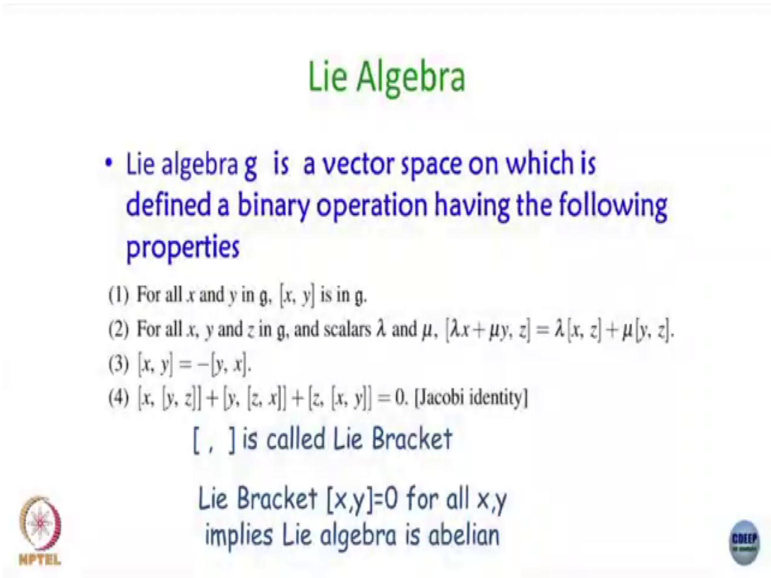
Group Theory Methods in Physics
Prof. P. Ramadevi
Department of Physics
Indian Institute of Technology, Bombay

Lecture – 41
gl(2,C) and sl(2,C) groups

(Refer Slide Time: 00:16)

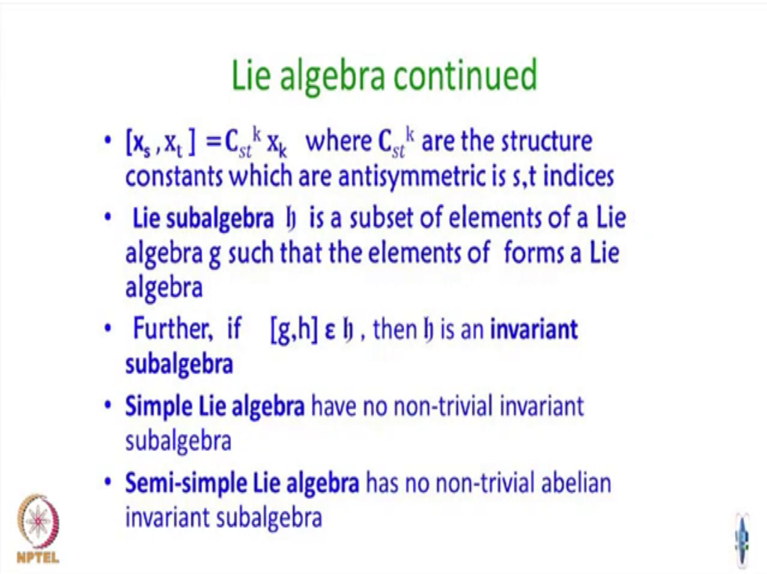


So, I will continue with Lie algebra and then Lie groups, as the warm up of recapitulating whatever we were doing in the last lecture. So, Lie algebra is also vector space and the operation is going to be a commutative bracket operation which we called as a Lie bracket and that vector space will have some basis states ok.

So, if x and y belong to that vector space and the commentator of x and y is also in that vector space that is one of the properties and then linearity, any scalars lambda mu. It could be complex or real depending on that you will call this as really Lie algebra or complex Lie algebra. They will satisfy the commutative bracket or the Lie bracket will also satisfy this linearity property. And of course, the Lie bracket by definition if we interchange the 2 elements, it is going to be negative of itself.

And, the Jacobi identity is a cyclic property which is naturally obeyed if you write it out, expand it out explicitly you see that every term will cancel with another term; so, that the right hand side will become 0. This your all familiar in Poisson brackets in classical mechanics and also the Lie algebra has this Jacobi identity properly ok, the elements of the Lie algebra. If the Lie bracket is 0 for all the elements then you say that Lie algebra is a abelian. Examples was your translation group, the Lie algebra was involving linear movement and that was an abelian algebra ok.

