Group Theory Methods in Physics
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Lecture – 04
Normal subgroup, Coset, Conjugate group

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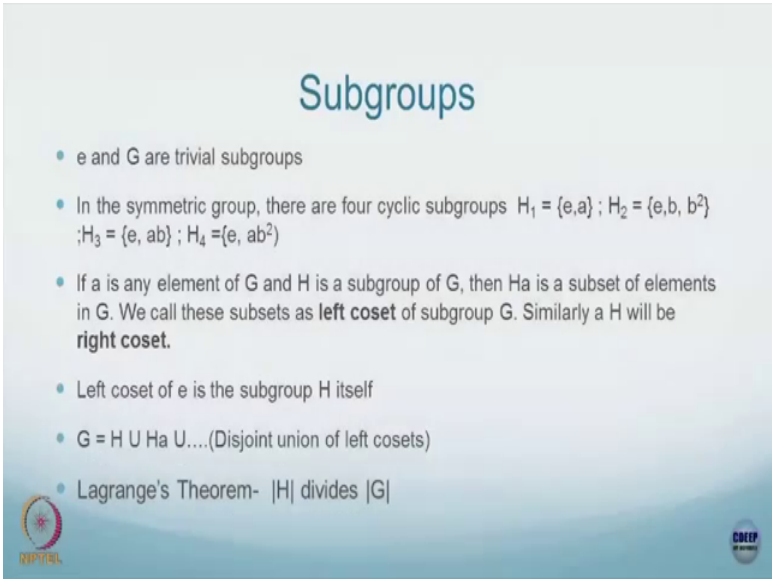

So, now get into the screen here. So, in this symmetric group which I discuss there are 4 cyclic subgroups which is what I did explicitly for you on the board. One is e, a; e, b, b square and then you have e, ab and e, ab square ok. So, now, what you can do is you can pick one subgroup ok. So, let us pick one subgroup.

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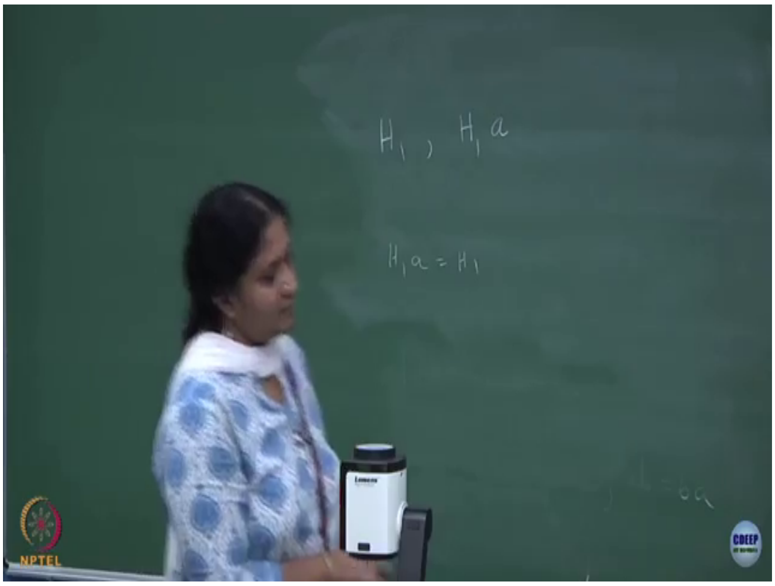

So, let us take H 1 to be the subgroup and then to H 1 you can multiply an a. So, H 1 has 2 elements ok. If you multiply an a anything happens? H 1 a is what? Is H 1 right, am I right? H 1 is e and a.

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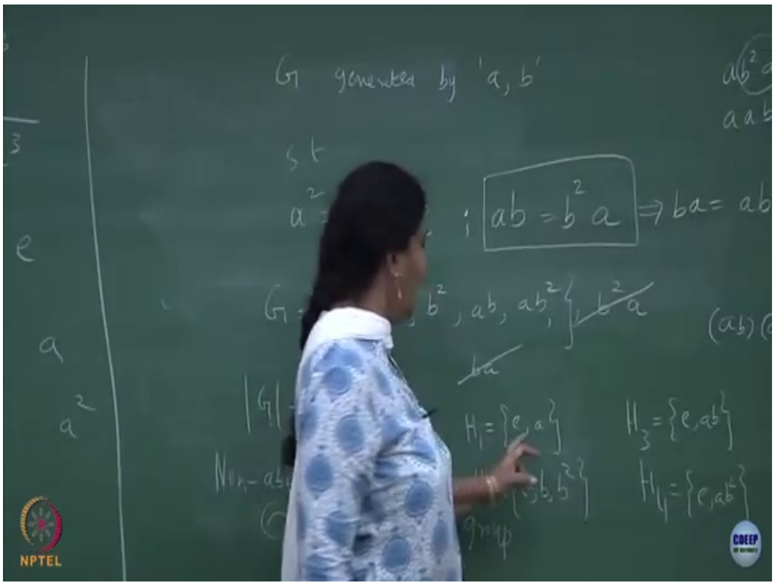

If you multiply a with that it will become a and a square will become e. So, you do not get anything out of it. So, let us remove the a.

