Group Theory Methods in Physics
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Lecture - 37
Introduction to O (3) and SO (3) groups

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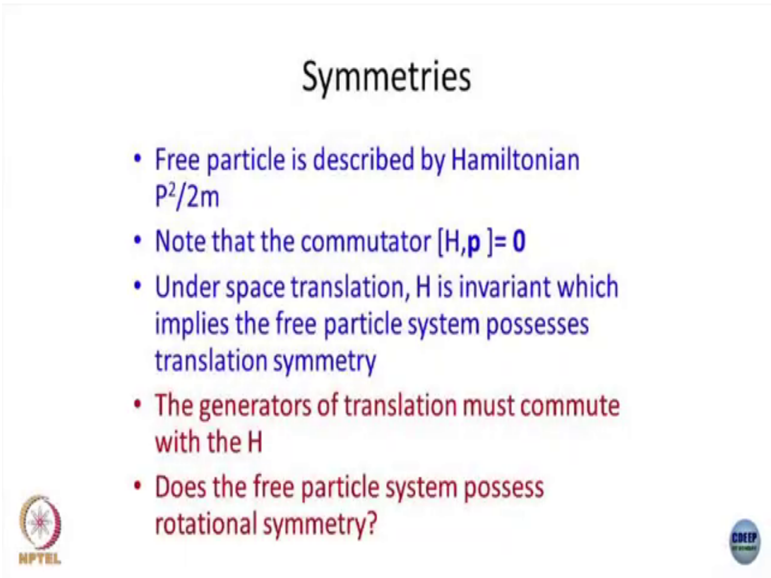

Coming back to some connections with what we were looking at in discrete groups; let us take a simple free particle system, simple system which is a free particle. Free particle is just given by there is no potential energy just a kinetic energy and the momentum P could have any value, it is not confined to a box, can have any value. What you can show here is that; if you take the commutator of Hamiltonian for a free particle, with momentum p x, p y, p z that commutator is 0 you all agree, right.

In the discrete group what did I say? If the Hamiltonian is respecting some discrete group symmetry, Hamiltonian should commute with every element of that discrete group, is that right. Now, Hamiltonian should commute with the generator of that symmetry. Suppose free particle, in this free particle you can see; if you do a translation x to x prime, P squared over 2 m does not change, you all agree?

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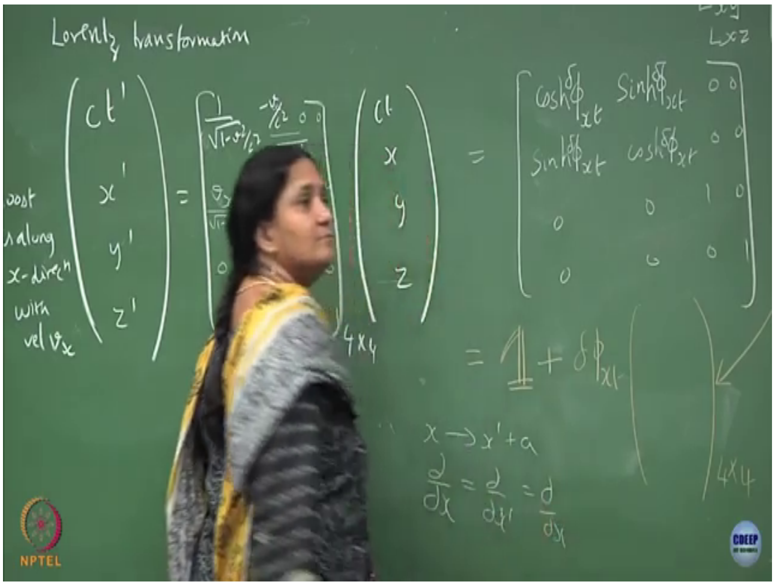

If you take x to x prime, del by del x is going to remain as del by del x prime ok, it is going to be same like that. So, free particle Hamiltonian respects translational symmetry; if it respects translational symmetry, Hamiltonian should commute with the generator of that symmetry and the generator of that symmetry is momentum, linear moment, clear.

So, lets take rotational symmetry, does this free particle system respect rotational symmetry? Yes or no? What will you do to check that, whether it respects rotational symmetry? Find the commutator with respect to the generators; if it is zero you know that should happen. But of course, you also know if you do a rotation, it is like p dot p; under rotation p dot p is same as p prime dot p prime, so it is rotationally invariant. So, your p squared should commute with L x L y and L z. Is that right?