(Refer Slide Time: 02:11)

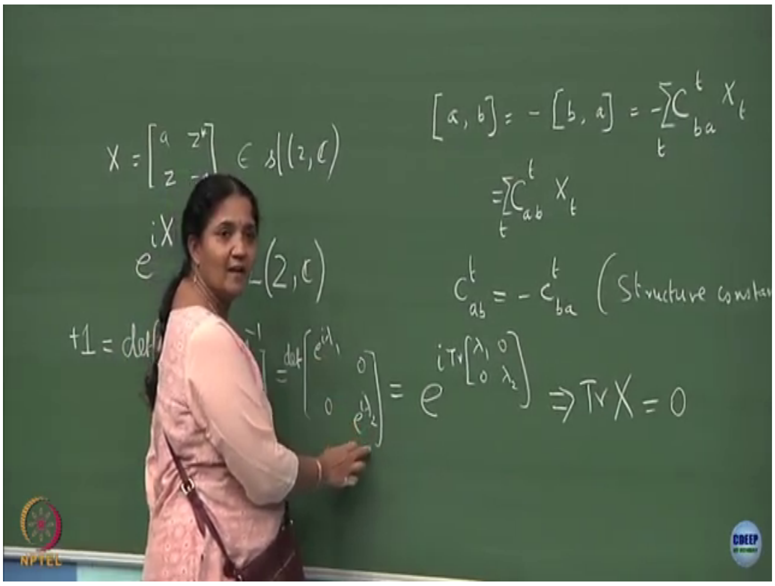


And, then I went on to say that if you take any 2 elements, this is the property 1 that it should give a linear combinations of the elements in that set. Because, this is a commutative a bracket which is antisymmetric you can show that when you interchange s and t, this coefficient which is what we call it as a structure constant is going to pick up a negativity ok. Its antisymmetric in s and t indices and Lie subalgebra is a sub set of those elements in that Lie algebra.

Just like this study Lie groups subgroups of discrete groups, you also have subgroups of Lie groups, you also have subalgebras of Lie algebras. And, the definition is that that subset if you write the commutative bracket within that subset, it will always belong to that set, that subset. So, a Lie sub algebra h is a subset of elements of the Lie algebras such that elements form the those subset element itself from a closed algebra. By closed algebra I mean that subset elements will satisfy this property ok.

So, then you call it as a Lie subalgebra is that clear and then I went on to say that there is the subalgebra could be an invincible subalgebra. If you take any element outside that subset and take the commutator with the subset elements which is subalgebra. This commutator should get back to the subalgebra, should be any element in that subalgebra. If that is happening then we say that the invariance, sorry the subalgebra is an invariant subalgebra is that clear. I did not add two more points, today I tried to add that also.

(Refer Slide Time: 04:35)



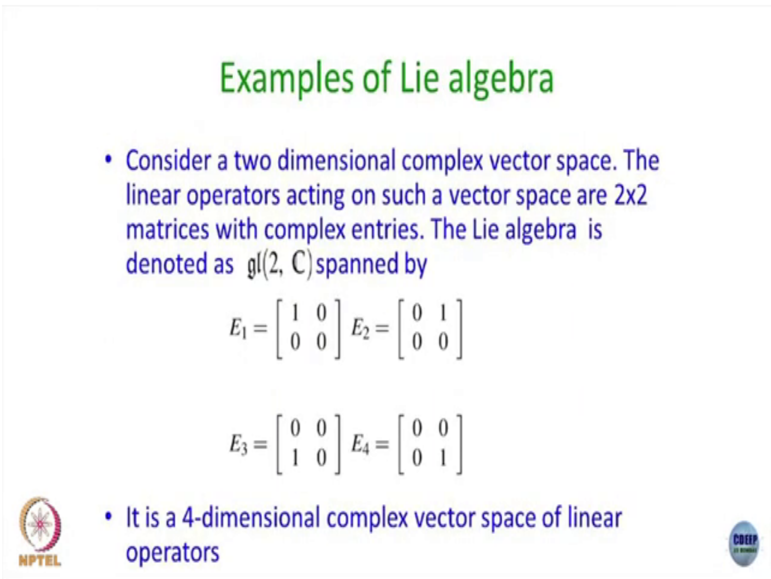
And, with a commutator if we expand it is right if you take a b; so, this by the Lie algebra as I said I can write it as C, the meaning is inbuilt that it is summation over t ok. But, what you can see is this one you would have written it as X t right, these two are equal. So, what does it show that C t a b is minus of C t b a. So, this is antisymmetric in the two indices ok.

