Group Theory Methods in Physics
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Lecture - 39
Generalised Orthogonal group and Lie algebra

Now, you can go further into abstract notation, which is called Generalized Orthogonal groups ok. So, let me write that also just like the way I wrote s o n.

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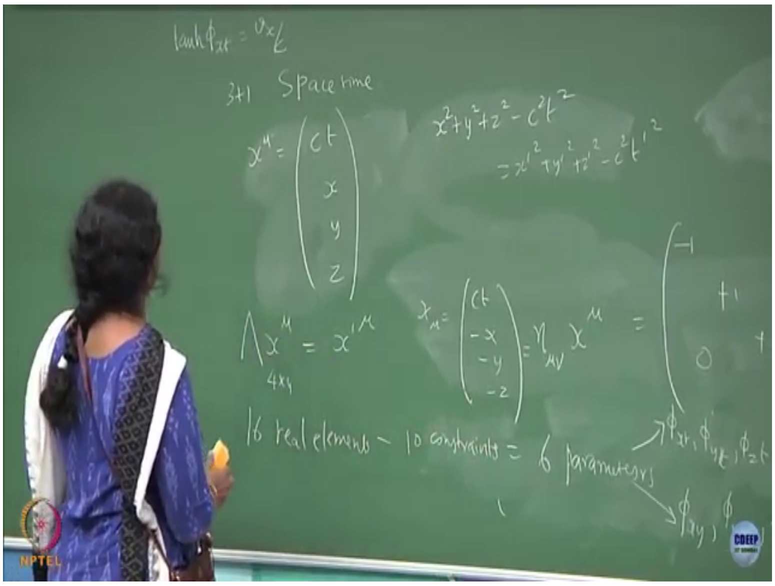

I want to do a generalized one. So, now, 3 plus 1 space time is gone, these meanings are gone, what I will do is.

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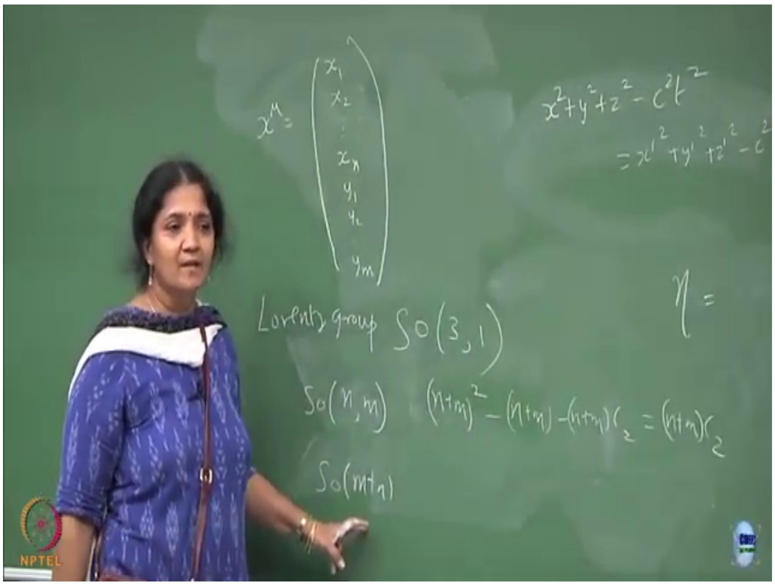

I will have x mu with x 1 x 2 x n then you have y 1 y 2 y n ok. What I am going to see is I am going to write a eta matrix.

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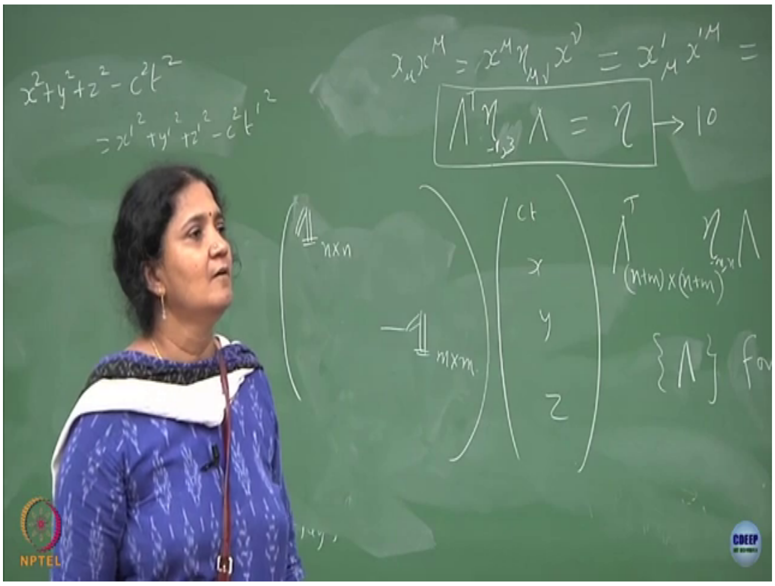

I am going to write an eta matrix with identity plus identity as n cross n, minus identity as m cross m. Earlier I had I put this as minus and this as plus, but you can do the. So, I will take such a matrix as eta. So, in our earlier case we had n to be 3 and m to be 1 right, 3 were positive elements 1 was negative element.

