is not arbitrary. You have fixed it, then the number of parameter is 0. Number of parameters is just nothing SU 1, the element itself parameter has forced to be 0 then there is no parameters space it just one point in the parameters space. So, SU 1 going from U 1 to SU 1, the number of parameters which was 1 got reduced to 0 can all see that, right.

So, now, let us do it for SU 2. So, let us do U 2, ok. So, let us take h 1 is an element of U 2. What you can show is that the element should only satisfy h 1 h 1 dagger to be equal to identity, ok. So, which means it will be Hermitean is that right Hermitean generators am I right?

So, we can write this as e to the power of ig 1 e to the power of minus ig 1 dagger which is equal to 1, ok. The determinant there is no condition, right. There is no condition on determinant which means the matrices could be the matrices of the generators could be some a, b right some c c start, clear? Hermitean condition only tells you that it tells you that what? These have to be complex entries with complex conjugates this has to be real. How many independent parameters are there in the generators? Four. U 2 has four generators, SU 2 has the determinant condition.

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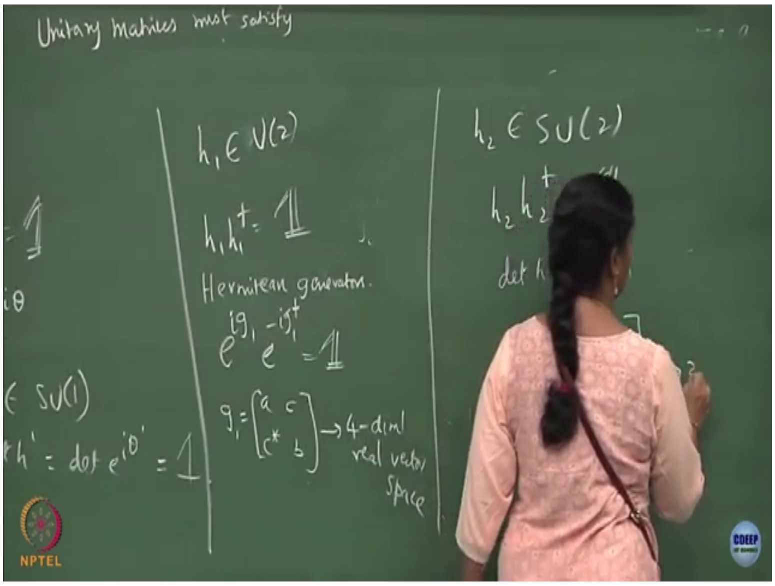

So, if you go to h 2 which belongs to SU 2, then you also have besides this h 2 h 2 dagger equal to identity, you also have determinant of h 2 to be plus 1, ok. So, that determinant puts it on the generator g 2 to be trace less ok; two real, these are complex and this is only one real. So, totally three real parameters, ok. Here you had four real parameters. So, this is a 4-dimensional real vector space you do.

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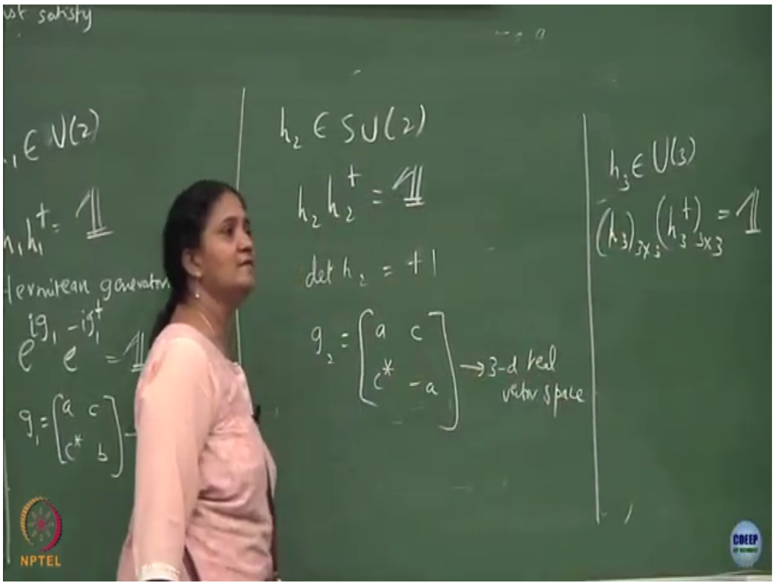

And, this one will be SU 2 is a 3-dimensional real vector space. Now, I want you to do it for SU 3, will you take it up and do it? U 3 how many elements are there independent?