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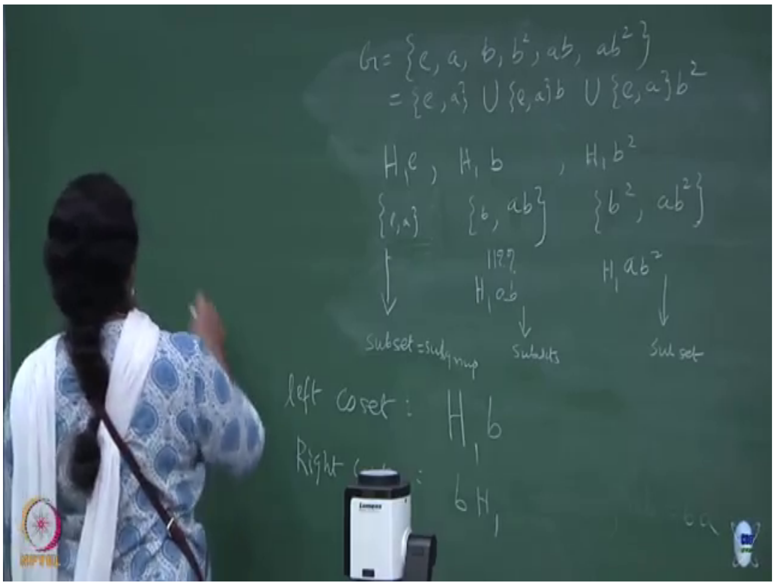

What about H 1, b? So, H 1, a is e, a; H 1, b is b, ba or ab, sorry that will be H 1, b. H 1, b square will be ok. So, you have cosets. So, this is a subset ok. Even though it is a subgroup the first element is multiplied by H 1 times identity. The second element is multiplied by H 1 times b, but this is arbitrary I could have done other things also. I could have done H 1 times ab also nobody prevents me from doing it. So, this one is same as H 1, ab is that right? Is it same or no?

Somebody? So, first element will become ab, the next element will be a square b; a square b is what? Anyway is a square is identity. So, you get back b, right. So, this is same. Whether I multiply H 1 with b or H 1 with ab you are going to get back the same set and similarly, you can show for the other case also. So, I am just given you one possibility of multiplying H 1 with one candidate, that one candidate can be anyone from this set ok. It should not really matter; like for example, here you could have taken another candidate as multiplication ok. Any other candidate from the set also will give you the same set.

What does this tell me? The choice of the element which I multiply from the right on the subgroup really does not matter. It does not take you from here. Suppose, I did b or I did with another element in that set I get back the same set, they are not talking to each other. They are disjoint sets ok. So, this intersection this is null always. So, this is a subset which is also a subgroup this H 1 alone anything multiplied with identity it is also a subgroup. These are only subsets. It is not a group, this is also a subset.

So, the group which had 6 elements the group which had 6 elements I can break the sub into e, a union e, a multiplied by one of the candidate element b which is what I am calling union e, a multiplied by another candidate b square ok. So, this is another way of writing it and once I write this that gives you a definition for a left coset.

Left coset is take some element b multiply with the subgroup. Take some element of the group, it need not be particularly b you can take it with the ab also. You can take it with ab square with a if you take it does not really give you anything new ok. So, this is what we call it as a left coset if you right multiply the group element with the subgroup.

What is right coset? Someone? You left multiply with the group element on the subgroup ok. So, suppose I try to generate b times H 1 will it be same as H 1 times b? Yes or no? Can we check?

