Group Theory Methods in Physics
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Lecture – 36
Generators of Lorentz transformation

I am sure you would have done conformal transformations. All done complex analysis and some math course at least, right? You have done, right?

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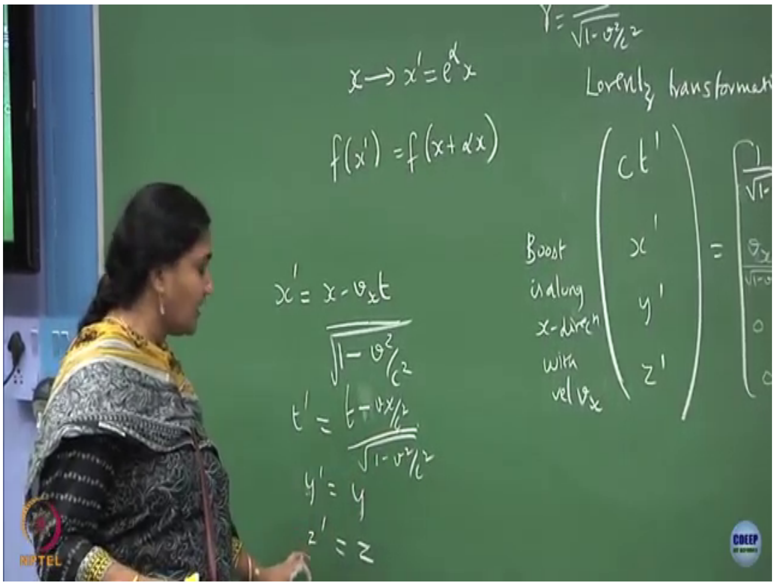

So, suppose I take my x to x prime which is e to the alpha x, what is this? It is confined to 1-dimension. What is this transformation? It is scaling, you just scale the vector by it is all for the x component, y component, z component I can do this. Now, can you find out if I write f of x prime which is let us lake infinitesimal it will become x plus alpha times x.

So, do it Taylor series expansion for this and see what is the generator and you say that this generator does the scale transformation, clear? Is the picture clear? Like, whenever I give you a transformation you can try and figure out what should be the generator or at least you can get a matrix representation for the generator or find an operator like this differential operator, but you will be able to find a form for the generator for every transformation, Lorentz transformation, let us find what is the generator for Lorentz transformation.

Let us take that the second inertial frame of an observer – one observer is stationary, another observer is moving with respect to the stationary observer with velocity v x then what happens to your transformation, like the way I wrote here you could write for that also right, you all know that.

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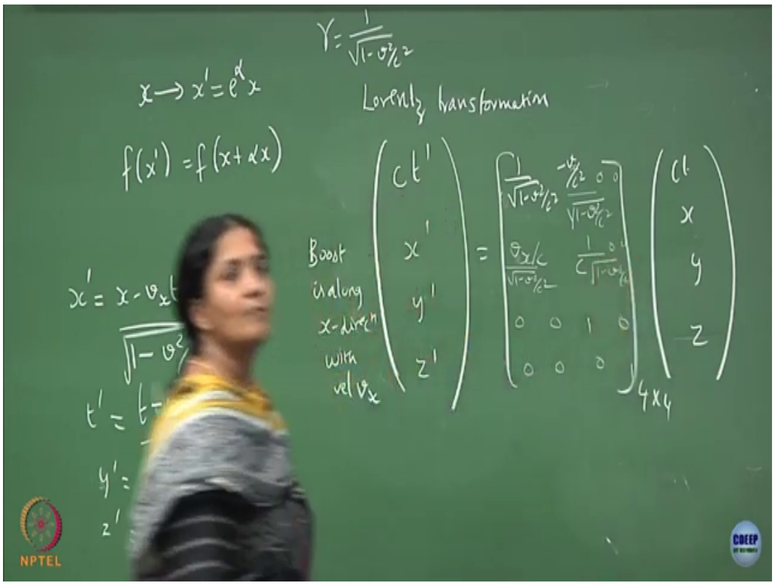

Lorentz transformation you can have c t, will be a 4 cross 4 matrix acting on right. Suppose, I want to do the boosting boost is a long x direction with velocity v x. So, you have done some flavor of spatial theory of relatively. Most of the engineering students have done and physics students have done and I hope you know these Lorentz transformation ok. Just like what I have written here you know how to write that x mine x prime will be x minus v x t by c, is that right? correct.