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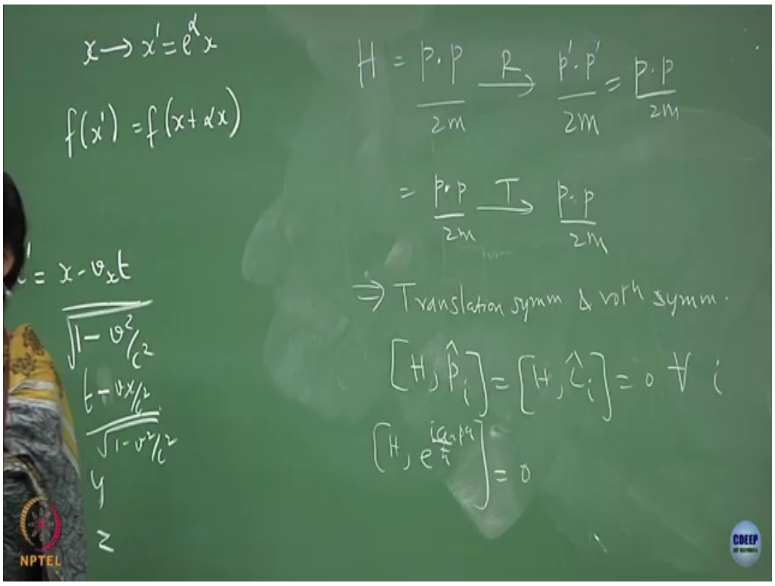

If you do a rotation it becomes p prime dot p prime divided by 2 m; but this is same as p dot p right, it is the dot product of two vectors, right. So, similarly under translation remains p dot p, nothing happens, its invariant actually ok. So, this implies, this Hamiltonian respects translation symmetry and rotation symmetry.

Now, you can check that, the Hamiltonian with p ok. So, consequence of any symmetry of that system requires that the generators have to commute with the Hamilton. In the discrete symmetry, it had to commute with all the elements; if it commutes with generator its good enough, because elements of the group are constructed using exponentiating 1 minus the generator. So, all the group elements will also commute. Like what I am saying is that, the Hamiltonian with exponential of i by h cross a mu p mu. What happens? This will also be 0, right, group elements will also be 0, ok.

So, it is good enough to see the commutator with the generator to get the implication about what is the symmetry of the given system described by some Hamiltonian. Hydrogen atom is it translation symmetry is there? Is there a translation symmetry? No or yes, think about it. Harmonic oscillator does it have a translation symmetry? No, ok.

Hydrogen atom you have to still think, there are two particle system; if you do a translation relative coordinate what happens, ok. So, please think about it. But harmonic oscillator, 1 D harmonic oscillator the potential energy if you shift x to x plus a; it is going to give you a new, it is not going to remaining invariant. Or equivalently if you take the commutator of the Hamiltonian with P; will it be 0, will it be is the next way of indicating whether it is respecting rotation symmetry or translation. Is this clear?

There is a parallel; whatever you have learned in discrete group, it is not completely disjoint from what you are doing now. It is just that it is going to be connected by a continuous parameter and we are looking at the essentially the generators; and once we know the generators we can exponentiate it and find any arbitrary group element.

Because the parameter is continuous, you will have uncountably many number of elements; that is why it is going to be a continuous group or we call it as a link group, ok fine.

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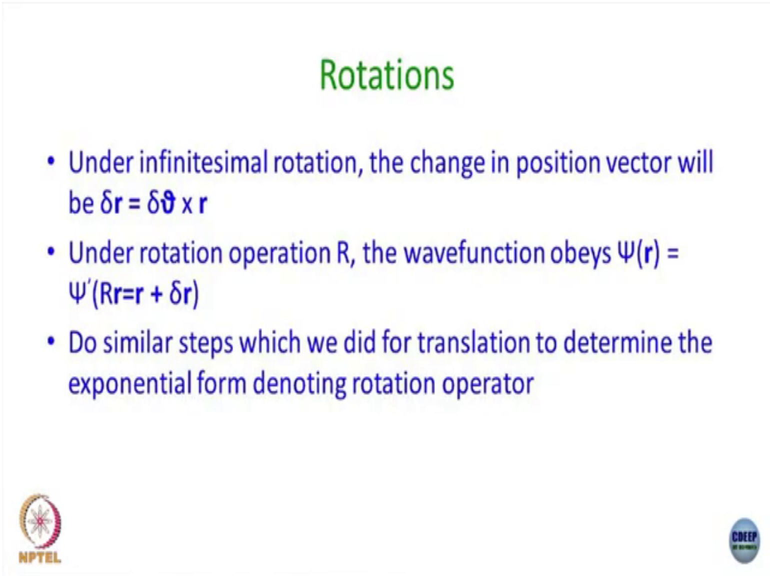

