Group Theory Methods in Physics
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Lecture – 40
Subalgebra of Lie algebra

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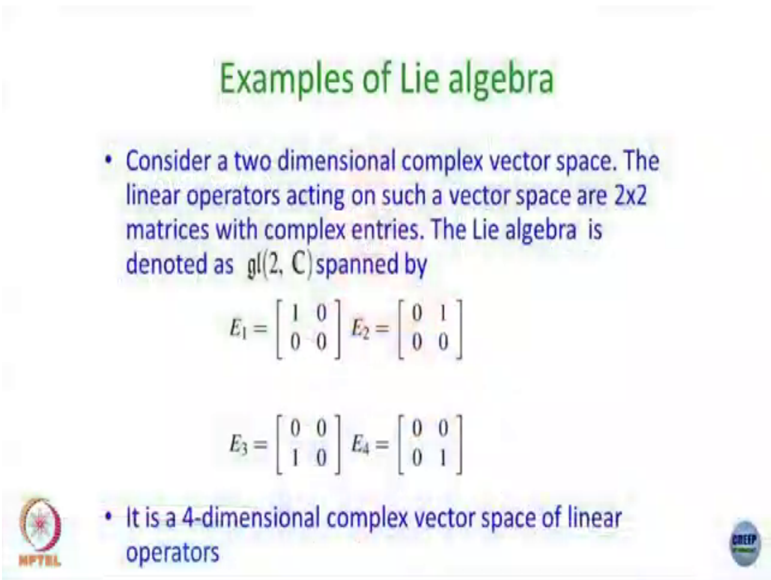

Now, let me come to some familiar examples of Lie algebras. You are all fine? I think many things are known to you in some pattern, I am just only fixing the notations that way you will start not getting scared with the notations. When you see an SO n or a SO n comma m in any book you will not get scared I do not know this you will not see.

So, take a 2-dimensional complex vector space. Till now I was trying to look at only a real vector space let us go to a 2-dimensional complex vector space ok. Then you can have matrices which are linear operators operating on this 2-dimensional complex vector space.

What should be the dimensions of those matrices? 2 cross 2 matrices, but the entries can be complex now ok.

How many independent real entries are possible in a 2 cross 2 matrices? 8 elements right 2n cross 2n. So, you will have 8 2n squared elements. So, there will be 8 independent real elements possible ok. So, now, you can try and look at what are the generators for such set of 2 cross 2 matrices with complex entries that Lie algebra is written as gl small g, small l 2 for 2 cross 2-dimensional matrices acting on a 2 dimensional complex vector space.

So, that complex vector space is put in here on the side as gl 2, C ok. So, I told you that whenever I am looking at Lie algebra use small letters. So, gl and 2 comma C. Just like you write bases states you can write 4 such base states, but you have to remember that you have to multiply with coefficients which are complex coefficients, instead you can also write 8 bases states with similar thing multiplied with an I fact and then there will be 8 base states for the algebra which corresponds to 2 cross 2 matrices with complex entries.

So, what do I mean by base states? Any arbitrary transformation we can write it has a linear combination of these base states. This is what you have done at some point in quantum mechanics, but as of now I am trying to tell you that it is these 4 and again you can multiply with I factor and say that it is an 8-dimensional real vector space the set of linear operators or 4-dimensional complex vector space. Take it as 4-dimensional complex vector space you denote it by gl 2 comma C.

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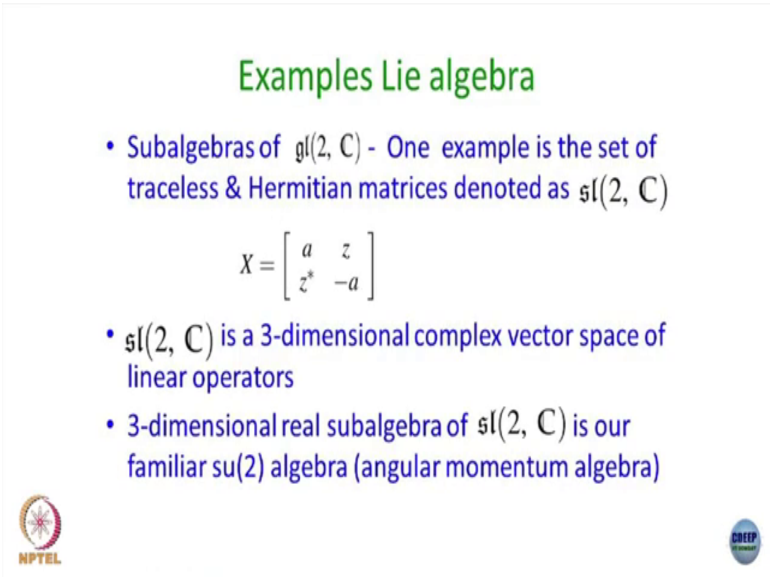

Subalgebras of gl 2, C you can look at a subset over on top of this general 2 cross 2 matrices with complex entries, you add some constraints traceless and it should be Hermitian ok. From where it rings the bell you use this traceless and Hermitian.