And, similarly you will have a t prime and then z y prime and z prime will be. What is t prime? v x by c square by 1 minus p squared by c square.

Student: t minus.

t minus sorry, yeah, thank you. Now, you can write this 4 by 4 matrix first term will involve is that right t prime will go to this and then it will also involve the x element v by c squared, probably this root of 1 minus v squared by c squared can be taken out. And, we generally define this one by root of one minus v squared by c squared is gamma right, am I right? This is what we did.

So, let me just give you a. So, this is the way you will write your you will write your elements 0, 0, 0, 0, 0, 1, 0, 0, 1, 0 and this element will be. So, I will leave it to you to check it out. So, this will be I think v x this should also be v x by root of 1 minus v squared by c squared x prime is that with t by c is there right, thank you.

Here also there will be a c. No, this c is not there ok. The other one will be 1 by c root of 1 minus v square am I right? Others are 0. You understand what I am saying? I am trying to do a Lorentz transformation where one observer is moving with respect to stationary observer with velocity v x. This is the transformation I want to find what is the generator for this transformation. How do you do that is, that the next question. You have to take v x to be really small and then find out the deviation away from identity ok.

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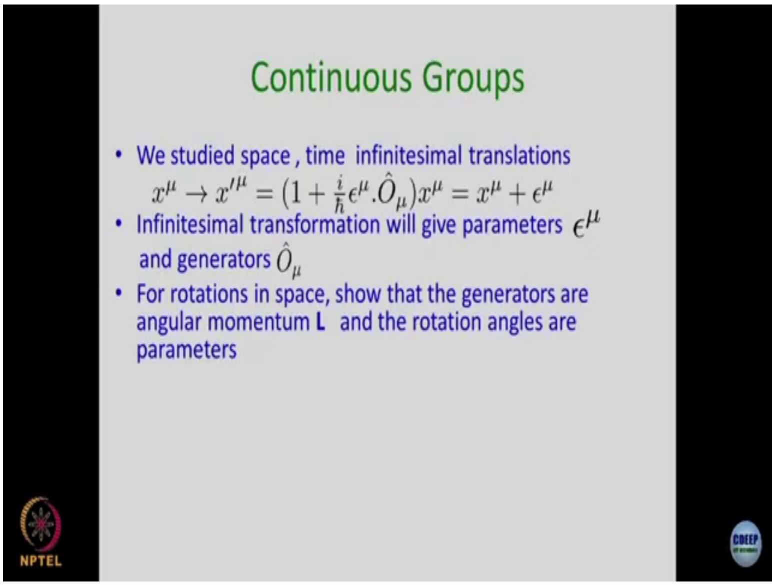

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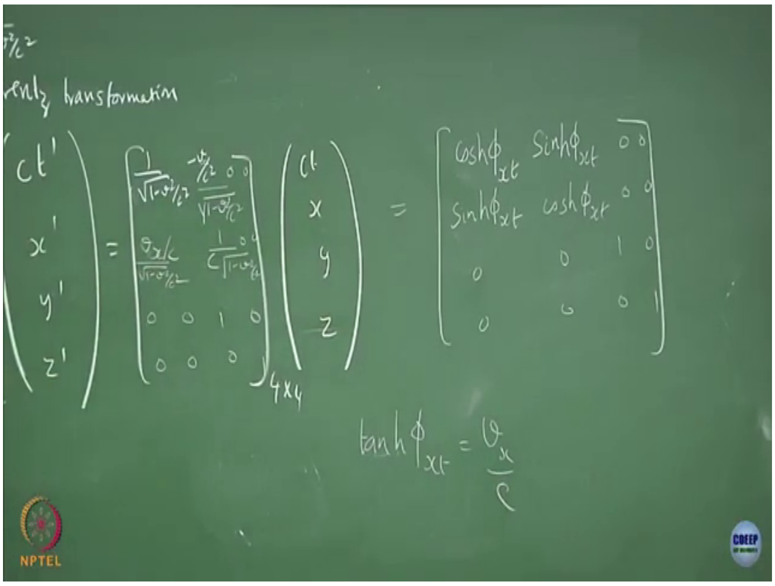