How I write is not important you can juggle around, but totally in that diagonal entries n of them are positive entries, m of them are negative entries ok. So, in the earlier notation which I did for space time 3 plus 1 dimensional space time c t the 1 entry was negative 3 were positive ok. So, in this case one is negative 3 is positive ok. Here n is positive n of them are positive, m of them are negative.

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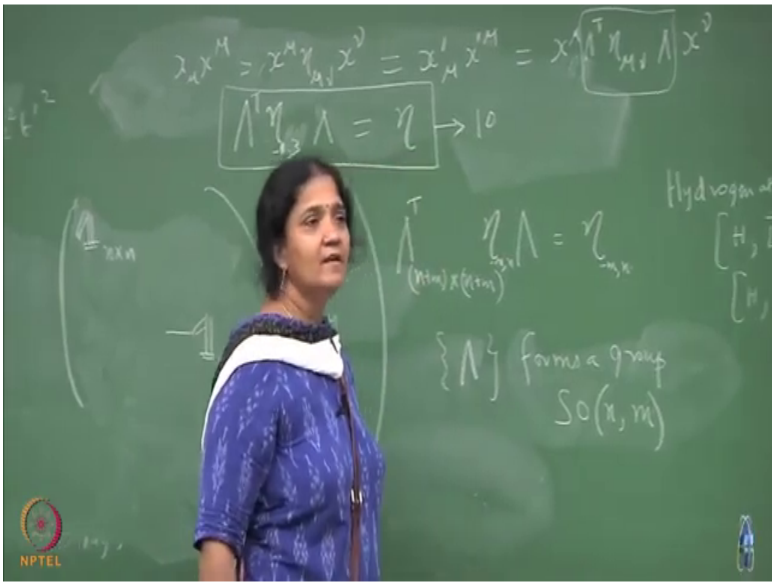

So, now I can write a lambda transpose, what will be the dimension in this vector space, n plus m cross n plus m and this eta matrix this is what I am calling it as eta, lambda should be eta where n of them are m of them are negative, n of them are positive, this keeping track of it. So, m of them are negative n of them are positive.

So, the set of these matrices is what I am going to call that set of these matrices are belonging to the set forms a group, which I am going to denote it as S o; S o n comma m, where n is the number of positive entries and m is the number of negative entries, but it is interchangeable you can overall negative sign you can put a new things, but this is the way I am going to define that it forms a generalized orthogonal group, which satisfies this condition and now what happens to your Lorentz group in this notation?

Lorentz group in this generalized orthogonal group notation can be written as clear. Number of generators of S o 3 comma 1 is same as number of generators for S o 4. So, S o m comma n what is the number of generators you find out, S o m plus n this is very different from this group why? This group will have all of them plus this one is m of them is different from n of them ok.

Now, there is a difference here ok. Sorry, I should have put n comma m n is positive. So, I am just saying that these two are not isomorphic groups, but number of parameters for defining these matrices will be how many? How many parameters you need for this?

So, you will have an n plus m the whole squared minus n plus m minus n plus m C 2, those are the off diagonal elements, these are the diagonal elements, these are the constraints, if you subtract you will get again n plus m C 2. This is not different from the number of parameters for S o m plus n, but as a group they are different, if you find the number of generators a number of parameters it is still n plus m C 2 for the generalized terms clear.

So, I took you from physical space and physical space time and I put you in a abstract situation of how to look at a group of rotations in an n dimensional real vector space, or a vector space, which is quote unquote like space time, where n of them will be like time and the remaining m of them is like space, but it is not to be viewed as something which you can see in your day to day life. Some of the systems may have such symmetries and you can exploit the symmetries and the algebra of them.

Student: How is this (Refer Time: 07:27) group of 2 of n comma just like (Refer Time: 07:31) S o 3.

Huh.

Student: (Refer Time: 07:33) to fix the data mining equals to 1.

Yeah.

Student: So, they are this o n comma n.

Huh.

Student: S o.

O of n comma m will be just a tensor direct product with the C S group; you can do that also ok. Just in instead of doing only inversion in space, which you would have done in S o n you will have a time translation sorry time reversal and space inversion both oppose clear ok. So, this is what I slowly took you of from on the slide from rotations on S o 3 as I said there physically we saw the parameters here.