Student: Poly matrices.

Poly matrices, but I am not doing that here. I am just trying to put in why a traceless gets the letter s, do you know? Remember, your rotations if I say SO 3 rotations and if you want to write this SO 3 rotation as exponential of I times something you have to see what condition determinant equal to plus 1 imply on the trace of the matrix.

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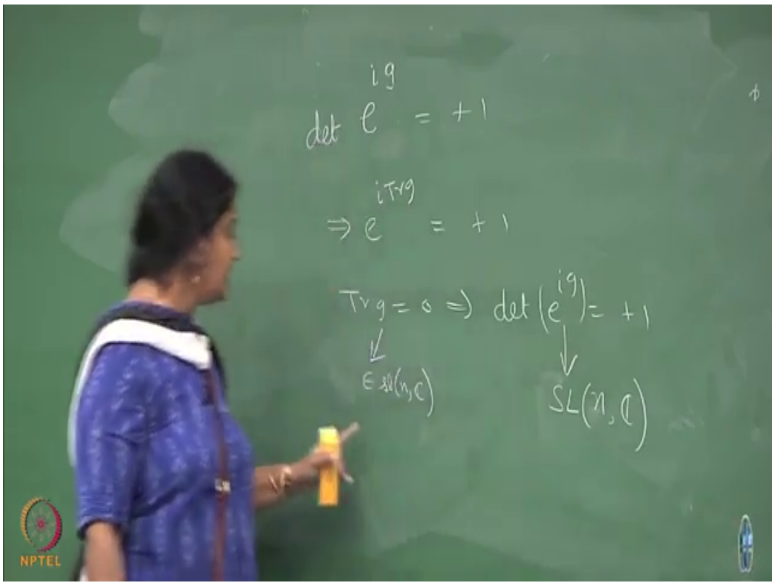

So, if I say e to the power of ig is I say determinant of this is the group element g is the element of the Lie algebra, if I want this to be plus 1 this can be re-written as these are acting on arbitrary vector space right, but you can always find that this is equivalent to writing it as. So, the generators which constitute the algebra if it is traceless, the exponential of those generators which gives you the group element will be having determinant plus 1, is it correct?

Trace g equal to 0 implies determinant of e to the ig equal to plus 1 and this is the group element ok. Group element I will denote it by here I will denote this group element by SL and if it is acting on n, C and these g's are elements of small sl n, C.

That is the notation we follow. Lie algebra elements are given by small letters and group elements are got by exponentiating the Lie algebra elements and they have to satisfy if it belongs to special this L for linear, S for special, g is for general. If it does not satisfy this then you will call it gl n's ok. So, all your SO 3 is a sub group of O 3 ok, similarly gl n, C to start with Lie algebra we have a sub algebra which is sl n, C because gl n, C can have determinant plus 1 or minus 1 right.

Sl n, C has to have determinant plus 1. The s denotes determinant has to be plus 1 and arbitrary elements which I write here in the Lie algebra is traceless and Hermitian ok. So, you will see how many independent elements you can write you can show that because you are going to be on a complex space coefficients will be complex there will be 3 independent elements.

A is one real, z has 2 real, but then you multiply with coefficients which are again complex which is real and imaginary. In principle it is a 3-dimensional complex vector space or it will be 6-dimensional real space just like what I said for gl n, C. Gl n, C was a 4-dimensional complex vector space or 8-dimensional real vector space same thing for sl n, C.

If you constraint at the coefficients have to be real which is what you will do when you do exponential of I theta orthogonal groups have real entries right. Similarly, if you constraint that you want the coefficients to be real then it will again be again subalgebra of sl 2, C you agree?

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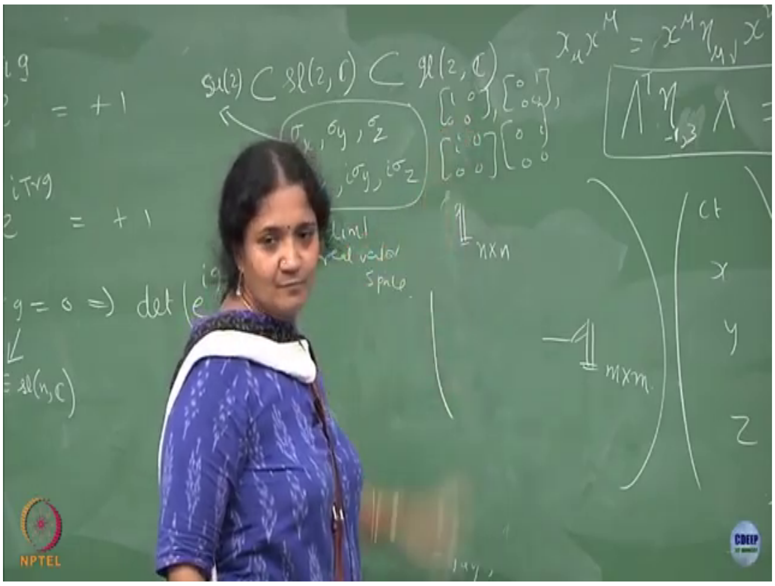