So, there is a nice neat way of writing this. By introducing this can be written as let me write it for you cos hyperbolic phi I will put an x here and t here to remember that the mixing is between the x coordinate and the time coordinate ok; sin hyperbolic phi xt, 0, 0, sin hyperbolic phi xt. So, what is that tan hyperbolic phi xt is v x by c. That is the definition. Please check whether the short hand notation helps.

The reason why I have done this is it resembles as if you are doing a rotation in the xt plane, but not the conventional trigonometric rotation, but your hyperbolic cos hyperbolic ok. This hyperbolic is also something we should bother you. The thing is if you have a time and x you know it is not like two spatial coordinates. You have to remember that a event 1 followed by an event 2 if it happens in one frame it cannot happen that event one is happening prior to event 2, like you know the causality cannot be violated.

If event 1 happened in frame 1 at time t 1; event 2 happened in frame 1 at time t 2, t 1 prime and t 2 prime so, t 2 is greater than t 1 here. So, t 2 prime and t 1 prime; t 2 prime should be greater than t 1 prime that is very important. Such things are not required if you are just doing spatial rotation and that is where your cos hyperbolic comes into picture. It comes into picture that you know it takes care of causality. A cause and an effect which happens in one frame cannot become an effect and the cause on another frame. It has to also be a cause and effect ok.

So, so let me write it in a fashion which is closest to looking like rotation in the x t plane, but not quite for boosting along x direction it is hyperbolic ok. So, similarly if you do boosting along y direction you will write cos hyperbolic which mixes the y and t and I will put the phi with yt and z direction. Besides this you can also have rotation in this 3-dimensional space sub space which is there in 3 space in one time that is also allowed. Earlier I have using it as an axis now I would like to rewrite it as if it is a suppose I want to say rotation about z axis I would like to write in the same notation as if it is a rotation in the xy plane, is that clear?

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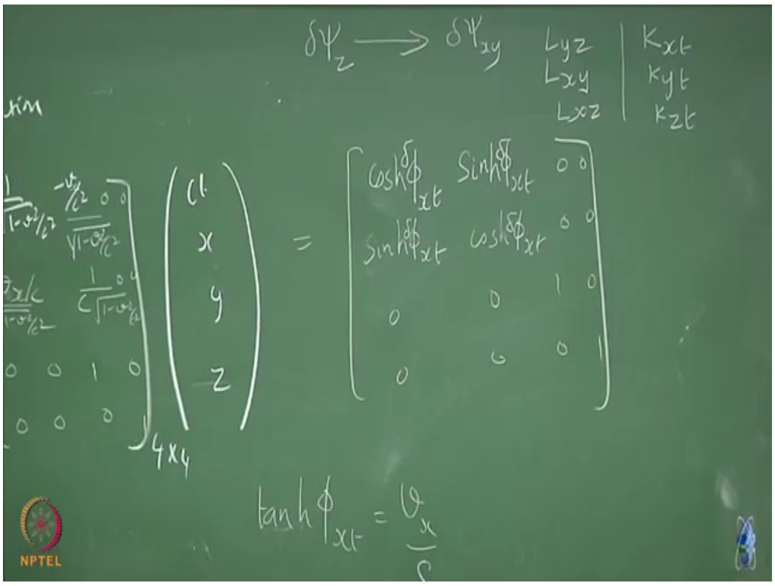

So, whatever I was writing delta psi z I will now replace it by a delta psi in the xy plane ok. When I go to more than three space there is no point in talking about axis. Why there is no point? There can be many axes right. If you are in 4 dimensions, as an abstract 4 dimension x, y, z and then let us put one more w as space what does an axis perpendicular to the plane xy plane? It could be z axis, it could be omega axis w axis right 2 two other coordinates can also become normal to this plane.