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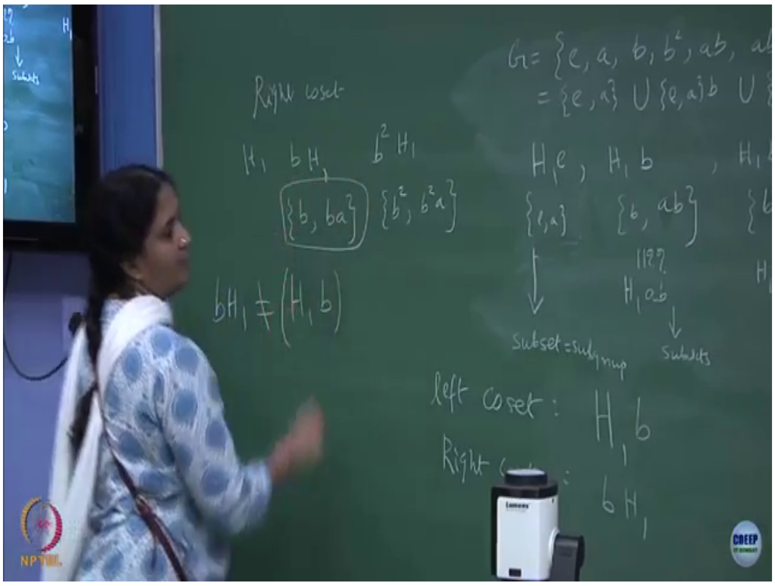

Let us do right coset H 1; b times H 1 what will that be? Is this same? What is ba? Ba is ab square right. So, what do we see that b times H 1 is not equal to H 1 times b. Left coset is not in general is that right? This one and this one are not same. One was a left coset where you multiply the group element to the right, the subgroup is in the left whereas, if you do the right coset vice versa you do see that left coset is not in general same as right coset for any arbitrary subgroup ok, is that clear?

So, I want you to do one more exercise for instead of H 1 I want you to do a similar exercise for H 2 cosets of a subgroup H 2. Can you do that?

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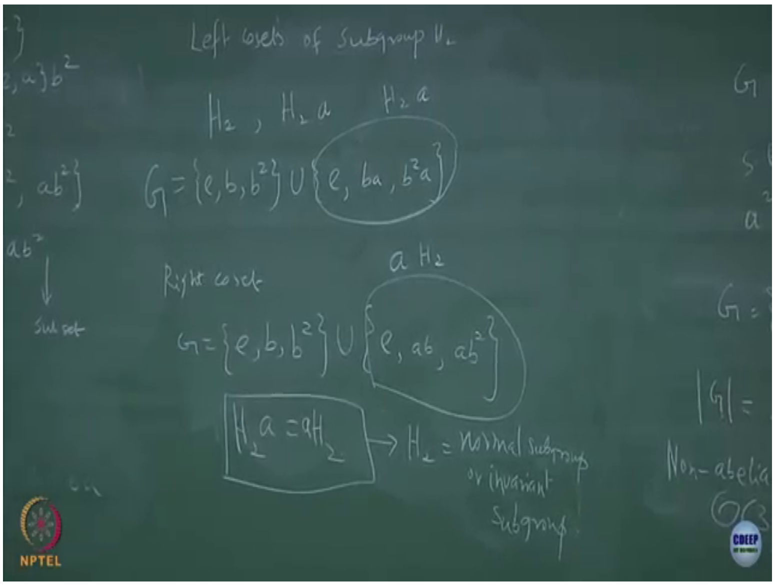

Left cosets. So, H 2 then you can have H 2 with b will give you anything?

Student: a.

It has to be with a and what else.

Student: That is it.

That is it. So, what is this? This is e, b, b squared, that is H 2. The next one is anything else? Totally there is a union of these 2 will you the group g. It is a disjoint union of these 2 clear? What about right coset? Are this 2 same? Same or different? This is a times H 2, this is H 2 times a as I set.

Student: It is same.

It is same right. So, there are some subgroups for which H 2 with any element is same as a times H 2. The left coset is equal to right coset for some subgroups. If this happens we call H 2 to be a normal subgroup or invariant subgroup. Here H 1 is only a subgroup, you cannot call it as a invariant subgroup because left coset and right coset are not same. Yeah, any questions? Is this clear? I am trying to confine to a simple examples so that the concept is clear to you ok.

So, what did we do? We initially took a non-abelian group generated by 2 generators which gave you the order of the group to be 6 and then we try to look at what are the possible subgroups in it and with one subgroup I try to do s left coset, I also did it with the right multiplication which is a right coset; I showed that in general if it is a subgroup does not mean right coset should be same as a left coset.

And, then I took another subgroup. In that subgroup we do see that the left coset and right coset are actually giving you the same subsets right. The subset order may change, I do not care about the order and the set, but you get the same elements ok.