So, sl 2, C it iself is a subalgebra. Sl 2, C is a subalgebra of gl 2, C and in SU 2 is your familiar SU 2 is the real algebra which involves real entries. So, this is going to be a subalgebra of sl 2, C ok. So, set of real Hermition traceless matrices forms a subalgebra and that subalgebra is sitting on a if this is a down each one is a subalgebra of ok. This is the broader one you have to have all the matrices with determinant non-0. Why non-zero?

It should be invertible, but you will have various subalgebras and exponentiating the generators of these Lie algebras will give you the corresponding groups ok. This SU 2 which you have been extensively using which you call it as angular momentum algebra is not very different from your orbital angular momentum algebra, but this was introduced because the Stern-Gerlach experiments showed that you either have a up state or a down state which is a

2-dimensional vector space correct and you need it the generators acting on a 2-dimensional vector space ok.

So, this is why you have the SU 2 naturally occurring to understand the 2 states, 2 spin states seen in the Stern-Gerlach experiment, but as an algebra you will see that there is no difference between SU 2 spin angular momentum and SO 3 orbital angular momentum algebra you know this. All of you know this.

Student: What is (Refer Time: 11:33)?

Which one?

Student: Sl 2, C.

Sl 2, C? Sl 2, C involves entries which are coefficients have to be C as well ok. So, what you can say is that you will have your sigma x, sigma y, sigma z and you will also have an i sigma s, i sigma y, i sigma z. So, this will be your 6-dimensional real vector space. So, but if you go here it will only be one sub set not this clear?

The base states could have any coefficient, but then you have to also allow that the coefficients could be complex if you are looking at the group with complex entries and that will force you too. Base state entries could have been complex to start with, but that is what happens in your poly matrix. On top of it you also want the coefficients to be complex real and imaginary that will multiply by a 2 factor if you are in sl n, C; sl 2, C, but if you go to SU 2 it only allows 3 of them not the other ones.

Fact here also I wrote the bases here. Basically, I am saying 1, 0, 0, 0 which I write I will also have a I 0, 0, 0 ok. So, there will be 8 of them coming. I am not writing the 8 of them, but you can write. So, that is why this will be a 4 dimensional complex vector space or 8 dimensional real vector space ok.

Now, to your familiar world of angular momentum which is your belongs to a unitary group in quantum mechanics all your inner products should remain preserve that is why your time evaluation is unitary you know. All you transmissions are all unitary is what you learn and you also know that such a unitary operator for the SU 2 algebra by SU 2 now you know that it is going to be the algebra involves Hermition traceless matrices with determinant plus 1 and it should belong to real know the entries have to be real the coefficient entries have to be real. You understand what I am saying?

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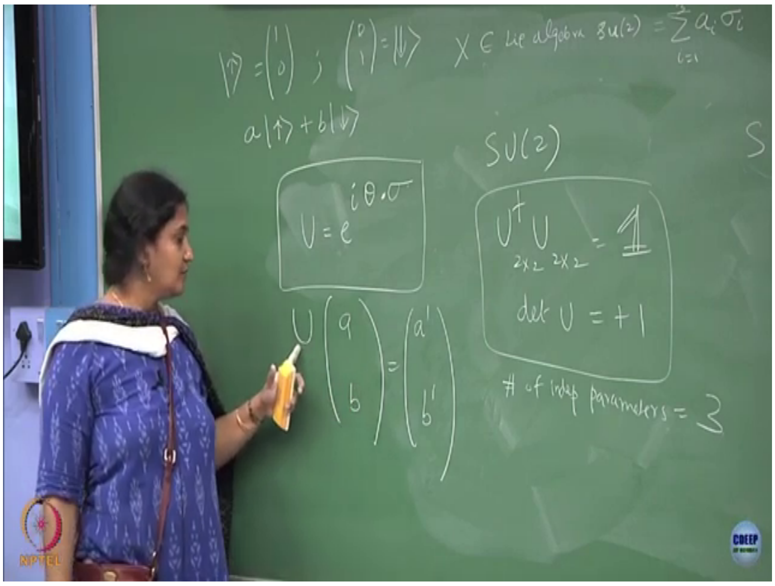

If I write in a arbitrary element x which belongs to the Lie algebra su 2 small s small u 2, it means that it should be summation over number of independent elements have to be this is real and then you have these poly matrices. This is 3-dimensional real vector space or 3 generators are required. Independently you can do it for just like I did for the orthogonal group you can define SU 2 as a group.