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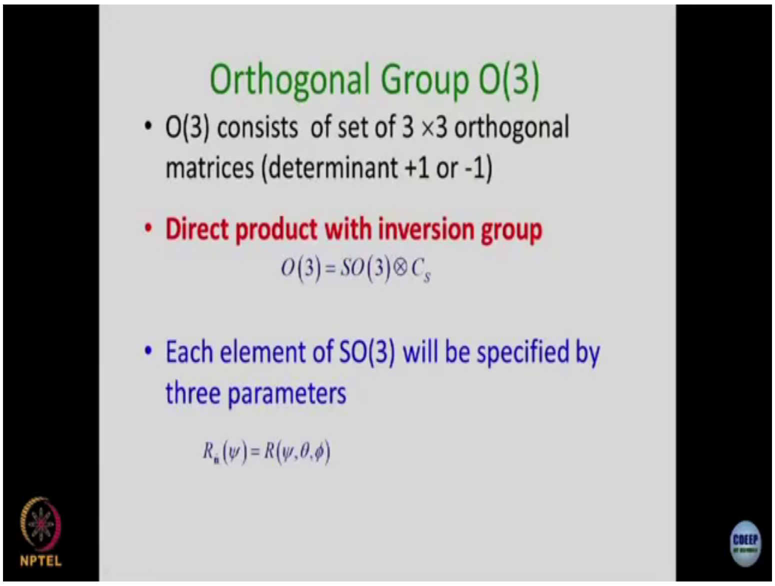

And, in the abstract way I taught you today how to see the parameters?. So, physically when we saw we said you have to fix a direction, which is hat n and with respect to the direction you have to do a rotation by some magnitude psi about this direction. And, that we compactly write it is what I said, but then I took you on to a generalized situation.

And, there was some confusion at the end of the class that some of you were unclear about this issue. Only thing is I wanted to try and redraw this parameter space in a compact fashion that is what I wanted to do? Only you remember that if you do a rotation by an angle pi using a unit vector let us say a long z axis, you could have also done it positive z axis you can invert it and make it negative z axis, it does not really matter as long as you do an angle pi.

If, you do arbitrary angle other than pi, then there is no such equality. It happens this equality happens only for angle pi that the diametrically opposite points on a sphere on a solid sphere of radius pi ok. Those two have to be identified is that clear.

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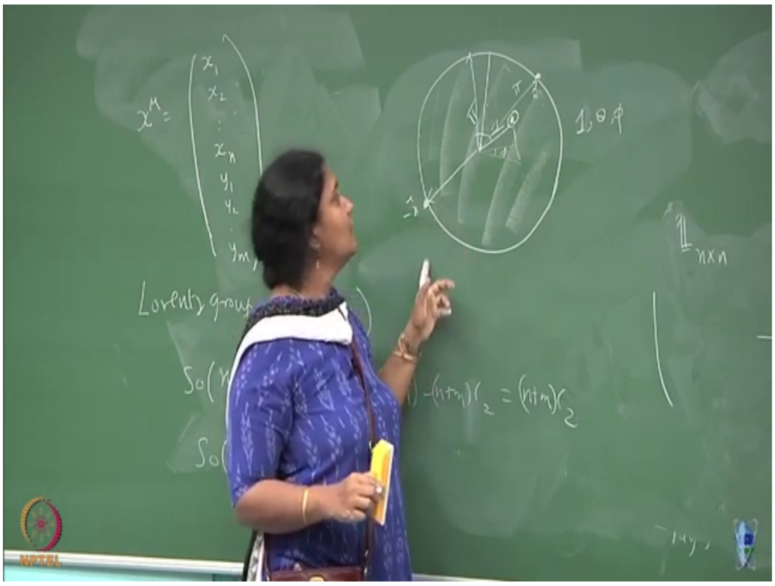

So, I took a solid sphere, this is solid. So, solid sphere the maximum radius it can have the radius can be only pi. So, the radial vector when you are doing spherical polar coordinate you will say 0 to infinity. The one which defines for as the parameter space in a compact fashion is a solid sphere with unit vectors like your theta phi, but the radial vector is taken by the magnitude of rotation which you do about such an axis. And, that rotation also should go from 0 to 2 pi or minus pi to plus pi ok.

So, this is a manifold parameter space which defines the space for defining your rotation in 3 dimensional space, any point here if you choose find the radius value suppose it is 5 units ok.

And, let us take it to be making an angle theta with respect to z axis and then on the projection let it make phi n phi with respect to x axis.

And, then you can define this point as someone at particular point you can write it as 5 for r and then whatever is the theta and phi as r theta coordinates ok. Phi is not a good number it should always be between minus pi to plus pi maybe I could call it as some 1 in radians ok.