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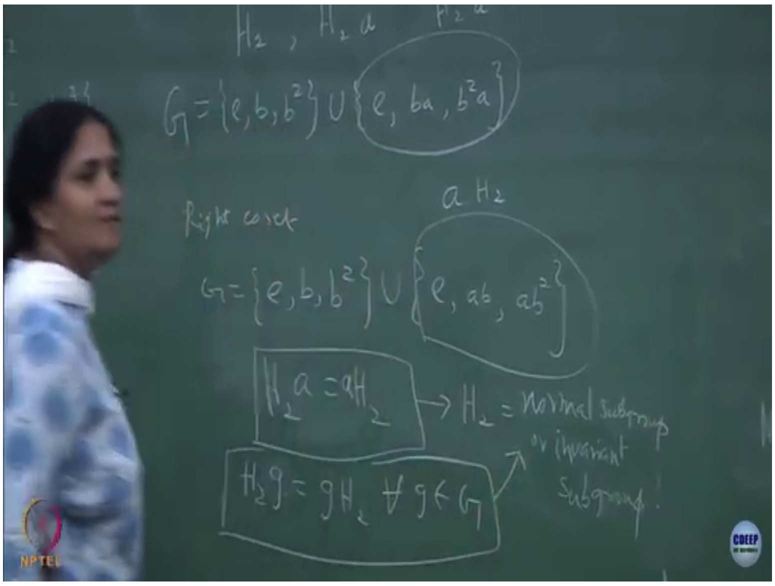

We can check that. H 2 with ab is that the quite confusion you have?

Student: (Refer Time: 12:41).

It will be.

Student: (Refer Time: 12:44).

It is for it is for all elements actually. You can say that H 2 g is same as g H 2 for all g element that g, that is true.

Student: Ma’am (Refer Time: 12:55).

Huh?

Student: (Refer Time: 12:56).

Correct. I think this is the most general requirement for it to call the H 2 to be a in this particular case that was the only non-trivial element that is why I put it in, but in an arbitrary group if the left coset and the right coset if. So, here if I multiply by g the same g has to be here, then only you can call it as the coset of the subgroup multiplied by g. The left coset and right coset gives you the same set then you call that subgroup to be an invariant subgroup. Good point

Student: (Refer Time: 13:40).

Yeah.

Student: But, they would not be equal for some other element, is that possible?

Then it is an invariant it is not an invariant, but like identity element for example.

Student: So, I will say is that also possible this is what I.

Identity element for example, ok; if you take g to be identity element left coset is same as right coset.

Student: Right coset.

So, there is no. So, there could be some trivial situations where left coset there are some commuting elements they may satisfy left coset equal to right coset, but not all the elements which will satisfy. Yeah, any other confusion is that fine, yeah?

Student: (Refer Time: 14:20) it will it should be enough to (Refer Time: 14:22) only for the generators?

Only for the?

Student: Generators.

You can do that also and then try to.

Student: All the other element.

All powers of these generators that is possibly what you are trying to argue for me in this particular case. I think that should also be true, but then you should have finitely generated group like you should have finite number of generators. This statement is most general that you know you can start finding out all the left coset and right coset and whatever you find with your particular element do it here and if these are satisfied for arbitrary g then you can say it is an invariant subgroup, good ok.

So, these two examples at least are going to give you some clarity that you cannot find left coset to be equal to right coset in general for subgroups. Cosets are writ10 for a subgroup that is a point number 1. So, they are first element is just multiplying identity element that will also be trivial subgroup even though it is a subset. The other elements are not any subgroups because it does not have identity element, but they are subsets they are called left cosets and you write your arbitrary group to be a disjoint union.

So, no 2 so, there cannot be one element from this coset over lapping with that coset, that is impossible. They are all the over lapped between this coset and this coset is null ok. There is no overlap between H 1 b and H 1 b square, but the group will be a union disjoint union of the cosets. From now on let me just confine to left cosets and I will call it as cosets ok. But, remember that left coset and right coset in general will not be equal. If it is equal for all the all the cosets then you call it as a that subgroup to be a invariant or a normal subgroup, is that ok?

So, let me get on to the screen to tell you what I am trying to say sorry, ok. So, in the specific example of the symmetric group you can see that there are cyclic groups 4 cyclic groups and then we could also construct left cosets, similarly you with a subgroup you can construct right coset. This is what I explained for H 1 and H 2 and if you multiply a left coset with identity element it is just the same subgroup, no change. You can write the group as a disjoint union of the subgroup H union left coset obtained by multiplying a on the subgroup elements it will be a subset union and so on.

And, it will be disjoint. By that I mean that there is no intersection between different subsets, there is no common elements between different subsets. So, how does this help you this way of writing explicitly gives you some kind of an indication about the Lagrange's theorem ok. What is Lagrange's theorem? Lagrange's theorem says that if you have an order G group for example, in the symmetric group you have order 6, the subgroup should divide the order of the subgroup should divide the order of the group ok. So, whatever order you find because it is an order 6 group the subsets could be 6 4 3 you know all possible subsets, but what you see is that it is only 2 and 3 which divide 6.