Student: 9.

9, good. So, h 3 which is a 3 cross 3 matrix h 3 dagger which is a 3 cross 3 matrix because this notation is set of unitary matrices which are 3 cross 3 and the entries could be complex, clear? Unitarity it does not put any condition on. So, what you can show is that this unitarity condition gives you constraints right. Put in these constraints and you will find that there are nine real parameters and you once to put determinant equal to one you get one more parameter removed. The reason for that one more parameter getting removed is because every element of SU 2, if you multiply by an element of U 1; U 1 is just e to the i theta, ok. You get an element of S U 3 or U 2.

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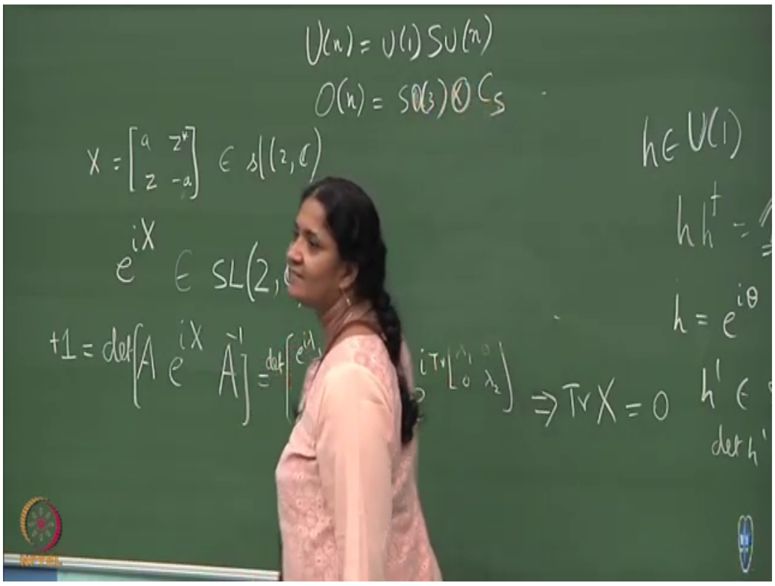

So, any U n you can write it as a U 1 eltim element multiplying an SU n element, SU n you can find by this argument how many number of real parameters are required. So, she said that there are 9 parameters here.

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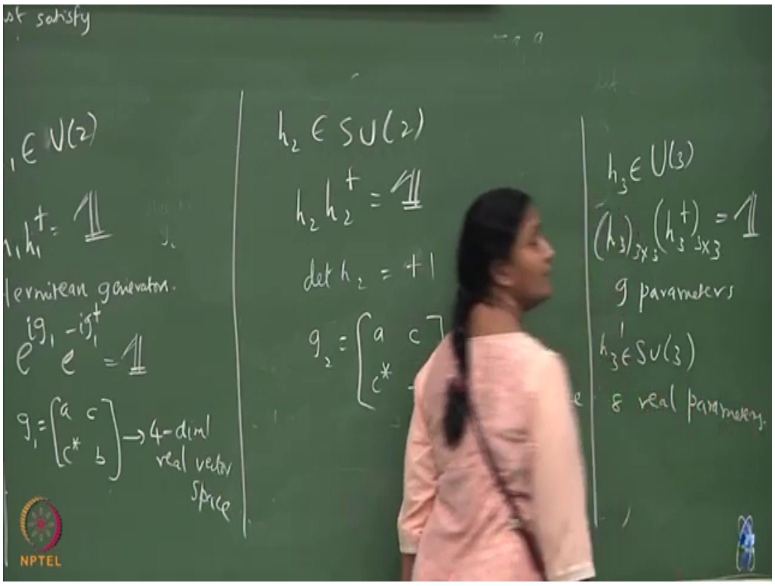

And, then you if you take on top of it if you want to look at SU 3 any element here and there will be 8 real parameters. I am not done this exercise for you, but I leave it you will check it what is that number and you can also find how many real parameters are required to specify unitary matrices with determinant plus 1 n cross n matrices. So, I will leave it to you to do this.

