**Group Theory Methods in Physics**
**Prof. P. Ramadevi**
**Department of Physics**
**Indian Institute of Technology, Bombay**

**Lecture - 35**
**Generators of translational and rotational transformation**

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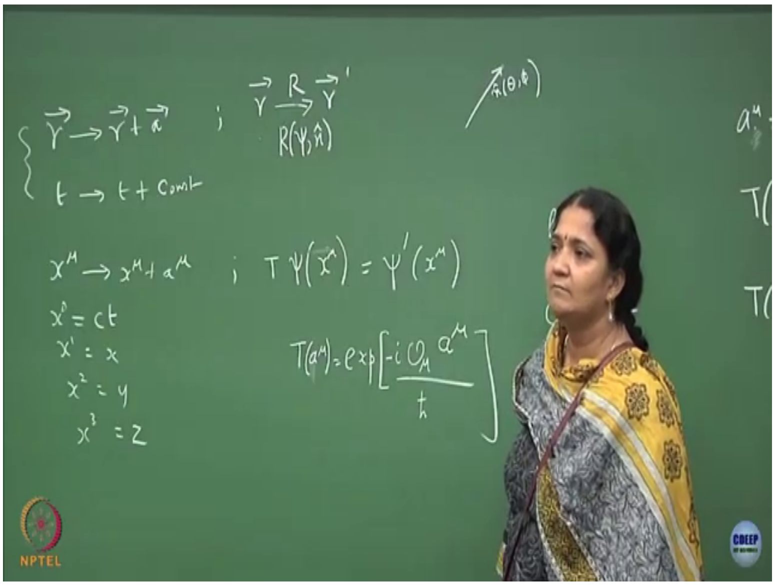


r going to r plus a and then time going to time plus some constant right. These 2 together we could write it as x mu going to x mu plus a mu where x 0 is like time with dimensions if you want to write you have to write the velocity of light multiplied by time and x 1 is x, x 2 is y x 3 is z ok. So, these are the compact way in writing in a space time notation. So, this is space translation. This is time translation, this is compactly a space time translation ok.

So, this space time translation be elaborated so much. There exists an operator right which when acts on x mu will give you psi prime of x mu and we determine what that operator was right. We did this for simple one dimensional motion in x direction and then I generalized to arbitrary r vector and I also said what will happen for the time translation. And what did we find? We found that T in general will be an exponential with an minus i and then you have an operator I am going to write it in a compact notation. You will have an operator associated with an operator there is a parameter divided by h cross.

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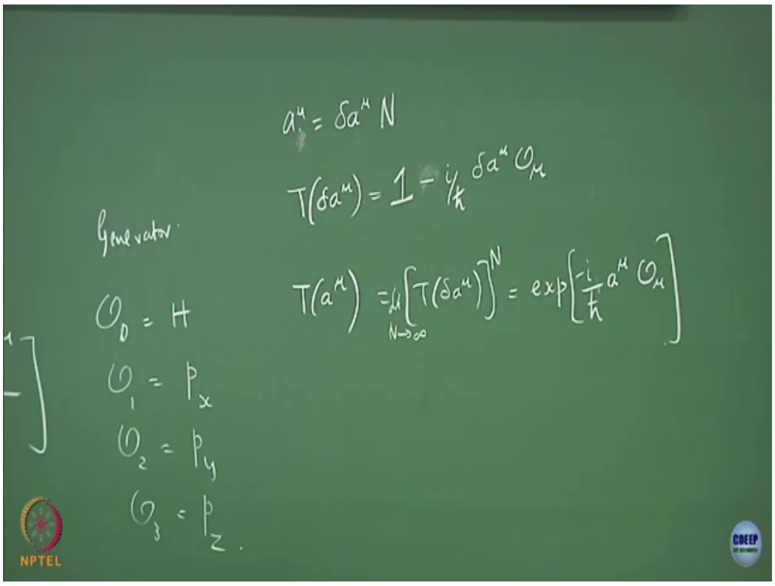


And what is this operator going to be O 1 will be O 0 will be Hamiltonian or the energy operator, O 1 will be your momentum p x. O 2 will be p y and O 3 will be p z is that clear.

The corresponding parameters can take any value between. It is not bounded you can take your time to be if you start the time as 0 then it is 0 to infinity, but if you say your initial time is minus infinity then it is minus infinity to plus infinity. The positions can also be between minus infinity to plus infinity. So, in principle this is going to give you a parameters in 3 dimensional space and one time direction.