So, you will have only an order to subgroup or an order 3 subgroup which is what is seen here. When we write the subgroups you see that these are order 2 and this one is an order 3 subgroups.

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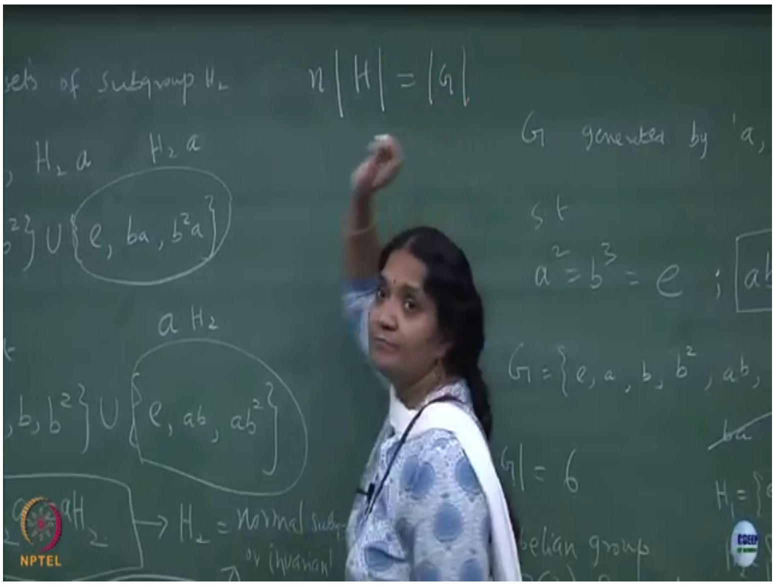

Which is telling you that the order of H should always divide so, some multiplication n times this should be order of G some integer n multiplying the order of the subgroup should give you the order of G. And, the only possibility in this symmetric group is only you can have either 2 or 3, is that right? Anything else? That is it ok.

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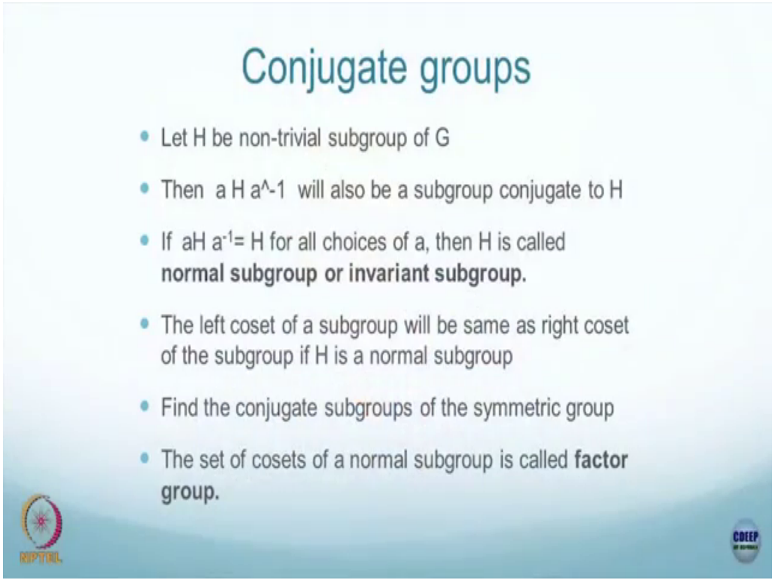

So, this also brings me to talk about conjugate groups. Let me spend some time on conjugate groups. One of the reasons why we are you know giving lot of importance to this conjugate things are even when you do quantum mechanics there are different frames which are related to each other by kind of a similarity transformation, like when you take operators when you do similarity transmissions; you know what is a similarity transmission?

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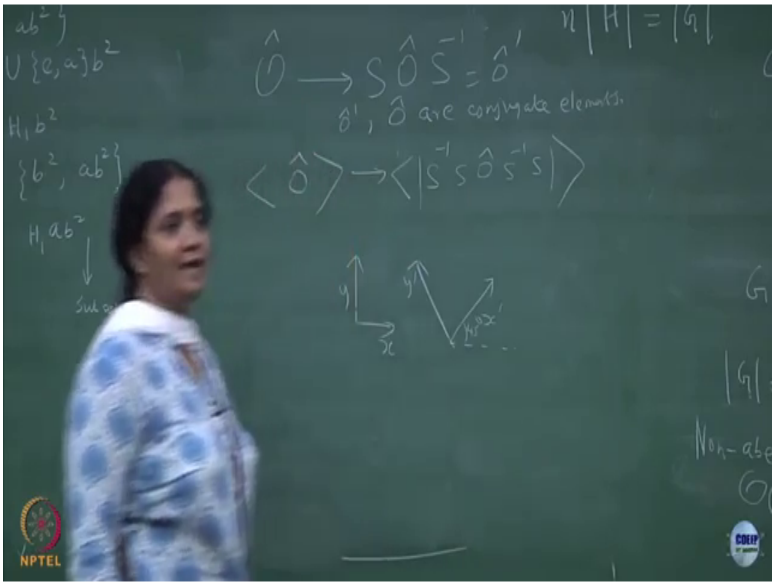