And, what I am saying is that every unitary element will be multiply by U 1 element. So, when you take the determinant of this, this determinant is one, but this determinant is non-trivial and that element has one parameter. So, that is why the number of parameters of U n will not be same as the number of number of parameters of SU n. In the case of O n what happened? It was SO 3 capital O, I wrote it as C S and C S was a discrete group.

In discrete group you do not have any continuous parameters. You just do a flip reflection is a flip you cannot continuously go by infinitesimal transformation to get a reflection. So, that is not going to give you any additional parameter, only for unitary groups you start see additional parameter. In fact, this is beautifully discussed in book by quantum mechanics by Schiff in the chapter symmetries in quantum mechanics. Anybody can take a look at it all these things which I am saying all explained there, ok.

So, once you have listen to this lecture it will be very simple to understand what is. Have you understood how the basis elements of lie algebra for a specific group given you, whether it is unitary group or orthogonal group how to find the basis set number of elements in the basis set that is given by the number of parameters. So, for the U 3 group there will be 9 elements in the lie algebra for SU 3. Sorry, U 3 and 8 elements in the lie algebra for SU 3, is that clear ok?

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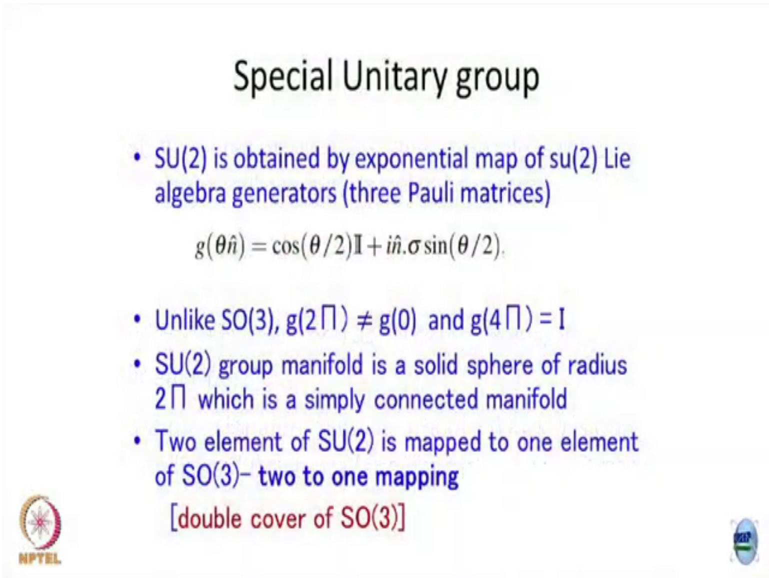

So, coming back to the slide now, this is something which you have been doing now because in the context of group theory I just want to project further what all you can understand for SU 2 and then take you to other groups by the same logic. That is why I am doing a warm up on SO SU 2.

So, SU 2 is something which in quantum mechanics if you want to write a group elements, there are nice properties of the Pauli matrices which helps you to write the exponential form in this form ok. So, all your SU 2 elements so, suppose I want to write g of theta, there will be three parameters which I can compactly write it as a vector theta right.

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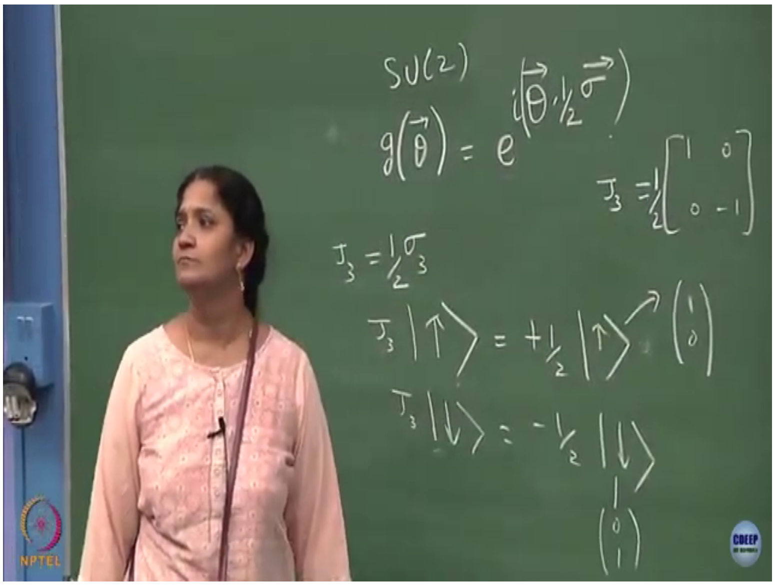