So, structure constants this is structure constants are antisymmetric with respect to a comma b indices is that ok. So, simple Lie algebra has trivial invariance sub group is just an identity element or the whole group ok. We should have a non-trivial invariant subalgebra, if you do not have a non-trivial invariant subalgebra, then the Lie algebra is called as a simple Lie algebras. Whatever you are studying, the angular momentum algebra, rotation algebra associated with the rotations all of them are simple Lie algebras.

You cannot find a subalgebra, you cannot find a invariant subalgebra, non-trivially invariant. Trivial is identity element in the whole set that is useless, you have to find something which is non-trivial ok. So, in that since all the Lie algebras which we are going to confine ourselves mostly here are simple Lie algebras. Then there is more variant, if you find an invariant subalgebra you have to check whether that invariant subalgebra is an abelian subalgebra.

So, if it does not have a non-trivial abelian invariance subalgebra, then the group the algebra is a semi simple algebra ok. So, these are some of the jargons which is which you will see in any books, but we are going to just confirm to simple Lie algebras where there are no non-trivial invariance subalgebras ok.

(Refer Slide Time: 07:31)



And, this I went through and some students said I went through very fast. So, I thought let me just take you again through this because, this is a concept which if you are the first time learners then it is nicer to get it understood. So, typically when you write 2 cross 2 you know invertible matrices. What do I mean by invertible matrices? Whose inverse exists right, otherwise if you do not have an inverse existing then you are getting into problems clear.

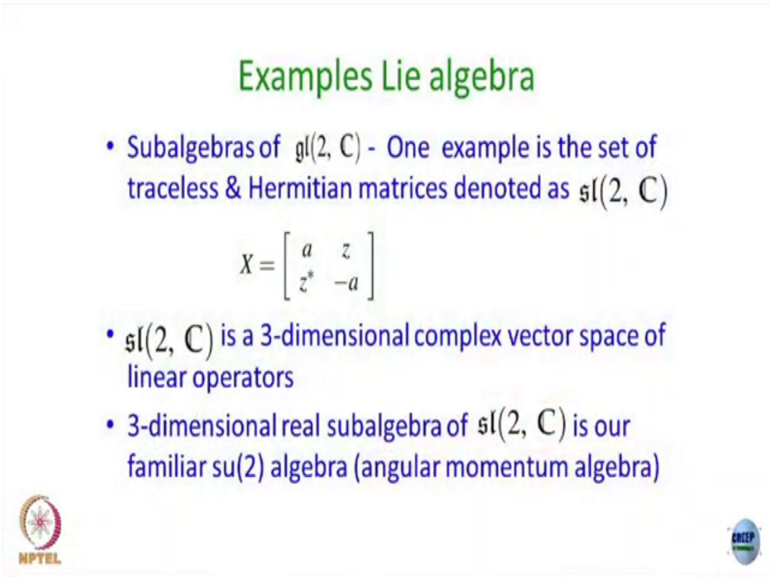
So, take a set of 2 dimensional complex vector space and let the linear operators acting on those spaces are 2 cross 2 matrixes. This is not new, I have taught you in discrete groups that depends on the dimensional of the vector space; then you have operators whose matrix representations will be 2 cross 2 if it is a 2 dimensional representation ok. So, the Lie algebra is formally it will have 4 elements and the 4 elements can be complex.

So, gl 2 C denotes 2 cross 2 matrix with complex entries ok. In order to generate an arbitrary element you can take these 4 independent basis and the coefficients which you can write on an arbitrary element is a complex coefficient which means it has real and imaginary. So, that is why I am saying that the corresponding vector space of these linear operators for the gl 2 C is 4 dimensional complex vector space or 8 dimensional real vector space ok.

Its equivalent 4 dimensional complex numbers will have real component and an imaginary component. There are 4 bases which is required to span the elements of gl 2 C, g for general, l for linear, 2 to denote 2 cross 2 matrices and C to denote the entries of the matrices can be complex ok.