So, suppose you have an operator in quantum mechanics if you do S O operator S inverse physics, physics is given by expectation values of these operators right, am I right? What happens to then you try and show that the states also change right there will be an S inverse S O operator S inverse S on the states right. Physics does not see this operation basis change.

If you take xy axis like this if you take another xy axis like this with respect to the earlier one it is let us say 45 degrees rotation, are you going to see any physic change when you do such a rotation? Whatever physical quantities you are going to compute in this frame or whatever physical quantity you compute here will be related by this, but any averages you going to find will not going to be bothered about such a transmission.

So, anything which you find by the similarity transmission we say it Is like they those 2 elements. So, suppose this is giving me O prime I will call O prime and O are conjugate conjugated. I will explain it in the group context, but just to give you a motivation that we want to keep track of these conjugate elements because not much physics information is obtained, but treating them to be distinct elements ok.

So, we want to we know it is a distinct operation, but if they are related by a similarity transmission then we should know that the physics is not going to be affected by them. So, to keep track for it I want to put forth what does the conjugate group ok. So, let me explain what is a conjugate group. So, take a non-trivial subgroup of H. Let us take H to be a non-trivial subgroup of G G and then take an element a. So, where a is an element of G ok. So, take an element belonging to G, a H a inverse. So, this is a subgroup ok.

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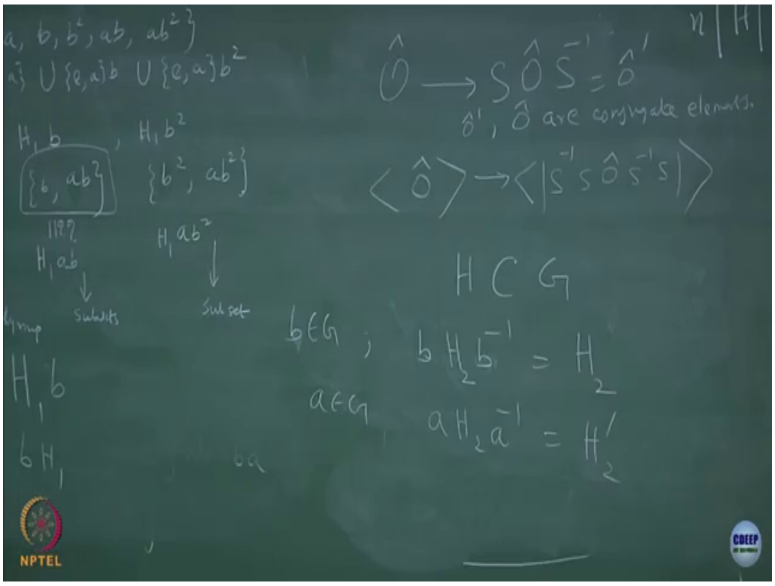

So, what you are going to get is a new subgroup. There are 2 there are some elements here right in the subgroup let us take a simple example like e, a, but do not confuse the same may let us take H 2 e, b, b square. If I multiply some other element outside that if you multiply within the elements of the subgroup you do not get anything is that right? If you take suppose I take this let me call that as H 2 and if I multiply it by b b belongs to H 2 as well as to G then what happens nothing happens right.

B multiplies b it gives you again the same set. If I take a H 2 a inverse where a is an element of G then what do we get? You could get a new element which I am calling it as a H 2 prime. Fine, is that right? So, whatever happens here will be a subgroup conjugate to H 2. Do we find anything like that here you can check it out which are the conjugate element.