So, this is a meaning of saying that I am going to take this point refers to rotation by angle in radians have taken it to be 1 and then direction is chosen to be making an angle theta with respect to z axis and phi in the x y plane clear. So, that is the parameter space which is drawn. After, I have drawn it I am only trying to use that property, that if you take it on the boundary this corresponds to pi right, any point here the radial coordinate is pi, and you can choose your axis which is also n hat axis along this. What will happen is that whether I do a rotation by pi with angle n hat I can still do it with minus n hat ok.

So, this is minus n hat. So, which means that this point is same as this point clear, these two points are rotation described by this point is same as rotation described by this point that is not true for interior points only on the boundary ok. So, the parameter space not only defines this whole space, but there are some identification which you have to keep in mind ok. It is not just every point on the boundary is independent ok, that is it.

And, then as a topologist or looking at the topology of this parameter space. Suppose, I take closed curves in some space, which has a hole then there are different ways of taking closed curves. Only thing is given this topology which is the parameter space for S o 3 now you do not associate rotation matrix with every point.

If, you ask are there only one type of closed curves are more than one type of closed curves, which you can draw on such a manifold, which I am drawn for you. You have to remember in this manifold diametrically opposite points on the radius equal to pi are identified. If you use that information, then the group manifold has two types of possible closed curves; one which does not even go to the boundary ok.

So, if you see these curves this is one type of curve, other closed curve can go from here to the boundary, diagonally opposite point is here and you get back. Similarly, this wavy line diametrically opposite point is same as this.

So, each of these curves which I have drawn this one curve by any deformation, I cannot make it look like this one, by deformation of this I can make it look like this ok. So, in that sense there is two types of non trivial closed curves, which I can draw on this group manifold associated with S o 3.

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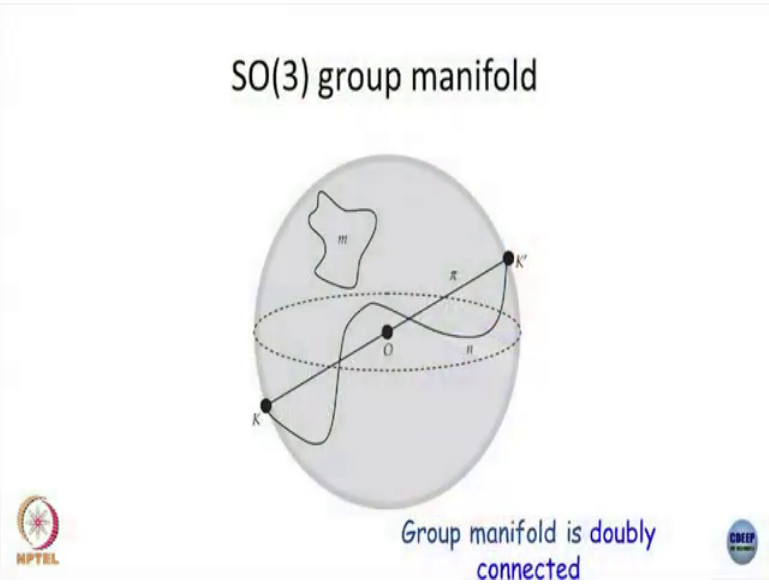

Where only 2 types, somebody came and said I could intersect the boundary twice, but you can nullify it and bring it inside you can try that out also. Only two independent non trivial closed curves can be drawn for the S O 3 group manifold ok. This nothing to do with seeing rotations parag you know every point is a rotation matrix, associated with every point you can write rotation matrix with that psi theta phi, but then once you draw closed curve it is just to look at the topology of the group manifold is that clear.

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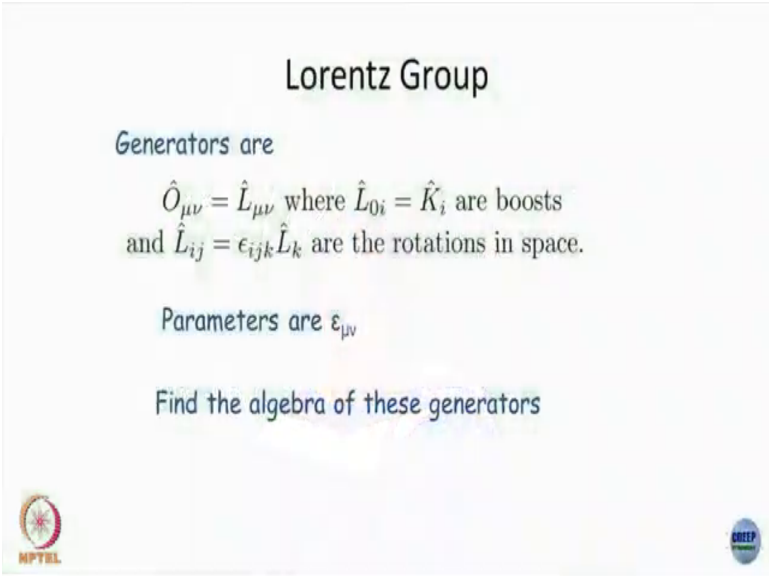

Student: (Refer Time: 16:31).