And, this can be written as exponential of i theta dot with generators, ok. So, those generators are the half of those sigmas. Those Pauli matrices which I wrote half of it and here you know that sigma 3 half of sigma 3 I can call it is as the J 3 the third component of your z component of your angular momentum.

Why, because if this acts on up spin what is that give you? I am suppressing the h cross anyway. If you want you can put the h cross, but remember that if h cross is 1, then you can write this as plus half up spin right and then J 3 on down spin is minus half down spin.

And, you can write this matrix form of J 3 to be 1, 0, 0, minus 1 you all understand this right? You can denote this up spin to be 1 0 and down spin to be 0 1, I mean this similarly this one and you can determine the matrix of J 3 which when acts on 1 0 it gives you plus one times that and which acts on to 0 1, it gives you minus 1 times that and that is this matrix represently with the half factor.

This is the only matrix which is diagonal by diagonal I mean these states are eigenstates, the J 3 on the plus minus will give you only a plus similarly minus will go to the minus what about J 1 and J 2 what about J 1 and J 2? Will that be true?

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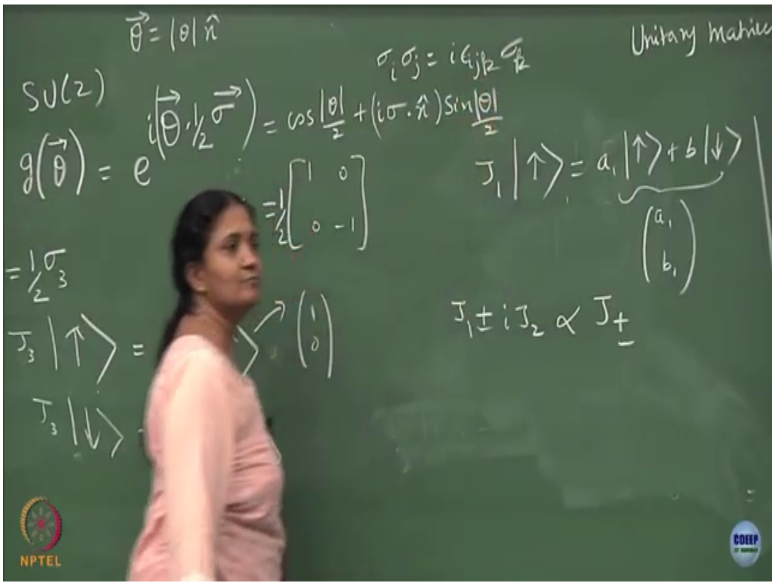

If you take J 1 on up I am do not going to say what exactly it is. It could be in general a linear combination of what is this meaning of this in the vector notation it is a 1 and b 1 just like what we did in the discrete groups, right. We did this in the discrete groups. What is this mean? I am not getting anything new which means this is an irreducible vector space. The generators are this Pauli matrices and I have written up to proportionality constant. The generators are still giving me states either as eigenstates or as linear combinations of the states, ok. So, the two-dimensional vector space is irreducible. Is this clear?

Is this concept clear that it this 2-dimensional vector space is an irreducible vector space the two bases states are this and this and any of the generators of the Lie algebra will only give you a linear combinations of these two states, it does not give you anything new, clear? So, that is why it is irreducible representation or irreducible vectors space. This is an irreducible matrix representation for the lie algebra this only, but I can write for J 1 and J 2 there are the Pauli matrices.

What is being done in this thing is that you do a convenient modification in this algebra, I am sure you would all done. This you write J 1 plus or minus i J 2 to be proportional to J plus or minus this is done for comfort. What do I mean by comfort? If I do that plus sign I take it as a raising operator minus sign I take it as a lowering operator J 1 and J 2 are Hermitean, J 1 plus i J 2 is conjugate 2 J 1 minus I J 2. So, J plus and J minus are not Hermitean, but it is just done for convenience and this methodology which is being done explicitly for SU 2 which you mechanically do it is very essential for other Lie algebras, we will come to it.