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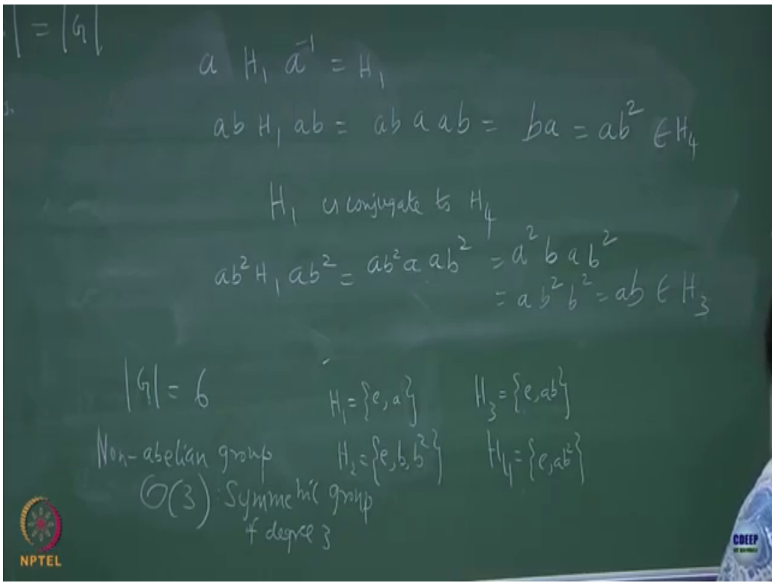

So, let us do it for. Somebody can work this out ab a, H 1 non-trivial element is only a right. What is this? Probably a raise the properties what do you get?

Student: b.

B.

Student: b.

You get b.

Student: ba.

Ba and that is same as ab squared right. It is an element of H 4. So, what does that show? H 1 is conjugate H 4. Instead of ab if you had multiplied ab squared what will that be? So, let us take this to be a right, a is the non-trivial element; b squared a is ab right and ba is ab squared. So, this is an element of H 3.

So, H 1 is conjugate to H 4 we have checked, H 1 is also conjugate to H 3. So, it means H 1, H 2, H 3 are conjugate to each other and with H 2 you will show that you will never find you will find it to be self conjugated you can check that all. So, what have we found?

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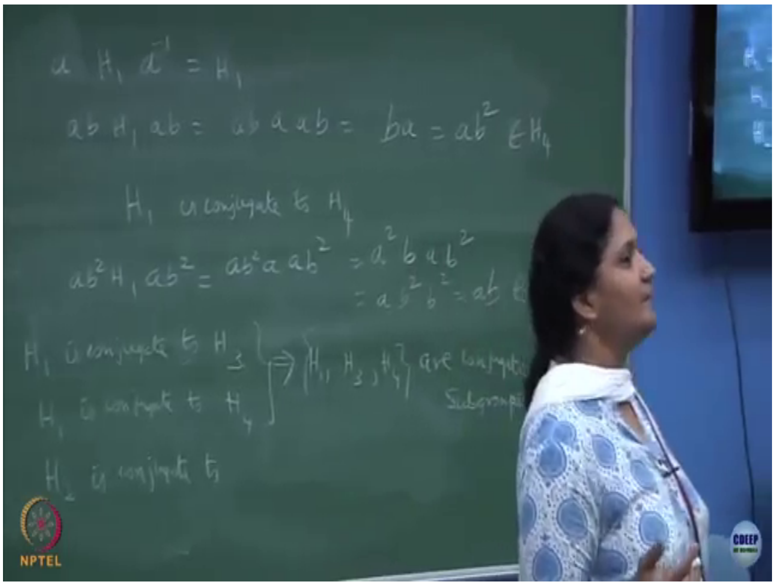

H 1 is conjugate to H 3 H 1 is conjugate to H 4. What about H 2?

Student: (Refer Time: 28:21).

So, these 2 implies that because they are related by some similarity transmission which takes you from one subgroup to another. This is also another way of generating the subgroups; you do not need to worry. If you see an order 2 element and order 3 element you write those 2 are subgroups and then start doing similarity transmission. Take H 1 element multiply with b H 1 b and see whether you get a new subgroup that subgroup will be conjugate to H 1 and then you can find the conjugate subgroups.

So, I listed H 1, H 2, H 3, H 4 you will generate H 1, from H 1 you can generate H 3 and H 4 by simply this similarity transmission which is called as conjugate. We did the converse because it was an ordered to subgroup it was easy, but in general you do not need to. You just take only the generate a subgroup and generate the other conjugate groups by these conjugation operation ok. What is H 2 conjugate do? Did you check?

Student: H 2.

H 2.

Student: (Refer Time: 30:03).

This one gives you anything new ab is b squared a.