But what you have to observe is that this operator is what we called it as a generator. So, what is the role of a generator is that if you do an infinitesimal transformation by infinitesimal I can write my a mu a mu to be delta a mu times n. You can do the finite translation in n steps each step having an infinitesimal step ok. So, these delta a mu is a very small and n times this will be the total finite translation.

So, if you do it this way you can write what happens. So, I should write that this T is dependent on a mu; dependent on a mu and now I can write what is T as delta a mu and this will have always the first term will be identity and then the next term will be some i by h cross the parameter times the generator which I can call it as some O mu. And instead of this delta a mu you could replace it by a mu by n ok.

So, because this non trivial transformation takes you away from an identity state. If you operate it on some state some vector the first one is like as if it does not do anything and this gives you a deviation or takes you away from its initial position and infinitesimal transformation is good enough to take you away from the initial position and that is why this is called generator for such a transformation. And equivalently you can write any finite transformation, you could do this n number of times ok. So, T of delta mu you can do this n number of times. Is that right? This is for infinitesimal step and you are going to do n steps for every step you do a T of delta mu.

So, this will be n number of times and this can be shown to be you can do that on this and show it to be a exponential. And you have to take the fact that limit of n tending to infinity because the number of steps for a finite transformation will be infinite delta a mu is really small this product will give you a finite value. So, this will give you an exponential form which I was telling you that minus i by h cross a mu operator ok. Fact this plus is also can be in general you can use you can observe these signs into your definition of a delta a mu. So, since I have follow this let me just keep it like this. Is this clear?

So, this step is straightforward this is the definition of your exponential right. You put delta a mu as a mu by n to the power of n, n tending to infinity will give you exponential clear. So, that is why we call this operator as generator.

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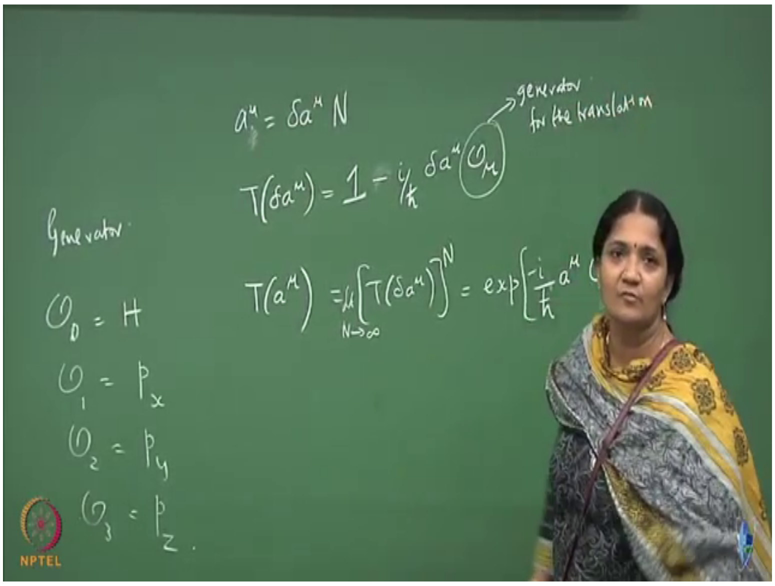


Generator for the translation in this particular case, but in general it could be some other transformation like rotation or other internal transformations which I could look at. So, right now let us just confine to some examples.

So, translation is something which I have discussed. You can even take it for rotations a deviation from identity which operator performs such a deviation that operator differential operator is what you call it as a generator of this such a transformation. Is that clear ok.

So, coming back to the slide.

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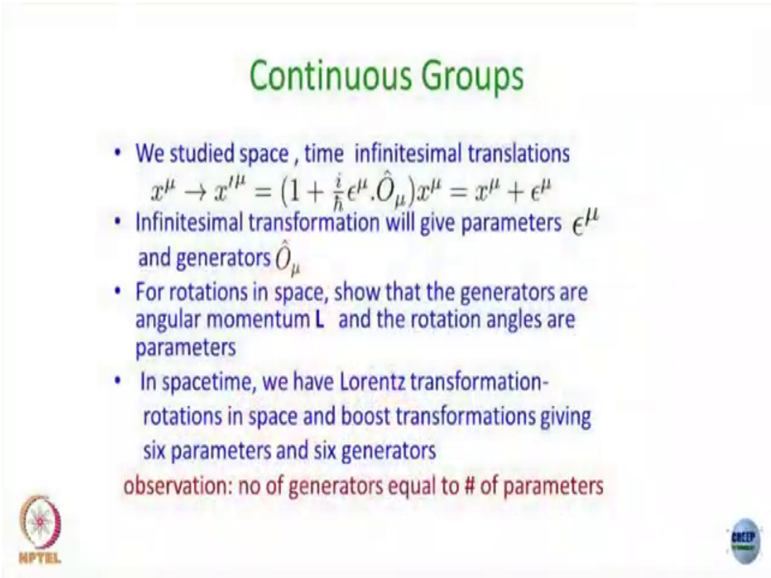