What is the meaning of SU 2? It is 2 cross 2 matrices with U dagger U to be identity and S to denote determinant of U has to be plus 1. You could do it from the group point of view and find out how many independent elements are there which will define for you number of parameters. Number of parameters will define for you the number of generators and those number of generators should be there when I write any element of your Lie algebra which is also vector space.

So, what is this condition going to give you? Unitary means you can try and figure it out from this condition that to start with 2 cross 2 entries and determinant will make that to be traceless. So, just work this out and see how many independent elements are there. So, number of independent parameters I will leave it to you to check this you will see that it is only 3. You can work this out for SU 3 also ok. So, do that also.

Now, I am writing group. So, if I want to see the element how will I write here, once I see 3 parameters then 3 generators I will write element U this U exponential of I theta dot; these are the bases which are the 3 generators. So, theta dot sigma clear?.

So, this will be your group element of SU 2 which when it operates on a 2-dimensional vector space a and b means linear combination what is this mean 1, 0 is up spin; 0, 1 is down spin; a, b means a times up spin plus b times down spin. What is supposed to do? It should give you some new a prime and a b prime right.

So, this is what is the transmission on a 2-dimensional vector space and this one will be the matrix and you all know how to write this exponential. So, you would have done it has an exercise in your assignment in your quantum mechanics right poly matrices have nice properties.

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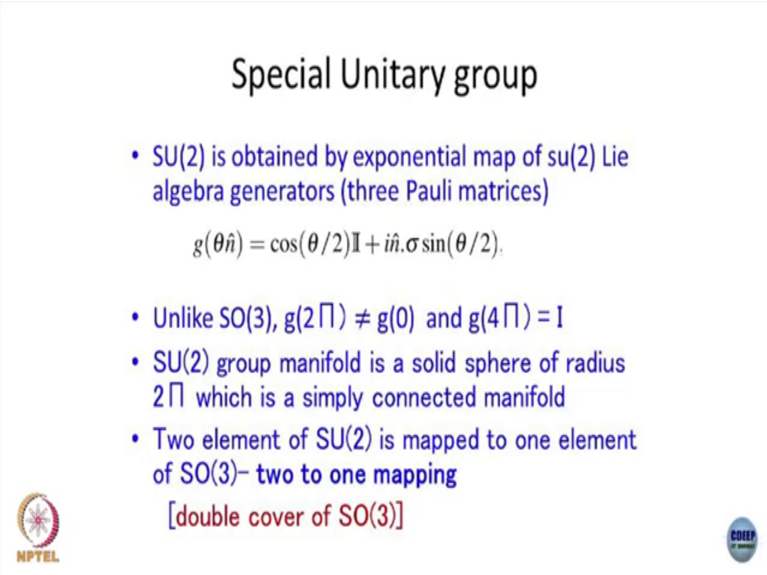

So, coming back to the slide exponential of SU 2 Lie algebra generators for example, the 3 poly matrices are good generators for traceless Harmition matrices. You do find that the group elements which is exponential of I theta dot sigma can be rewritable as cos theta by 2 identity plus I n sigma sin theta by 2. How many of you have not done this? Anybody who has not done it except for one person whom can try it out later by others? You all done this right?

So, this what is the group elements which can be rewritable in terms of sin theta and cos theta. Why am I doing this is mainly because I want to show the group elements group parameters space if you put theta to be 2 pi, what happens? If you put theta to be 2 pi you get g of 2 pi in rotation when we did rotations if you do a rotation by 2 pi it is equivalent to not doing any rotations. But, if you do a rotation by 2 pi in this 2-dimensional spin vector space it gives you a negative sign right you get an negative sign.

So, what does that mean? Means if you do a 4 pi rotation you will get back to identity ok. This is not new fermions are spin half particles, bosons are integer particles; when you exchange to fermions the wave function can pick up a sign, when you exchange to bosons the wave function does not pick up a sign you all know this. This is what is nature's way of doing things and this is what happens here ok.

So, explicitly when you write now tell me what will be the group manifold for SU 2? It is no longer a solid sphere of radius pi, it better be a solid sphere of radius 2 pi going from minus 2 pi to plus 2 pi and g of 4 pi is always identity which means all the points on the boundary are all identify not diametrically up (Refer Time: 21:44). So, that is the parameter space for your SU 2 group.

So, let me stop here and.