So, this I have already explained to you under infinitesimal rotation, the change in position vector; I put a boldface to remember it is a vector with a rotation by an angle theta cross r and then you try to use how to write the relation between the wave function obeys this. Two similar steps which we did I have already done it for you and shown you what is happening to your generator for rotations. And then you have to do n number of steps of infinitesimal transformation and you will get the exponential ok.

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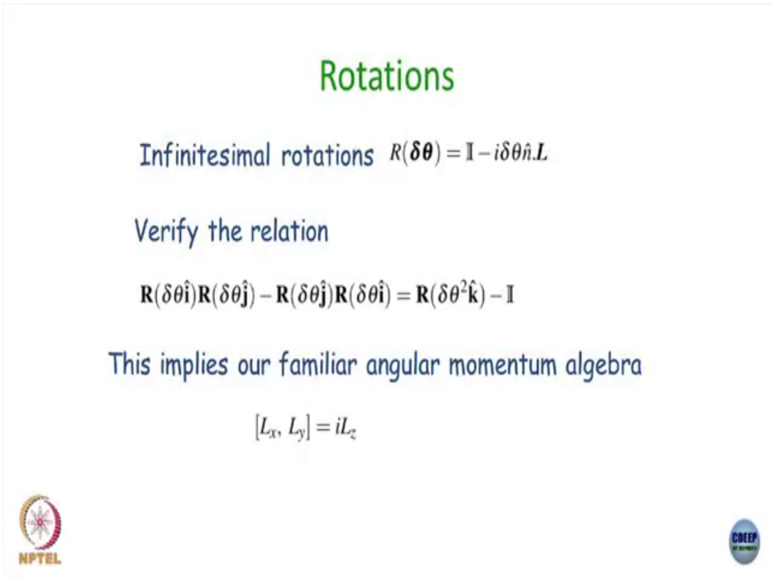

So, this also I have explained that, if you take infinitesimal rotation and then you can rewrite it as; I have put the h cross to be 1, you can rewrite it as if it is by an arbitrary angle in hat n, magnitude is delta theta times L. This is something which I want you to check. Can you please check this that, if you do two consecutive rotations about x axis and y axis, infinitesimal rotation by the same magnitude angle; do the other way round, the difference between these two, you know it is nonzero, but it is going to be of order delta theta squared.

So, the difference between these two rotation is going to be of order delta theta squared and it is as if it is a rotation about the k axis; if it is x axis and y axis, this result is about the k axis. So, please check this. And from here you can explicitly use the first line to determine that, L x L y is i L z. So, these two line implies that it is a non Abelian group rotations; if it was Abelian what would have happened, right hand side would have been.

Student: Zero.

Zero ok. So, then fact that the right hand side is not zero at order delta theta squared ok; if you are doing. So, many books they take infinitesimal and ignore delta theta squared. If you ignore delta theta squared and look at it as a very small angle transformation, you can still treat it to be process to Abelian. But if you keep up to order delta theta squared, then the right hand side is not 0 ok. And this deviation is what gives you automatically the algebra of orbital angular momentum, ok. Now, if this summarize whatever I did on the board, for completeness on the slide and please check this part, verify this relation,

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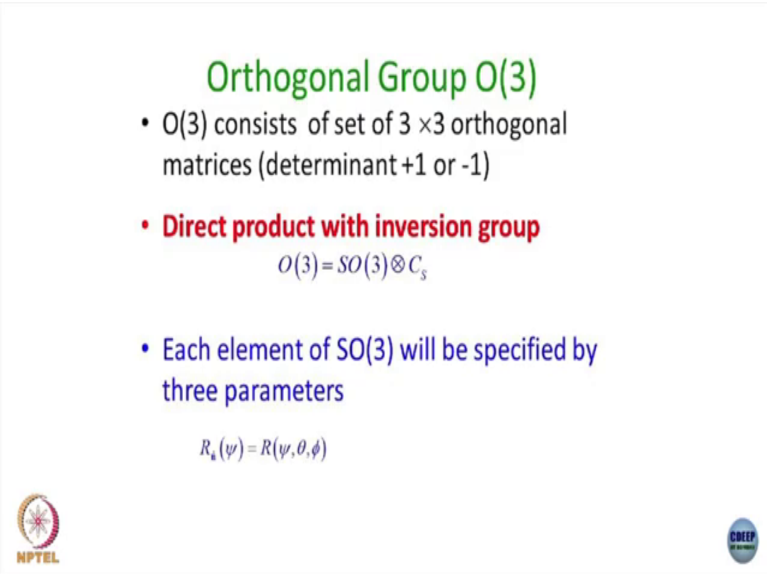