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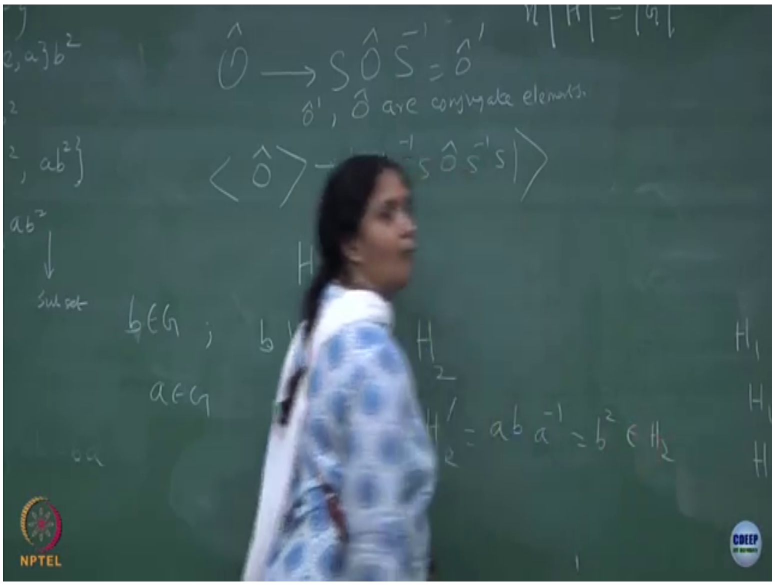

Right? So, which means H 2 is self conjugate.

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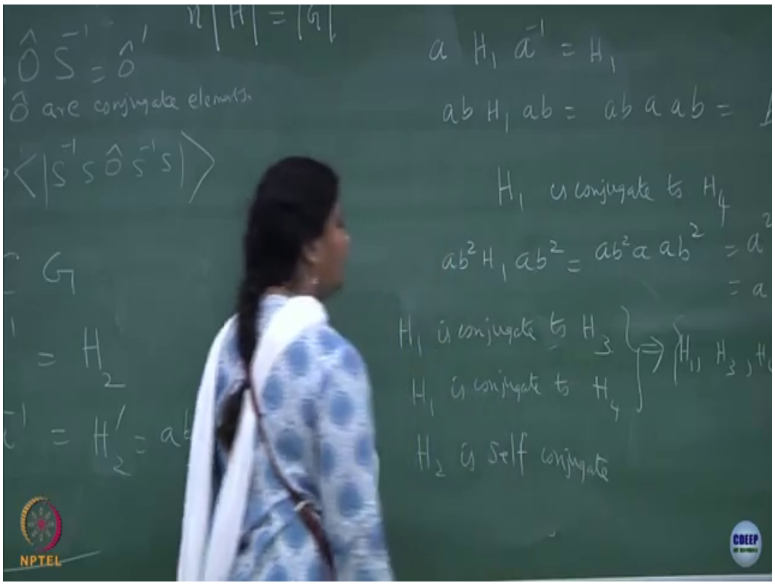

Another we have seen that it is self conjugate you should know now using normal subgroup, H 2 was also normal subgroup. What does that tell you? Let us do by the normal subgroup argument.

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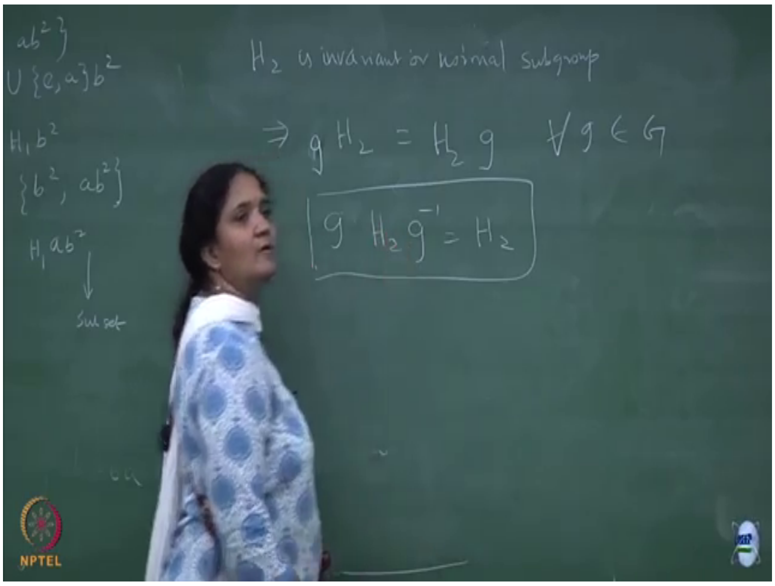

H 2 is invariant or normal subgroup. This implies a H 2 should be equal to or even g H 2 equal to H 2 g; left coset should be equal to right coset for all g element of G right. And, now you can right multiply g inverse on both sides you will get sorry, right multiply g inverse on both sides, what will you get?

So, this implies H 2 is self conjugate subgroup which also implies for self conjugate subgroups left coset is equal to the right coset, is that clear? So, they are all intertwined. So, whichever property you want to use it. So, you ultimately get that it is a self conjugate subgroup.