So, this also we went through. So, let me not get into this and this is left as an exercise for you to do the algebra of generators ok.

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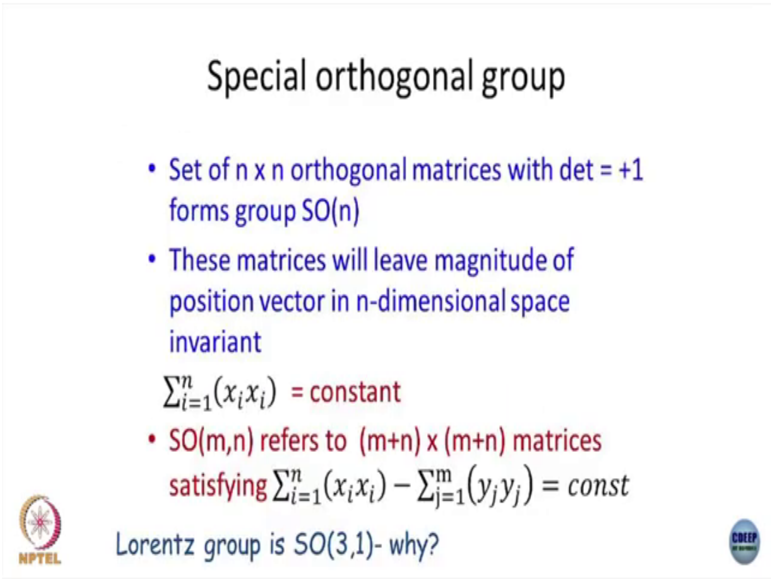

And, this I have already explained to you that set of n cross n matrix with determinant plus 1 forms a group S O n and these will leave the magnitude of the vector invariant, which you can formally write it as x i squared and so on ok. I have not introduced the eta here, but you can introduce an eta. So, that minus sign is automatically taken care of. So, S O 3 comma 1 is the Lorentz group and the reason is that c p has a sign different from x y z ok.

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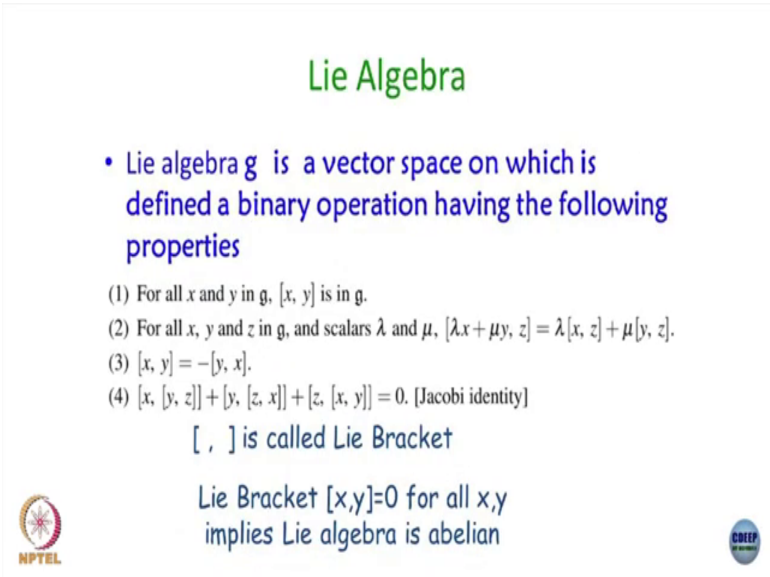

So, now I am going to introduce the formal abstract lie algebra. So, lie algebra is also a vector space ok. On which you define an operation which you call it as this lie bracket, this bracket with comma which we call in quantum mechanics has committed a bracket in lie groups context it is called lie bracket ok.

So, you have a set inside this lie group just like rotation group I had L x L y L z, your Lorentz group had L x L y L z and the boost transformation 6 of them. So, these x y are the generators of that lie algebra or the generators of the transformation, which will constitute to define the commutator brackets ok. For all x and y in g x y commutator should also be in g, that is what you see L x L y commutator can be 0, but it can also be the remaining generators.