The diagonal generators are untouched those generators which are not diagonal you will do this linear combination with an i factor, so that they are complex conjugate or Hermitean conjugates of each other they are not Hermitean, J plus dagger is J minus is that clear? This you do it here, but you will keep doing this when we do the corresponding Lie algebra for SU 3. For example, and then you start seeing a similarity. What you have learnt as ladder operator has some meaning when you start going to SU 3 and in general SU n or in general for any arbitrary simple Lie algebras is that clear, ok.

So, I just want to say that this pieces because of this Pauli matrices, Pauli matrices are additional properties. What are the additional properties? You can write sigma I sigma j as epsilon ijk sigma k right. Using these properties the group element of this special unitary 2 cross 2 matrices there can be written as cos of mod theta by 2 plus i sigma dot n hat, I will tell you n hat is this theta vector is mod theta h hat ok, direction you can pull it out time sin theta by 2. This is only the magnitude. This most of the quantum mechanics students middle of the sleep I am sure you will be able to say this yes or no? No, I can give it in exam then, ok.

So, this is something which you all know, ok. So, that is what I have shown it on the slide for you – exponential map can be rewritten as cos here I am taking theta to be the magnitude and putting the unit vector by the side, ok. I did not write it as a theta vector. So, that is why I used a magnitude to be theta cos theta by 2 times identity. In fact, I should identity here, 2 cross 2 identity and sigma dot n hat is a 2 cross 2 matrix. So, this is the one which was useful to show that when you take theta 2 pi, 2 pi it was not identity just like in your rotation group you found r of 2 pi. So, let me write the map between SO 3 and SU 2.

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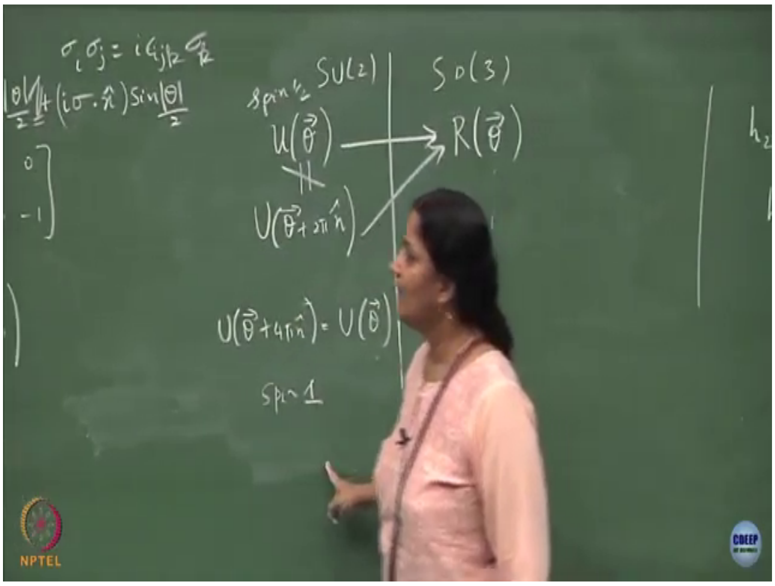

So, SU 2 and SO 3 here you can write a rotation by an angle theta, here you can write a rotation let me call it as some u by an angle theta. u by an angle theta is given by this exponential of theta dots half sigma by 2. So, there is a one to one correspondence between rotations in physical space and rotations in an abstract two-dimensional spin space and this was necessary because experimentally Stern-Gerlach experiment showed this that there is a two-dimensional spin vector space.

The spin information about spin the first experimental evidence was from Stern-Gerlach experiment that is why we started doing this, right. So, it is a two-dimensional vector space here, but that does not matter, but there is a one to one correspondence rotation in the internal spin space is given by this kind of exponentiation and rotation in the physical space by the same angle is you know what is that rotation. Is this one to one on 2? If it is 1 to 1 on 2 you will call it is an isomorphism right. Is it one to one and on?