So, we studied this under space time infinitesimal translations ok. You have a slight change away from identity as I said this plus or minus i is just a matter of definition you can observe it into an epsilon ok. So, this keeps as the epsilon mu is infinitesimal and x mu is the initial position and it gets shifted by an epsilon ok.

So, infinitesimal transformation will give parameters epsilon mu and the corresponding generators is ohm. For rotation and space you have to show that the generators are not linear momentum. So, we already saw for translation the generators were linear moment. So, redo this exercise for rotation. What is rotation going to do? You have a vector r and you do a rotation operation and gives you an r prime right and the rotation will depend on an angle and a choice of an axis right.

So, r will depend on some rotation angle let me call it a psi just for uniformity with my slides and a choice of a direction unit vector direction. You have to choose z axis, x axis, y axis or arbitrary axis and had denotes the arbitrary axis about which you rotate by an angle psi is that clear, you have an arbitrary axis. Usually a unit vector will be given it can it can denote your theta phi angle in your spherical polar coordinates ok. Take a unit sphere the direction of it on the unit sphere will give you different direction corresponds to different theta and phi. So, unit vector on the sphere will generally depend on theta and phi.

If you take the z axis theta is 0 ok. So, similarly you can take other axis also theta equal to pi by 2 will be your x axis and so on. So, you can choose your axis accordingly. In general the rotation can be will be fixed by an axis of rotation and the amount by which you are going to do a rotation clear.

So, this is the rotation operator and it is going to change your vector r to r prime. The next question you will ask us how do we write the infinitesimal transformation. So, just for simplicity let us take the rotation about z axis ok.

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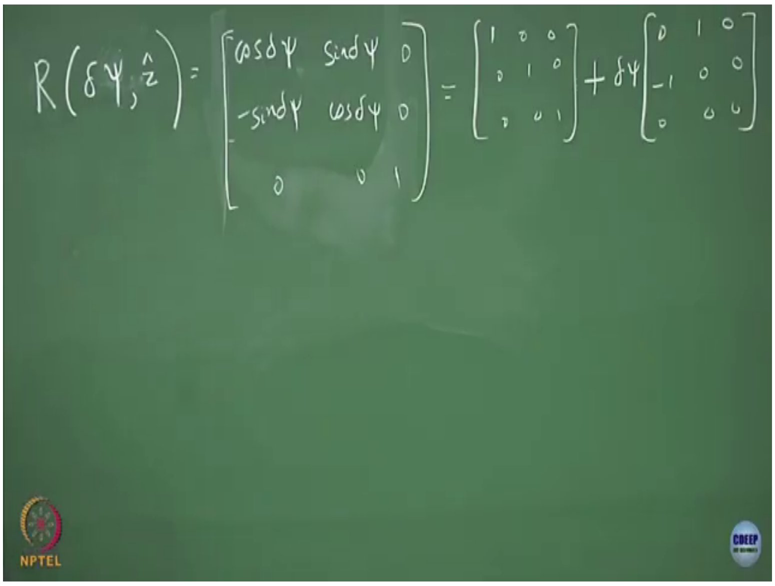


So, let me take it to be rotations by some infinitesimal angle which let me call it as theta or if you are getting confused let me put it a psi about z axis. So, this is z cap direction. You all know what this is. That is first one you can write this as cos delta psi sin delta psi minus sin delta psi cos delta psi because I have written it in the x y z basis. So, which means I have to make it into a I am looking at rotation of vectors in 3 dimensions. So, this is going to be clear and because delta psi is small you can rewrite this as identity plus a delta psi ok. Is that right? Now, you tell me what is the generator up to some normalization and sin.