And, any scalars multiplying them that is why it is a vector space, you can rewrite that commutator as lambda times this these are known to you from your quantum mechanics commutator bracket, it is exactly the same properties. And, commutator will be minus off if you exchange operator a and b then you get minus sign and you also have this Jacobi identity satisfied.

Jacobi identity is just cyclic permutation of the 3 elements of the lie algebra ok. So, this vector space which has elements x y z the set, if it satisfies all the 4 properties you call that to be that vector space to be a lie algebra under operation which is the lie bracket clear. Suppose, the lei bracket is 0 for all x y when does this happen in our case translation group it happened, translation generators of translation you found that p x p y p z forms a is constitute a lie algebra, but it also be an abelian lie algebra because p x p y is 0 and so on.

So, in general if all the elements of this vector space commute with each other, then you call it as an abelian lie algebra. Now, I am looking at lie algebra not the group, if you want to get the group you have to exponentiate the generators of the lie algebra clear no ok.

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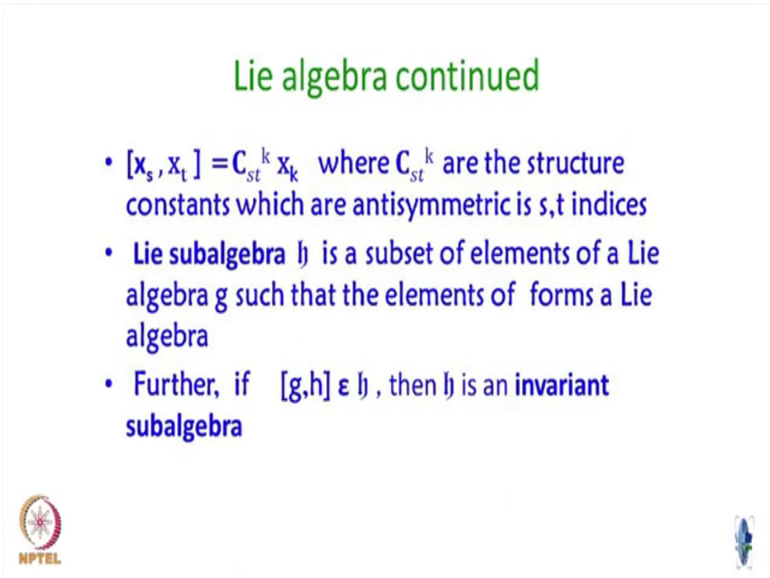

So, as a formal definition of a lie algebra x x x t commutator if you had a set of all the x s x is now an x abstract notation and for an abstract transformation what is the lie lie algebra and so on. So, in the abstract rotation you will have x s x t should give you a linear combination of x k. And, what is the requirement? Because, this is a commutator kind of lie bracket with anti-symmetry property, whenever you interchange the s n t these coefficients should become negative of itself, but also you have seen L x L y will give you L z, L y L x will give you a minus L z right.

So, this is this k is summed up that is the assumption here, the repeated index is summed up and then s and t are anti-symmetric. This coefficient should be such that s and t are anti-symmetric C st k are called the structure constants is that clear, what is the structure constants in the case of your orbital angular momentum algebra.

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Student: (Refer Time: 22:04).

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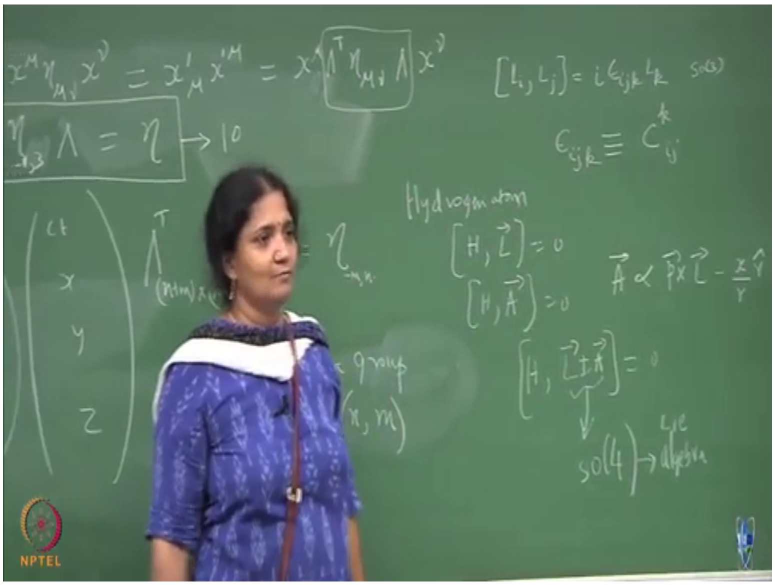

So, I can write this as L i L j is let me suppress the H cross i times epsilon i j k L k right. So, your epsilon i j k is your C ok. So, these are the structure constants for your algebra. And, this algebra is what I will denote it as s o 3 small s small o 3, because this is the algebra of generators for rotations and it is satisfying all the axioms.