That is where I am trying to tell you that U of theta plus 2 pie times the same unit vector if you do it on the theta is theta times unit vector, what do you get? You get it to be a distinct element from here whereas here it is same. It is same as R of theta plus 2 pi times n hat. This is not equal to, that is not equal to. So, what is that tell me? A physical rotation by an angle theta you cannot distinguish in the spin space it could be theta or theta plus 2 pi. These two elements are map to one element, clear? Not just single element.

It is not 1 to 1 on 2, SO 3 and SU 2 group elements are not an isomorphic group. They have three generators the algebras ditto for both you know that commutative between l x and l y gave you I l a z, sigma x, sigma y commutated gave you i sigma z the algebra is the same. You would have that the group should be a isomorphic. What you see is that explicitly when you write the group elements, the group elements when you do theta by theta plus 2 pi in a same direction, you should have got back the same g of theta, but you do see that these two are not same come that here expression.

Student: Should we from the other way on that R theta and R theta must be going back to (Refer Time: 27:17)?

R theta and R theta there is no distinction. So, I do not even need to write that it is equal.

Student: (Refer Time: 27:25)?

Yeah, because physically in this physical space a rotation by theta and rotation by theta plus 2 pi are one and the same, but in the vector space in spin space they are two distinct elements, ok. So, the two distinct elements are mapped to one unique element in the physical rotation space. So, that is why he said these two are not same as far as this list is going this element is distinct from this element, but if I want to see what happens in the physical space that rotation is not distinguished. So, you will get the same element ok. So, that is why I wrote that.

So, in fact, it is a set any arbitrary theta. So, you can try and say that two elements here will always get map to one element there and this is one and you also know that U of theta plus 4 pi times n hat will be same as U of theta, ok.

Student: Now, this is applicably redefine the as then greater sigma of sigma, right?

Yeah.

Student: So, for the SU 2 in general we cannot (Refer Time: 28:50)?

That is a good question, but yeah. So, I am making some kind of analogy with what we know in the experimental data that they are half or integral particles that information is going in here. So, that is a good question. So, what is happening is that instead of taking half sigma suppose, I did not put that then I will not get this half and minus half. But, what I am trying to say is that these particles are spin half particles; if I look at spin one particle then there is no distinction. So, let me write that also here.

So, this is for spin half; if suppose I write spin 1 ok, there is a slight modification when I do spin 1. What is the modification?

Student: (Refer Time: 29:44).

So, this is not Pauli matrices, but you should write a 3 cross 3 matrix satisfying the same algebra same SU 2 algebra, just because you wrote 3 cross 3 matrix it is not SU 3 algebra, ok. So, what I am saying is let me write that.

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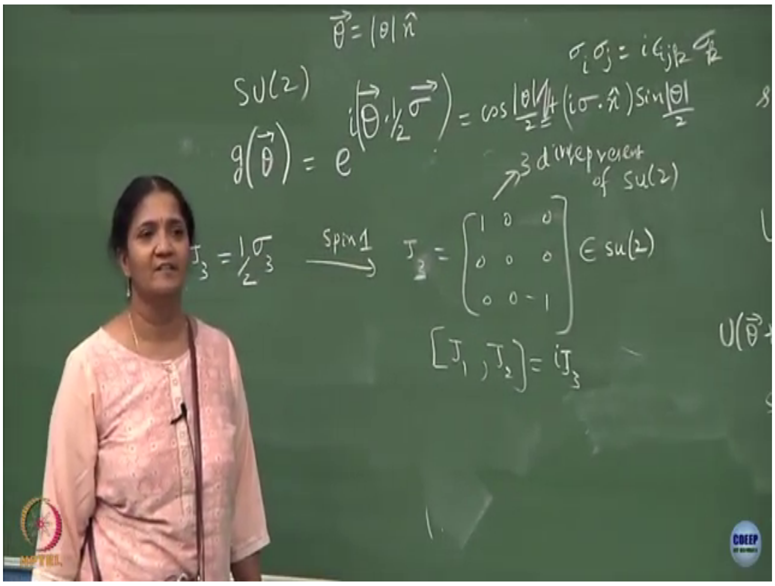

So, if you go to spin 1 then J 3 will become 1 0 minus 1, you agree? But, this J 3 is a 3 cross 3 matrix, but it is an element of the smallest small SU 2 only algebra. Why is that? If you take J one corresponding 3 cross 3 matrix J 2 corresponding 3 cross 3 matrix you will get only i J 3.