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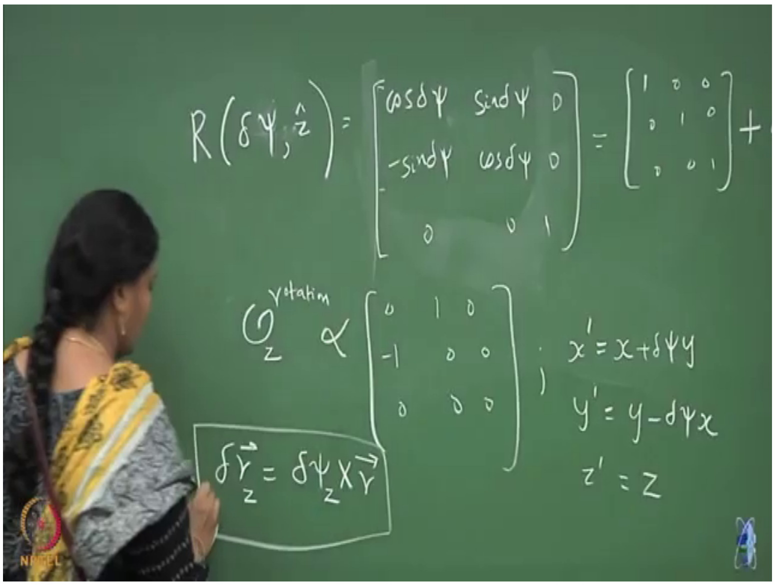


The generator let me call it as O subscript z for rotation will be proportional to 0 1 0 minus 1 0 0, 0 0 0. This is a matrix representation for the generator executing a rotation about infinitesimal rotation about z axis, clear.

But just like the way I did here I would like to write this also as r vector plus some infinitesimal quantity delta r was del a delta a mu here. I want to find what this delta r is. Can you help me out with this using this? So, just check that delta r for rotation about z axis can be written as delta psi z cross r vector. Can you check this is that right?

So, if you use this you can show that x prime is x plus delta psi y right. Using this ro r matrix x prime the components x prime y prime and z prime will be ok. I could compactly write that as the change in vector for a rotation about z axis is this, but in general I could write this as if it is arbitrary axis by putting a vector rotation here and r of course, is a. Is this clear? Because this is the z component cross product gives you only either x or y the z component cross z component will be 0 ok.

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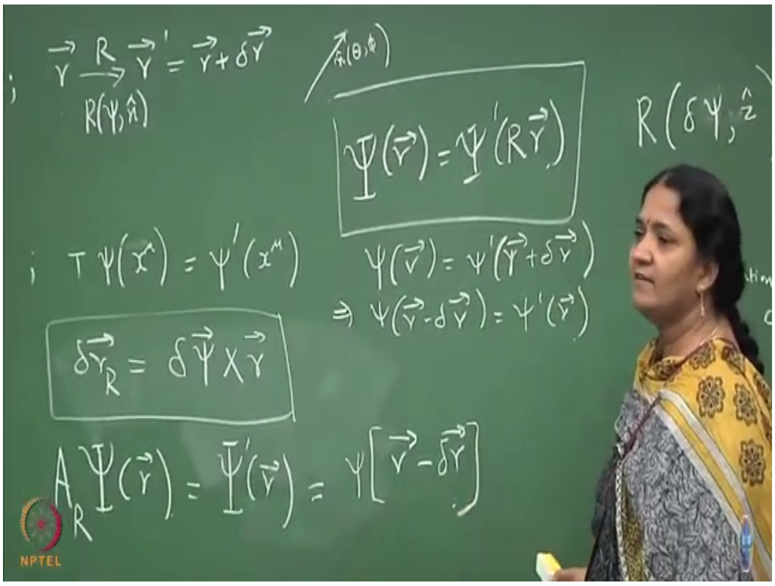

So, that brings me to writing this delta r under rotation about arbitrary axis can be written as delta psi. This is now an angle angle which is a vector in the sense that you can resolve it in 2 components about z axis, x axis and y axis ok.

So, this is this psi will be on an arbitrary hat n axis cross r vector ok. You all with me. Is this clear. I just given you explicitly an operation where it is done with respect to z hat. So, this has this is this z component other components are all 0, but you can choose it to be an arbitrary n hat where it will have z, x, y, z components and then you can write the change in the vector in 3 dimension under the infinite symbol rotation by an angle psi the psi is. I am using the psi to be the angle of rotation about arbitrary axis. Arbitrary axis is given by the unit vector or unit vector will give you theta and phi coordinate. Is that clear ok?