So, it is definitely a lie algebra. And, is it abelian or non abelian? It is non abelian, because it is not going to commute. You can also have a subset of elements of the lie algebra g such that the elements obey all the rules of a lie algebra ok. So, if you do your Lorentz transformation.

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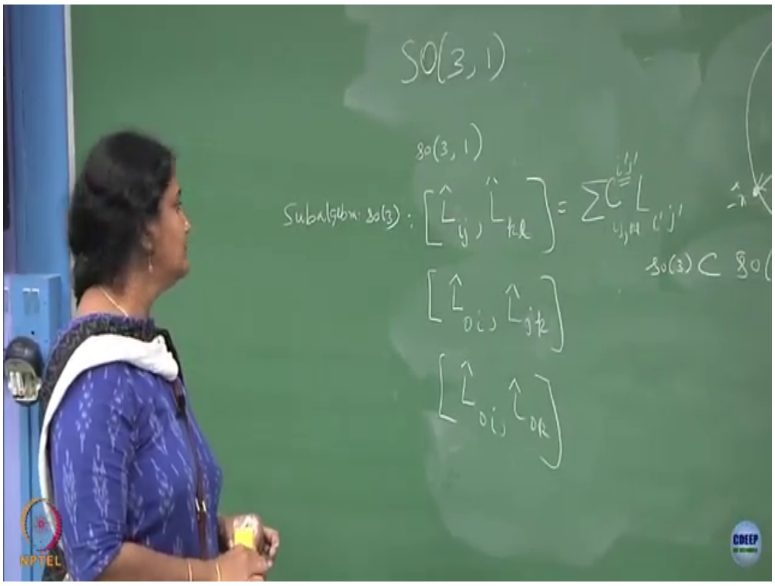

So, S o 3 1 the algebra will be s o 3 1 so, this o and all is capital. So, that is the group this is the algebra and the algebra will involve your L i j L k l ok. You can also have algebras which involves L 0 i with L i j or let me put this to be j k, you can have L 0 i with L 0 k, you understand what I am writing this 0 denotes the time coordinate ok, which is of a different minus sign or plus sign your eta matrix, i j k are like your positive coordinates ok.

And, amongst the positive coordinates if you try to write this bracket what will it give me? Will it give anything which involves the time coordinate; these are the generators of rotation in your physical space what space time. So, if you try to write this out it will turn out to be some linear combination with some L i prime j prime I am not writing what it is with some coefficients, you agree?.

I am putting a double index because each one is a double index you will have an i j and a k n ok. Will this have a 0 j prime; 0 j is for boosting 2 consecutive rotations whatever you do? Are you going to generate something that is the one frame is moving with other frame with velocity v x, can you generate by that no you can never do that ok.

So, in that sense there is a subset there is a subset or S o 3 which I have written here is a subset of S o 3 1 clear. S o 3 are pure rotations in space even in C squared t squared minus x squared minus y squared minus z squared, you have a subset of rotations x y z. And, when you do such a transformation of x y z, it is going to only give you these; it is not going to take you to the other three generators which are like boosting of one frame with respect to the other.

And, if you have a subset, which satisfies a closed form amongst that set itself amongst that subset; you call it as a sub algebra just like subgroups. If you have a subset of a group with product giving you elements which belongs only to that subset you call it as a subgroup. So, this implies that these are elements of s o 3 ok, because any lie bracket if I write, it is going to be only with respect to these special components.

So, in that sense this is a sub algebra sub algebra is s o 3 and you can denote this as s o 3 is a sub algebra of s o 3 comma 1 ok. The next definition is just like we did in variant subgroups, you also have an invariant sub algebra, h should be a sub algebra of to start with ok. And, if you take any element of the lie algebra with any element of the sub algebra the lie bracket, if you find it to be inside the sub algebra then you call it as an invariant sub algebra clear. Can you think of an example for this where you can get this?.

So, let us take our Lorentz transformation to that let us add translations also space time translation, because I am doing Lorentz transformation. So, space time translation will have p x p y p z and energy, which are also generators if I say that I want to look at the group transformation, which is not just Lorentz transformation, but also space time translation. So, how many generators will be there Lorentz transformation will have 6 generators and space time translation will have 3 row, 3 translation p x, p y, p z and the corresponding time translation whose corresponding generators the Hamiltonian.

When you have to write a lie algebra for this complete set, which is what we call it as a Poincare group ok. Poincare group allows space time translation symmetry, rotation symmetry, and boost symmetry together that forms a Poincare group ok. If, you look at this what will be the algebra between p x, p y, p z and Hamiltonian, that will be an abelian algebra it is a sub algebra.