So, now, coming into the jargon of what we call the rotation groups as; you all know that this rotation matrix which takes you from r to r prime.

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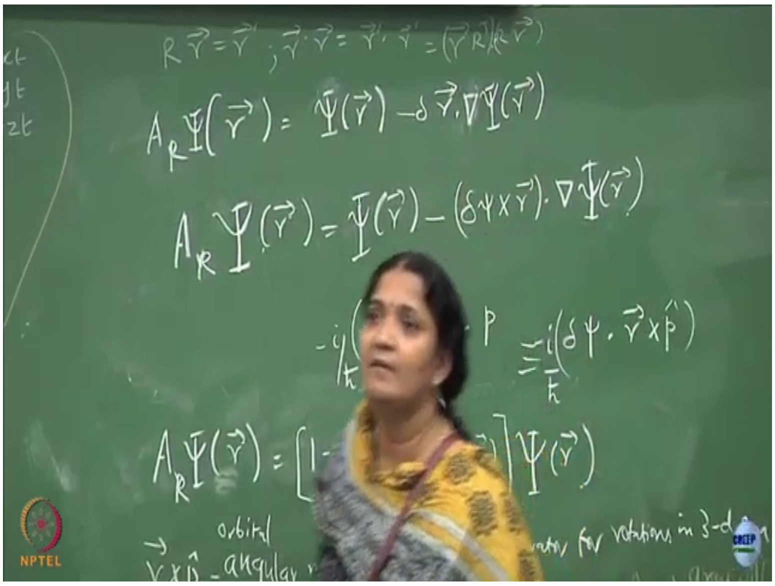

This rotation matrix is a 3 cross 3 matrix right and they are orthogonal matrices. Why orthogonal matrices? R dot r r prime dot r prime which can be written as r R transpose R r, right, am I right? So, R transpose R has to be identity. The set of matrices which satisfies that property R transpose R to be identity are orthogonal matrices. So, the corresponding group is what we call it as an orthogonal group and minimal non trivial dimension which we have to look at is rotation in three spatial dimensions is denoted by O of 3.

So, this is set of orthogonal matrices in three dimensional space; the determinant can be plus or minus 1 in general, need not be only rotations, it can be rotation, times reflection, inversion ok. So, the all three means it could be orthogonal matrices, determinant could be plus or minus 1; but S O 3, S O 3 requires that the determinant has to be plus 1.

So, this has proper and improper rotations possible; S O 3 will allow only proper rotations ok. And you can show that O 3 is nothing but S O 3 times an inversion element ok, direct product clear. So, the C s group will have an identity element and minus 1, minus 1, minus 1 diagonal element. So, any element which you find for S O 3 with identity will be S O 3; with the inversion element will give you the elements which does not belong to S O 3, but belongs to O 3. This is the same notation which I use for the discrete groups, clear.

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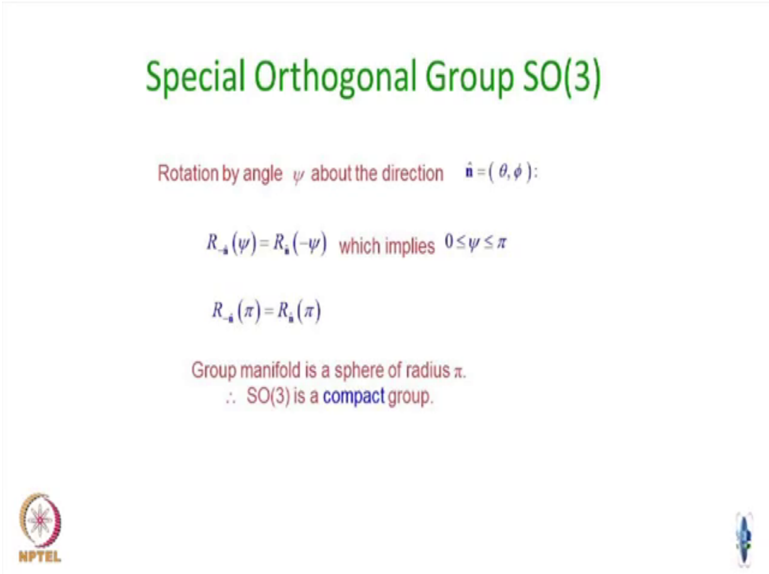