So, if I want to say what is it in the notation of real notation, I will write E 1 E 2 E 3 E 4; on top of it I will write another 4 where 1 is replaced by i ok. Using those 8 matrices I can generate any arbitrary general linear 2 cross 2 matrices with complex entries clear. So, in that sense it is a 4 dimensions complex vector space.

(Refer Slide Time: 10:39)

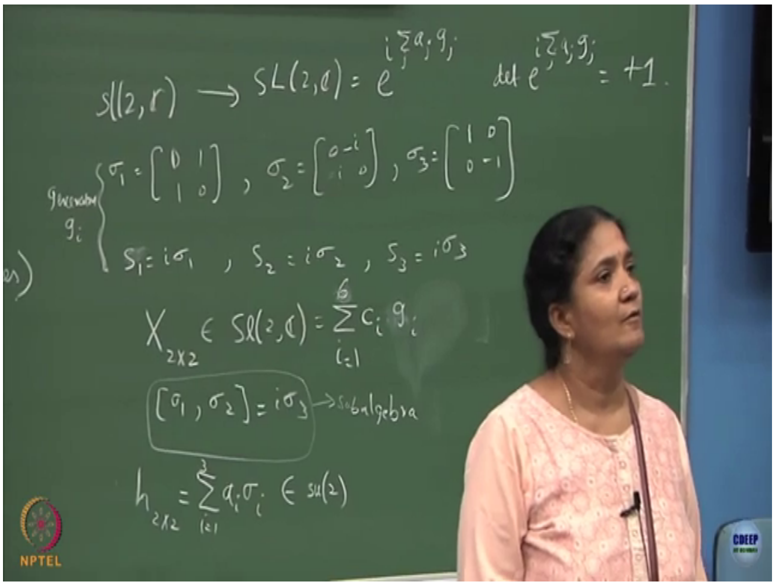


Then I went on to saying that I wanted to look at subalgebras of these 2 cross 2 matrixes with complex entries. One subalgebra which I could look I am not given why I am looking at traceless Hermitian, suppose I look at traceless Hermitian matrices. For example, suppose I want to write down a matrix which is a 2 cross 2 matrix tracelessness will force you that this has to be opposite to this Hermitian will force the diagonal elements to be real right and, off diagonal element should be complex conjugates of each other ok.

So, essentially you have only 3 independent possibilities here, earlier you had 4 independent possibilities because there was no restriction. But, once I put a restriction you do see that I have to write only 3 independent basis clear; I cannot do anything more than that.

So, this essentially tells me that sl 2 C, this is what we call it as a sl 2 C algebra or the elements of the sl 2 C algebra involves set of 2 cross 2 traceless Hermitian matrices ok. And, these matrices the number of independent basis you can have its a 3 dimensional complex vector space because, you can still put complex coefficients ok. On top of it I want to see the sub algebra of this sl 2 C's ok.

(Refer Slide Time: 12:33)



So, sl 2 Cs so, you can write generators are your poly matrices.

Student: (Refer Time: 12:41).

Yeah.

Student: (Refer Time: 12:42) other two (Refer Time: 12:48) are the components.

Which other two?

Student: From the.

Can be the (Refer Time: 12:51) factor.

Student: sl 2 C if we define a and z, the other two (Refer Time: 12:55) completely different (Refer Time: 12:57).

Yeah z is complex. So, it has two more non-trivial entries in it, that is the meaning of it ok; a is real, z is complex. After you write any arbitrary element will be a linear combinations their basis states which have complex coefficients you understand. So, that is why it will become either 6 real dimensional vector space or it will be a 3 complex dimensional vector space is that clear.

So, sigma 1 sigma 2 sigma 3, did I make a mistake on this i? Is that right or 0 minus i and i 0? All minus i right ok. So, this is one set of generators, all of them are traceless and Hermitian right and then as I said that it will be you can also write s 1 which is i times sigma 1, s 2 which is i times sigma 2, and s 3 which is i times sigma 3 ok. So, it is 6 real dimensional vector space ok..

What I mean by that is any 2 cross 2 matrix which belongs to sl 2 C we should be able to rewrite it as 1 to 3, some complex coefficients of these objects which I can call it as sigma 1 plus s 1 kind of sigma i plus s i clear. So, this will be a you can show that these will again be these are Hermitian and traceless, any linear combination with coefficients will also make it Hermitian and traceless. So, it is not a problem ok, now you look at a sub algebra of this.