So, now once I give this what is the next step? I need to find an operator a with respect to rotation which acts on the wave function sorry I am using the same thing, but I am putting a capital psi here to give me psi of psi prime of r vector just like what we did for translation. We want to find an operator where you can also use the additional fact. What is that additional fact? Psi of r vector is same as psi phi prime of r of r vector; r is the rotation operation on this vector and these 2 are one and the same.

I am just doing whatever we did for translation in the context of rotation. You rotate this wave function state as well as the space nothing changes. And you want to find an operator at the same point how the functional form of psi changes yeah. Any questions on this?

So, tell me now you will use this fact to write the r which I have as. So, you can do so do this and tell me what will happen. So, you can write this as r inverse of r prime right. Use this fact to write it and then do the translation. So, let me write it for you. This one is same as. Is this right? This one is r prime r prime is r plus delta r right.

So, same as this for infinitesimal whatever I call it as r plus delta r let me call it as r then the delta r comes here with an opposite sign right, this implies ok. Are you all with me? I do not see anybody nodding their head sleepy tired fine good. So, this is the expression which we get. Now, do a Taylor series expansion. So, do a Taylor series expansion for this.

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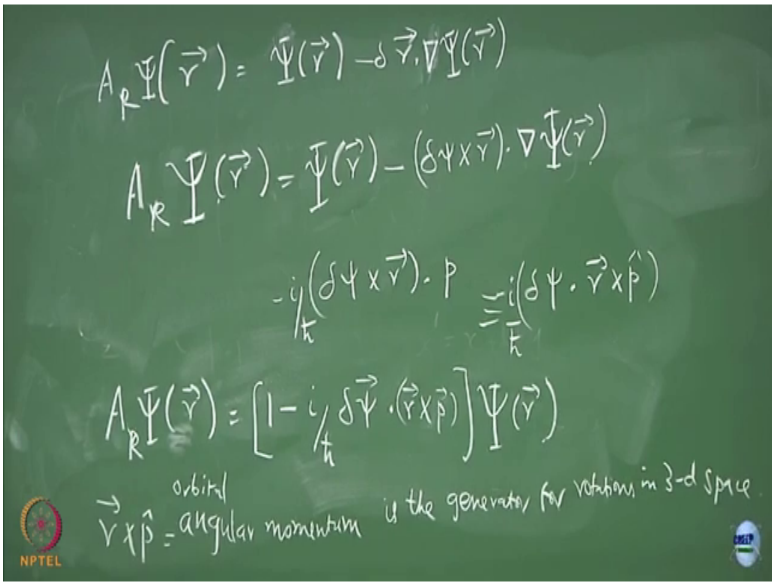


So, the first term is psi of r minus delta r times, sorry this psi should be a capital psi so. In fact, I should write technically as a dot product with the gradient ok. Just a Taylor series expansion and what is delta r? Delta r is delta psi cross r ok. So, this is ok. I am putting a capital psi for the wave function ok.

So, what is that one? That is a scalar triple product, I can play around the scalar triple product and write it as tell me. So, this let me call it is p with a i by h cross introduced here ok. So, this is equivalent to z right. Maybe I made a mistake on the signs, but you understand what I am saying ok. So, this is a scalar triple product and it can be rewritten as delta psi dotted with r cross p. So, I have given you now the generator ok. So, a r of psi of r, s identity minus i by h cross delta psi dotted with r cross p on. Are you all with me?

Now, tell me what is the generator for rotation now? Infinitesimal rotation which I am doing, this r is for the rotation. This is the operator which takes you away from identity. The one which takes you away from identity for an infinitesimal rotation is this piece and this piece is what we call it as a generator for the corresponding transformation which is the rotation ok.

So, r cross p which you all know what it is; it is angular momentum orbital angular momentum. Orbital angular momentum is the generator for rotations in 3 dimensional space 3d space. I am confining myself to rotations in 3 dimensions ok.

So, how many components are there? Is r cross p is a vector. So, there are you can have l x, l y, l z like px, py, pz. There are 3 generators and the corresponding parameters are your delta psi z, delta psi x, delta psi y depending upon your fundamental axis of rotation which is like choosing x axis, y axis, z axis ok.

So, what am I told you now, always you will find number of generators will be equal to the number of parameters that is why you can do this dotting dot product of parameter with generator otherwise you will not be able to do that. And you can determine what is the generator for simple cases like this sure.