What is this? Can you tell me what this is? We did test in discreet groups. This is the higher dimensional irreducible representation of the same algebra. SU 2 the lowest non-trivial algebra will be 2 cross 2 matrix, for the same SU 2 you can have higher dimensional matrix representation and they will also be irreducible. What do I mean by irreducible? If I write the basis state for this J plus and J minus will only take a linear combination of the basis states, ok. So, this is a three-dimensional representation. In fact, I should say irrep of SU 2, SU 2 algebra. So, that is why I am using a small letter s and u 2.

Student: Ma’am (Refer Time: 32:18).

Yeah, 2 represents the non-trivial lowest dimension where you have constructed the algebra which means the nontrivial lowest dimension is 2 cross 2 matrix after that you can construct by tensor product. You can take tensor product of 2 spin half particles that will give you reducible representations, is this familiar the notation and then you have projectors and the projectors should be such that it will give you direct sums of irreducible representation, we will do that also. That is where the (Refer Time: 33:07) and coefficient curve plays the role of a projector.

So, this was the binary basis sorry, primary basis up and down and then you go here I will do take a tenser product of two up and down; let us do that anyway, I think that will give more clarity. Anyway this is clear to you that it is an two to one mapping irrespective what arbitrary spin you have you can use SU 2, ok. So, it could be fermionic or bosonic for any arbitrary thing could be a fermionic 1 the mapping is 2 2; 2 to 1. So, in that sense the SU 2 group is sometimes the parameters space when we draw that also we did this here.

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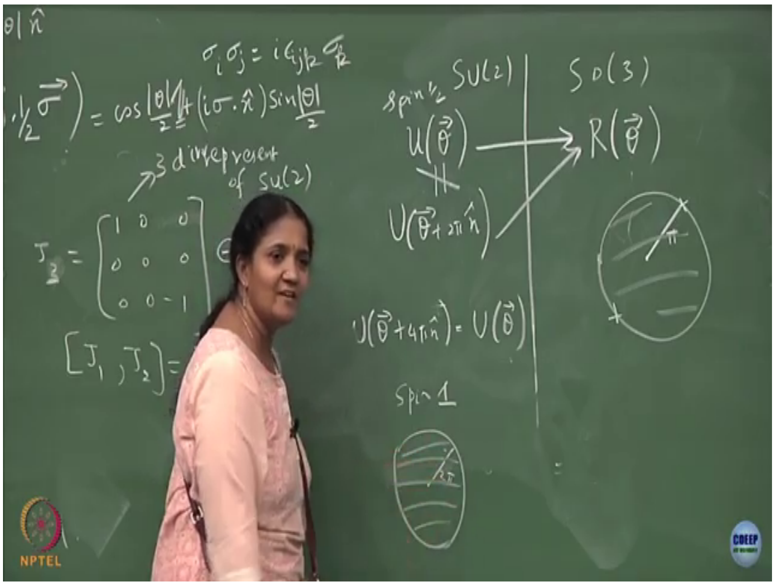

What was the parameter space? It was a sphere which goes up to radius pi solid sphere with diametrically opposite points identified. Here because of this property, it will also be a solid sphere, but the radius will get minus 2 pi to plus 2 pi. So, you will have 2 pi as the radius solid sphere and all the points on this are identified with each other, because all the points are identified if you try to make any non-trivial curve it is going to be only simple, it will not be double connected kind off parameters space ok. So, this is what I have tried to put in a slide, let me get to the slide.

Now, unlike SO 3 g of 2 pie is not equal to g of 0, I just put that theta to be 0, but g of 4 pie is equal to 1, this identity. So, SU 2 group manifold is a solid sphere of radius 2 pi which is simply connected that manifold the topology of that manifold is simply connected. And, two elements of SU 2 gets mapped to one element of SO 3 for an arbitrary spin, ok. You should not specifically look at integers spins and argued it is one to one on two spin is not specified for an arbitrary spin you do see that in general two elements can get map to one element.

So, this is why you call SU 2 as a double cover of SO 3, ok. Many books will say it is an double cover of SO 3, but this is the meaning of it that there are two elements in general for any arbitrary spin which gets mapped to one element here. So, it is also homomorphous, it is not an isomorphism, is that clear.