Student: The coefficient will be real.

Coefficient will be once I write it as this and this ok. So, I should be little careful here this plus sign is confusing; so, let me write it like this. So, it is coefficient C i; so, let me right 1 to

6 C i times these generators, let me call them to be what is the best notation I can use. Let me call them as generators as g i let us say, all the 6 r. So, it is C i times g i ok, then you can keeps C i to be real.

This is probably a better notation, adding those two may not be a right thing with the common coefficient is not right ok. It is a 6 real dimensional; so, these coefficients are real now, 6 real dimensional vector space ok. So, these 6 are the once which is going to generate for you the algebra of sl 2 C clear. So, the algebra of this sl 2 C involves sigma 1 sigma 2 sigma 3, you have s 1 s 2 s 3. And which one is a sub algebra? Both of them are sub algebras of it you agree, sigma 1 sigma 2 sigma 3, it closes amongst itself right.

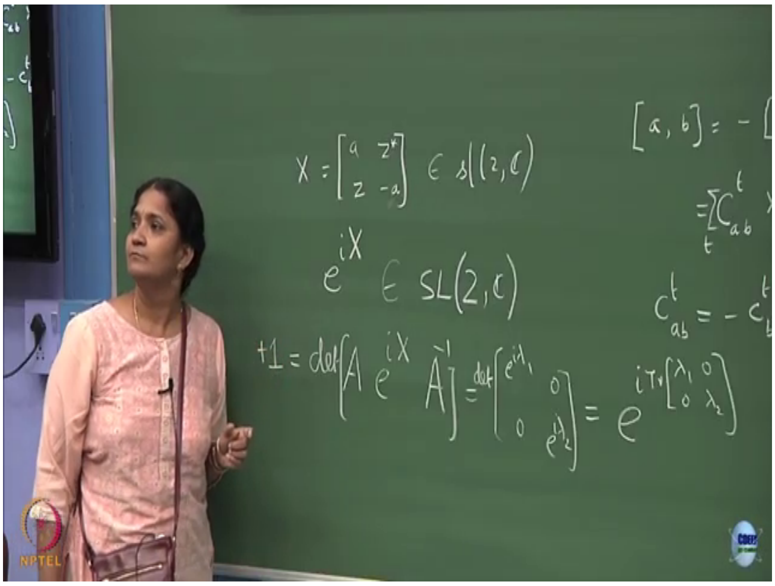
Sigma 1 sigma 2 is i sigma 3. So, this is a definitely a sub algebra ok, it does not mix these two. So, it is definitely a sub algebra and you can look at elements generated using the sub algebra. So, those are what I call it as elements which let me call it as some h which is 2 cross 2 will be summation over a i sigma i ok.

So, these are arbitrary elements involving only the subset of those Lie algebra elements of sl 2 C. And, they will generate for me your familiar angular momentum which you have studied which is going to give you an element of Lie algebra which I am going to denoted as small s small u under 2. So, any element of this Lie algebra which is a linear combination of sigma 1 sigma 2 sigma 3 will be traceless, will be Hermitian ok.

And, why we had introduced a u will be visible when I do the group elements ok. So, we will see why the u has been introduced. In the earlier case I passingly said that tracelessness means determinant of the corresponding group element will be 1 ok. So, I said that this is Lie algebra, if you go to a group which is sl 2 C; this will be an exponential of i ok. Any of these are linear combinations of them, if there are 6 generators, then you will have 6 real parameters.

So, you can write it as summation over j a j times all this g i g j’s, g j’s is what I am calling it here. So, this will be the element of the group which is obtained by exponentiating the generators of the Lie group which are going to determine your Lie algebra. And, this element must have determinant to be, this has to be plus 1 that is the meaning of saying it special linear group ok, that is the important requirement. And, once I put that condition on the determinant of an exponential try to think of it as matrix representations, determinant when you do a diagonalization ok.

(Refer Slide Time: 20:56)



If you do a diagonalization of this matrix; so, let us take a 2 cross 2 matrix. Let us take one element in the Lie algebra as 2 cross 2 matrix, only thing what we remember is that it should be traceless right and z star and z. This is an element of Lie algebra sl 2 C you agree. Now, I want to exponentiate this, let us take one element; I am just taking an exponential of that.