So, there is no point in talking about hat direction axis direction perpendicular to the plane on which rotation is happening, it is not unique. Instead it is better to say that there is a rotation which mixes the xy coordinate rather than writing it as rotation about z axis. Rotation about z axis is good if you are just doing only three space, but if you start doing event space time it is nicer to write them as rotation which mixes xy coordinate rotation which mixes yz you can also call the boost as quote unquote and what exactly rotation, but which mixes the x and time coordinate, clear. So, how many generators and how many parameters now, tell me?

Student: (Refer Time: 13:28).

Independent.

Student: 6.

6 right. You will have analog to this you will have L xy, L xz, L yz. These are like your angular momentum which you did and then you also have K xt, K yt and K zt. I am using a different letter K just to remember that the transformation involves hyperbolic – cos hyperbolic and sin hyperbolic which mixes those index notations. The subscript tells you that it mixes, x and t in this case and that is why this notation transformation ok. Just like I wrote for rotations you can now right 4 cross 4 matrices for the generators for Lorentz transformations ok.

Can you do that? Can you write what is K xt now from here? So, let us take this delta phi to be small and then what is K xt can be read off from here? Yeah.

Student: (Refer Time: 15:07).

This one you understand right it is only mixing x and t, this is mixing only t and x; y is not touched, z is not touched. Just to keep track that it mixes only x and t due to boost boosting along x direction I am going to call this parameter phi with that xt notation to keep track that that is the fundamental transformation I am doing, that is all. Is that clear? So, I can do fundamental transformation which mixes x and t, which mixes y and t and z and t, but they will involve cos hyperbolic functions because they are all Lorentz transformation.

Besides that you can also do rotations in your physical space and in the same notation which I am writing I would like to write an arbitrary transformation in space time as delta phi dotted with operators with the number of phi's is same as the number of operators, to do that I have introduced that. Instead of saying I am rotating about z axis, I would like to say it is a rotation which mixes the xy coordinates. Yeah?

Student: (Refer Time: 16:33) z only, but we are using a different notation (Refer Time: 16:38). It is same as z (Refer Time: 16:40) different.

It is just the systematic generalization which helps us to keep track of what coordinates are getting mixed, yes.

Student: We are not mixing the actually in the rotation like what we did in (Refer Time: 16:54) we are not using anything.

We do mix know. When you do that rotation about z axis x and y coordinates only gets mixed x prime is x plus theta times y y prime is just y minus theta times x. So, it is mixing x and y, it is consistent. We are not putting at all. It is just said that the axis was very unique in the case of three dimension if you want to look at it as if it is an access perpendicular to the plane on which I am rotating it could have also been a t axis and there is a confusion.

So, we do not want to get into that confusion. Instead we want to use the parameters not with the subscript z x y we would like to use the parameters subscript parameter was subscript which tells you which coordinates are getting mixed, is that clear ok?