And, if you take p x with any of these rotation generators what is that going to be? You all know what happens when you take L i with P j.

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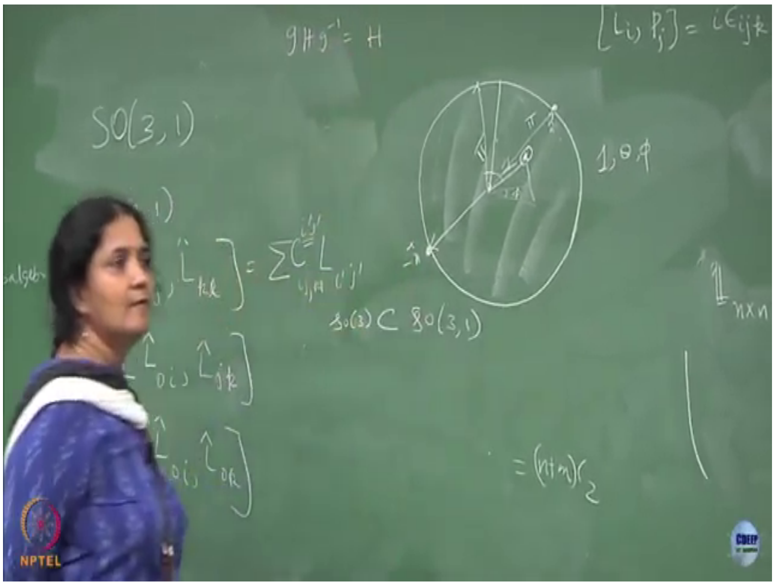

Student: (Refer Time: 30:33).

Will it be L i or will it be p it will be p right. So, you can write it as apart from this i factor ok. So, I said that p i p j commutator is 0 it is an abelian sub algebra, over on top of it the elements of this abelian sub algebra with the other rotation generators. For example, is giving me back the generators in the abelian sub algebra.

Such a thing happens then you call that sub algebra to be a invariant sub algebra ok. It is not obvious from here, how the group elements will satisfy the properties which we learnt in the discrete groups? Right. What did we learn in the discrete groups? If, it is an invariant sub algebra subgroup what is the requirement is that right. Then only you call the subgroup H to be an invariant subgroup.

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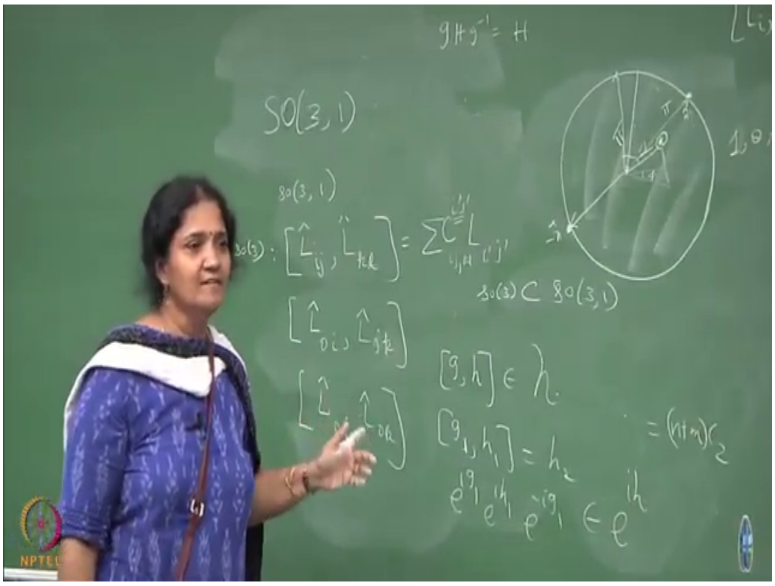

Now, I am trying to tell you that an invariant sub algebra is one, where g with h is an element of that invariant sub algebra right. How does this imply that, how do you do that?. So, let me take a g 1 element, with a h 1 element let me call it to be h 2 suppose.

Corresponding to the element in the lie algebra you can write group elements, how will you write that e to the power of i g 1 will be the group element, e to the power of i h 1 is the group element and I need to show this piece is going to be an element of e to the power of i h right. This I will give it as an exercise and we will discuss at some point, but these two are equivalent ok.

So, when I say that the sub algebra is an invariant sub algebra, it is commutator or lie bracket with all the elements of the lie algebra, this is not group element lie algebra elements, which is like generators, they have to belong to that sub algebra. How this reflects to what you learnt is to redo the whole thing explicitly and see that it gives me this condition automatically gives me this condition ok.