And, I want to say that this is an element of the group sl 2 C ok. Suppose, I do a diagonalization here, I find a matrix A e to the i x A gives me a diagonal matrix. So, let me call it as e to the i lambda 1 e to the i lambda 2 right A inverse, I can always do this on a given matrix. If I take the determinant of this; determinant of this what does this mean?

This is e to the power of i trace of lambda 1 0 0 lambda 2, you all agree. This expression is a same as that expression ok. So, what I have tried to show is that the determinant of the matrix if it belongs to sl 2 C, the determinant of this has to be 1 ok. So, this belonging to sl 2 C means that the determinant has to be plus 1.

Student: (Refer Time: 23:08).

What does that reflect on the e to the i x?

Student: (Refer Time: 23:16).

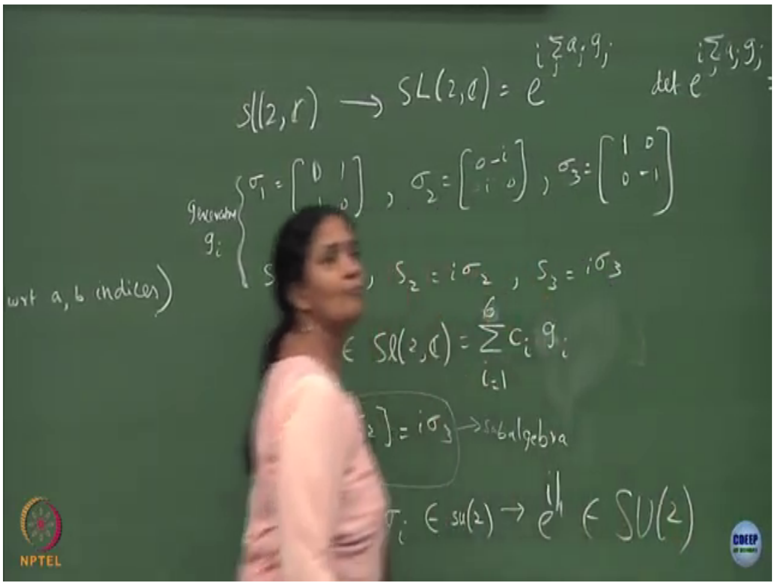
It has to be.

Student: Traceless.

Traceless. So, that is why automatically they put in a small letter s and said I am looking at traceless matrices ok. So, this is the implication between the path from the Lie algebra to the Lie group, the exponential map which I am doing.

And, this clearly shows that this implies trace of X, because you can map this e to the i x right, trace of X is equal to 0 yeah, is this clear ok. So, now the additional thing is suppose I want to look at unitary groups SU 2, what do I mean by SU 2? So, let us now take this one.

(Refer Slide Time: 24:23)



The corresponding element of this e to the i h will belong to SU 2 clear. Now, whatever I said here tracelessness condition will be satisfied again; h should be traceless because the S denotes determinant has to be plus 1 ok. On top of it U denotes what? Just like in rotation orthogonality relation now U denotes unitarity.

(Refer Slide Time: 25:05)

So, I have to make sure that e to the i h; e to the i h dagger, it should be equal to identity that is unitary; unitary matrices must satisfy is that clear fine. So, now, you can because these two are same elements in the exponential you can add them. So, e to the i h dagger i will become minus i ok; so, minus i h dagger has to be equal to 1. What does that tell you?

Student: h has to be (Refer Time: 26:08).

h has to be, this implies h should be h dagger that is Hermitian ok. So, I have just driven you through some complex path starting with arbitrary linear 2 cross 2 matrices with complex entries. I tried to say what are the basis states which forms a vector space on which you can write a Lie algebra. Then I said sub algebras with determinant plus 1 will sit inside this general linear algebras and you can exponentiate these elements.

And, if you impose the determinant is plus 1 then you can say that the Lie algebra elements have to be traceless ok. And, then we also saw that from the generators a sl 2 C you saw that you also have a sub algebra and you can generate the corresponding Lie algebra for those sub algebra which is denoted as SU 2. Why it is u is by exponentiating it and seeing it as a group special unitary group involving 2 cross 2 matrices which is determinant plus 1 and it should be unitary. Just given a unitary matrix to you with determinant plus 1 how many independent parameters you have?