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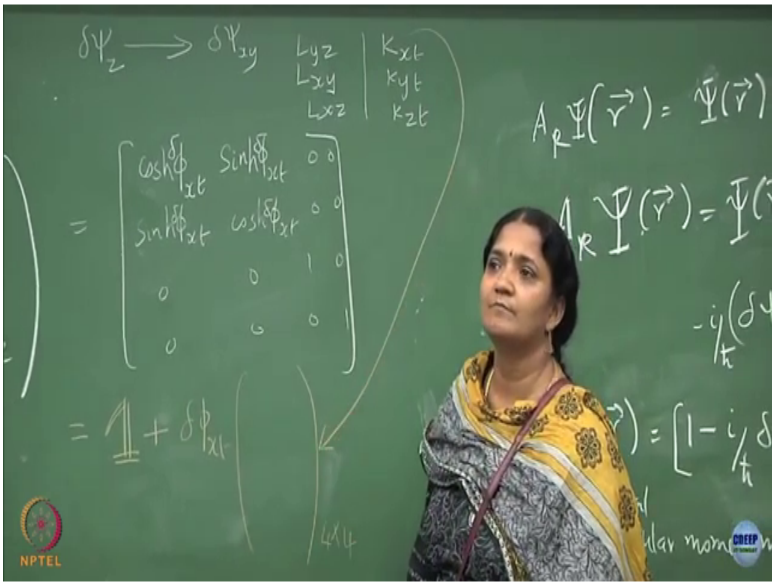

So, let us write it out. Can someone help me out? It is identity plus delta phi xt by the way this I have written it here, but please check that tan hyperbolic theta is giving you v x by c ok. And, then you have a 4 cross 4 matrix and that is the matrix which is going to be proportional to which one?

Student: (Refer Time: 18:27).

Louder. Which one?

Student: K xt.

K xt. So, this is the one which is this ok. So, you can write matrix representation for all the six generators this way. After you write all these six generators what is the next step we need to do? Amongst these generators we want to see what is the.

Student: Commutation.

Commutation relations and we will come to why we are doing this commutation reaction so far. So, amongst these generators let us look at the commutation relations and figure out what is happening.

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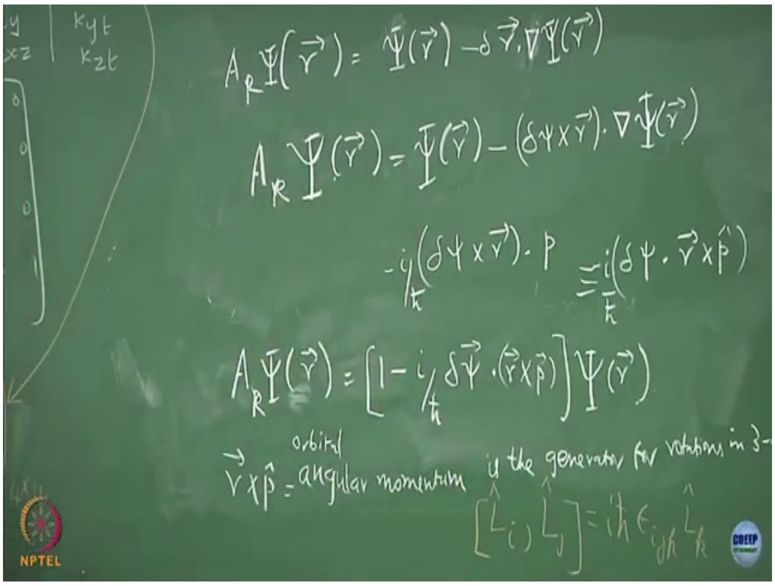

Incidentally, here the commutation relations everybody is familiar right L x, L y, L z have a nice commutation relation L i-th component, L j-th component will always give you epsilon ijk L k. I am not going to do this you would have done it in your quantum mechanics course.

In translation I said p x, p y commutator 0 imply something what did I say? p x, p y are generators for translation along x direction and y direction; if the commutator of p x, p y is 0 it means that you can do translation along x and y in whichever order you want to do right.

Similarly, here these are generators for rotation about direction i, rotation about direction j, but the order matters. You all know that right if you do a rotation about x axis, do a rotation about y axis followed by a rotation about y axis it is not same as doing a rotation about y axis followed by rotation about z. So, that information is contained in the algebra generated by the algebra of generators which satisfies non-trivial commutator. So, the rotation elements which I am finding it out will always be abelian or non-abelian?