So, each element of S O 3 will be specified by axis and the rotation is what I was saying. And special orthogonal groups, rotation is by an angle psi about a direction which I am calling it as hat n given by theta and phi unit vector. Couple of observations you can see; if you take a rotation about z axis let us say ok, by an angle do a opposite clockwise or anti-clockwise rotation.

You can show that the axis can be inverted and the rotation angle can be made; if it is clockwise, it can be made anti-clockwise, there is a symmetry like this, ok. So, what does this mean; if you do a rotation by pi, pi and minus pi R equivalent ok. So, what you see is that, the rotation by pi about positive hat n axis or minus hat n axis gives you the same value; but for arbitrary psi, it is psi goes to minus psi.

Why am I putting all these things? I was always saying that I need to look at the parameter space, right. When I did the space translation, I said the parameters a x, a y, a z are from minus infinity to plus infinity. Now, this rotation I am saying you choose an axis; the axis can be on a sphere.

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So, let me take a sphere and let me call the z axis a psi. So, I am going to use, sorry the radial vector to be psi. So, any vector which I am going to take, so this, let this radius be pi. I am going to fill the whole sphere, ok. It is a solid sphere with the maximum radius being pi; any particular point I take, I will call that point as psi theta and phi. Theta phi specifies for you the direction of your axis; the radial distance from the origin, this is psi ok. The radial distance from the origin specifies the angle, the magnitude of the angle by which you are doing a rotation.

So the parameter space analog to a x, a y, a z for the rotation can be compactly put in on a solid sphere with psi, magnitude of size should not exceed pi, right. This angle always from minus pi to plus pi, but this magnitude will only can go up to pi, right. So, this is the parameter space which is compact unlike your space translation, where I said that it is R 3; a x, a y, a z can be anywhere from minus infinity to plus infinity is the parameter space for space translation.

What is the parameter space for rotation? You can nicely draw it as a solid sphere with the radial coordinate ranging from 0 to pi and every point in this corresponds to a rotation element. Suppose I take this to be psi prime and theta 1 and phi 1, this will give you a new rotation element.

So, the parameter space is compact that is the first observation; unlike here, where the parameter space is non-compact. I want you to think about what happens for Lorentz transformation. It has phi x t, phi y t, phi z t and phi x z, x y z and what happens there, you please think about it.

So, group manifold, the parameter space which defines for you the group S O 3, is a sphere solid sphere of radius pi. So, once you have this parameter space to be compact, you say the corresponding group is a compact group.

Student: Compact means.

Compact means, it is all within this finite region; whereas here, there is no bound, ok.

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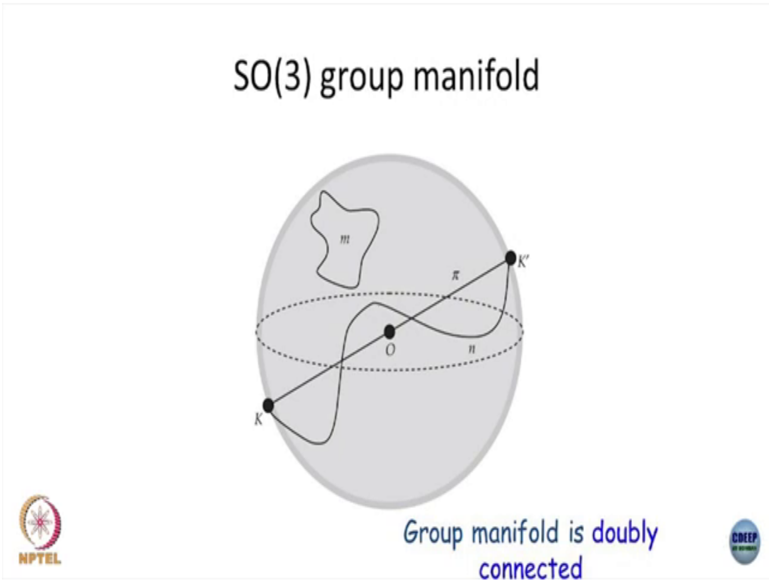

There is also one more small thing you can observe here, ok.