Student: (Refer Time: 27:45).

Given a 2 cross 2 matrices you have 4 complex entries and then you have to try and see putting a determinant condition and unitarity condition. Just like we did it for orthogonal groups, you should do it for even for unitary groups. If you do that you will find that there are only 3 independent parameters for SU 2 is that right clear.

Student: Yes, (Refer Time: 28:21).

Traceless even in the generator language you can see, you do not need to see in the group elements, group elements also you can do 2 cross 2 matrices. There are 4 complex elements, determinant plus 1 is one constraint, unitarity is another constraint ok. So, unitarity will have off diagonal, diagonal everything and then the determinant will be one more constraint ok. So, you can check that out and see that the number of parameters is 3 for SU 2, yeah.

Student: A special unitary can be (Refer Time: 29:00) condition.

Special unitary is a group.

Student: Special (Refer Time: 29:06) just for sl 2 do we need (Refer Time: 29:08)..

S l we do not need Hermitian condition we do not need it, but if I want to look at SU 2 as a sub algebra there; I am just trying to look at if you take a sub algebra then some of the properties have to be continued. So, in that sense here this turns out to be a Hermitian, I do not have a reason why it should be Hermitian.

Student: sl (Refer Time: 29:31).

For sl in general only determinant has to be plus 1 or in the algebra it should be traceless yeah.

Student: Here sl like if you take s 3 and sl its not Hermitian (Refer Time: 29:42).

What is s 3?

Student: s 3 is i sigma 3. It will actually be anti-Hermitian.

Yeah. So, these are in the real direction base of writing it, if I do not put this in I will take these 3 only and take the coefficients to be complex; you understand what I am saying. If I take it be real then you see it to be anti-Hermitian I agree with you, but this equivalent statement between complex vector space and real vector space is if it is a n-dimensional complex vector space its 2 n dimensional.

So, if you put the I factor automatically it becomes anti-Hermitian, I agree with you yeah. But, you start with only saying I am looking at a complex vector space and then these are the 3 generators with complex coefficients ok, in that sense it will be even there coefficients can be a.

Student: Complex.

Complex right. So, then Hermitian things you need to worry about it there, I agree. The generators are traceless and Hermitian that is all I can say ok. So, I do not think I need to put in the condition of Hermitian for the sl groups, but when I come to SU groups Hermitian comes up natural, is that clear?

Student: (Refer Time: 31:11).

Yeah. Any other question? Yeah, that is the good point sl groups you only want determinant to be plus 1, there is no Hermitian requirement. But, SU because of the unitarity you do see that you get that h and h h should be equal to h dagger where, h's are the generators of the Lie algebra. So, are you all sinking the fact now what is Lie algebra, what is Lie group?

Student: Elements.

Elements of Lie group are exponentiating elements of the Lie algebra and the Lie algebra elements, if you take any 2 elements you do a commutator. You have to get a linear combination which satisfies those same properties ok, that is why it forms that Lie algebra; is that clear? I have given you some examples just to clarify things for you.

Student: (Refer Time: 32:08) sl 2. It was for sl 2 determinants plus 1.

Yes, once I put a letter s for the Lie algebra it is traceless which is equivalent to the group capital S to be determinant to be plus 1, that step is what I have shown you here; that you can always diagonalize the matrix. When you diagonalize this one will have eigen values because, these elements are e to the i x; I have written e to the i lambda 1 and e to the i lambda 2, are the 2 eigen values. So, determinant of this can be also written as e to the power of i. So, determinant is e to i lambda 1 into e to the i lambda 2. If you take a trace it is lambda 1 plus lambda 2, it is e to the i lambda 1 into e to the i lambda 2.

Student: (Refer Time: 32:58) s o also (Refer Time: 33:00). So, what is the (Refer Time: 33:03) s o s and.

Yeah s o will also be some sub sub algebra of this sl.

Student: s l (Refer Time: 33:10).

Yeah. So, only thing is there it is a orthogonality conditions and the coefficients have to be real and there will be some sub algebra of these sl groups ok. So, there you have to look at 3 cross 3 matrices, even though I have consider 2 cross 2 you can do this for gl nc; I will I will just mention those notation ok.