Student: Non-abelian.

Non-abelian. So, the rotation elements which I am going to find is always going to belong to a non-abelian group that is because of the algebra of generators the commutator brackets of these generators are (Refer Time: 21:40) ok.

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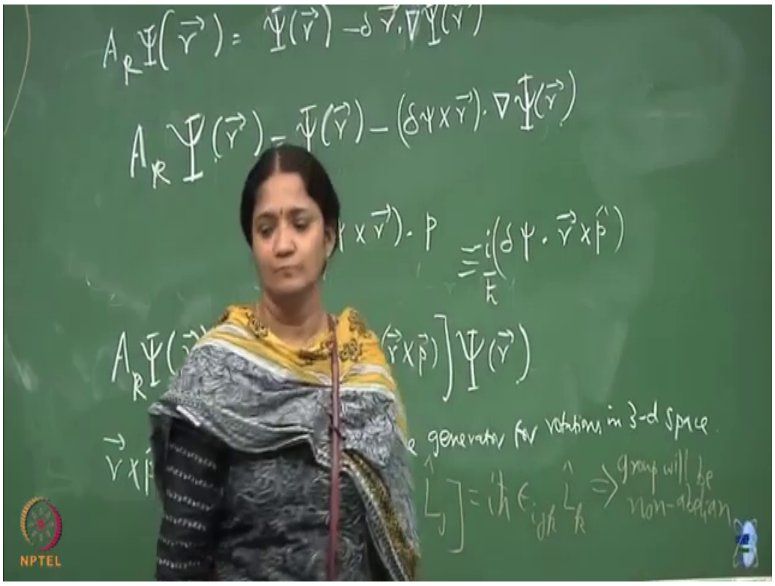

This implies the group will be non-abelian ok. So, I have given you some warm up. Let me go onto the slides and then we will come back to more staff ok.

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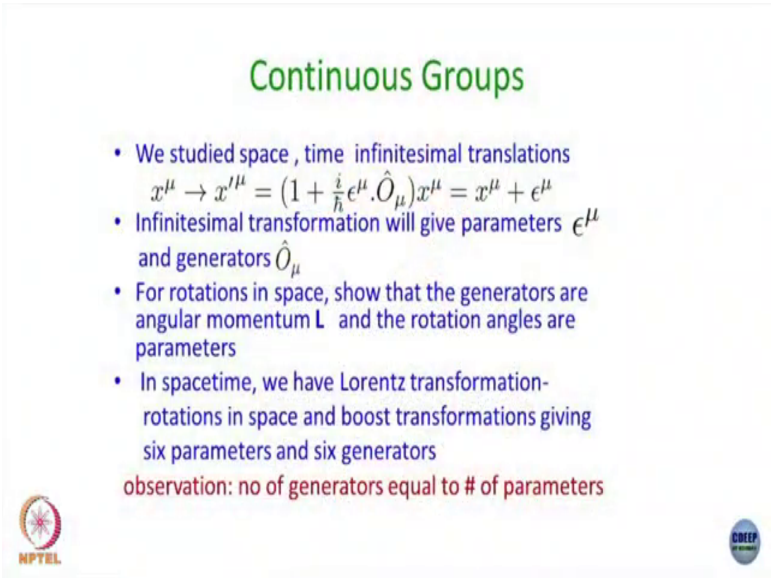

So, for rotation and space show that the generators are angular momentum and rotation angles are parameters. I have already given you the explanation on the board. In space-time, we have Lorentz transformations rotations in space and boost transformation; boost is what is the jargon which is used for moving one frame with respect to another frame with velocity v x ok; v x, v y, v z. So, clearly you have six parameters in six generators, I will associated with each one you will have a parameter.

So, what is the observation from whatever I have been saying? For any transformation you can actually sit down and figure out what is the corresponding generators, how many generators are there and how many parameters are there and both the number of generators and the number of parameters will be actually same, clear.