Student: Radius is concurrent by psi pi, is radius will be infinite by psi parameter.

Yeah and it is not infinite, it is only goes up to pi.

Student: Psi parameter.

Psi only, the psi is maximum value of psi is only pi.

Student: It is not a, because radius will be sphere know.

No it is a sphere, but the one which is applicable to rotation I explained it on the screen for you; in the screen I have explained that psi has to lie between 0 and pi.

Student: You talk about spherical coordinate system, it has one radius and two angle. So, angles would lie between pi that is fine; but the radius also should be bounded, radius could go from.

Yeah.

Student: Zero to infinity.

So, if you try to match every element of rotation to a point in this space, the point in this space the radial coordinate is taken by the amount of rotation magnitude of an angle rotation about an arbitrary axis. So, this radial coordinate is replaced by your magnitude of rotation angle and the theta phi gives you the direction about which you are doing the rotation, ok. So, this psi is not exactly the radial coordinated in spherical polar coordinates, this is a way of mapping it onto a region to show that it is a compact bounded space.

Every point here corresponds to, associated with this point you will have a rotation defined ok. If you want to go from point a to point b ok. If you want to do a rotation psi and then rotation another psi one, so you can keep doing that. And then you can start you know going around and coming back to the same point also; this will give you, at every point you will have rotation multiplied.

And if come back to the same point you expect that you have done nothing right, it is like an identity operation. Interestingly this point and this point, diametrically opposite points are all identified. This is one of the property that; if you do a rotation by an angle pi, rotation by an angle pi about the opposite diametrically opposite axis are one and the same ok.

If you do a series of rotations, you could either do a series of rotations and come back to the same point which is like as if done nothing; you could also achieve that you go from here to the boundary, go to the diametrically opposite point and do a series of rotation and come back here, clear.

So, there are actually two non trivial paths of achieving an operation, either you can be inside the sphere; this is a group manifold; manifold or the parameter space with the requirement that diametrically opposite points are identified. So, you can have various paths; this path you can make it smaller and smaller and make it as if it is identity. This path you cannot break it into smaller and smaller into a point, there are two paths ok. So, these are the ones which tells you that the parameter space is a doubly connected space ok.

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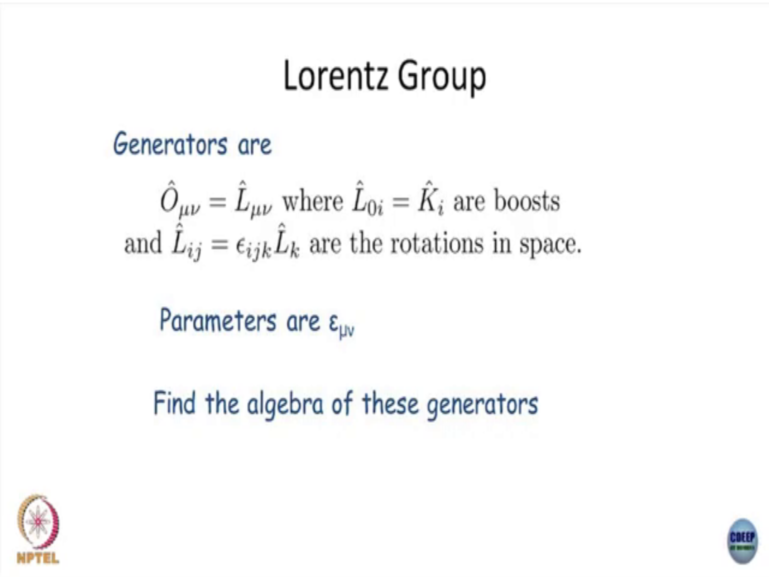

So let me stop here, Lorentz transformation also I explained it to you; that you can write it in a compact notation as L 0 i for your k x t k y t and this is what i said. And you can write your L x, L y, L z in terms of L x, y L x z and L y z ok.

So, this I will leave it you to do the algebra of these generators; there will be one more assignment which will come to you and that will be the algebra for Lorentz transformation. So, we will come to the L i algebra formal definitions in the next class.