Group Theory Methods in Physics
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Lecture - 34
Introduction to continuous groups

So, now let me just warm you up with this piece today.

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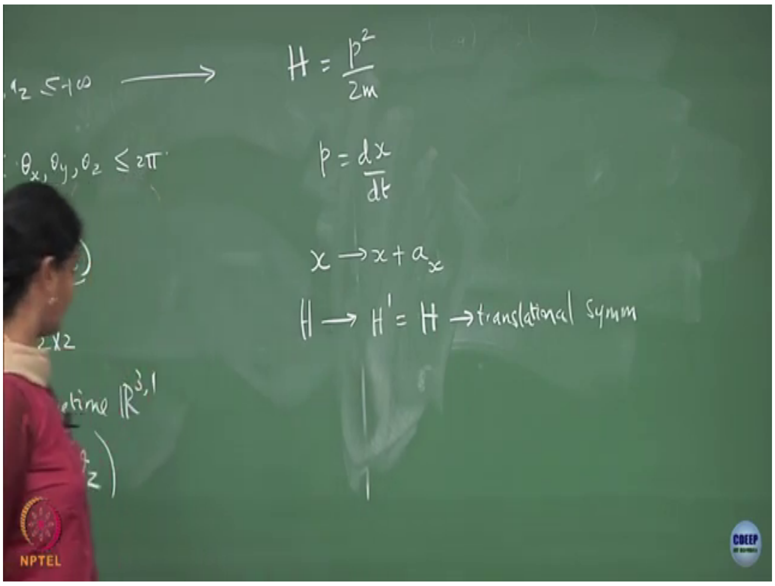

If you take a free particle ok, what happens to free particle? Free particle is described by a Hamiltonian, which is given by p squared over 2 m, where p is nothing, but dx by dt right. If, you take x let me confine myself to 1 d problem, let us take the motion in 1 d and understand translation. If you take x to x plus a technically it is a x. Hamiltonian is it is a constant.

What do I mean by a constant, time derivative of a x is 0 ok. Hamiltonian goes to a new Hamiltonian, which is same as the old Hamiltonian, which means this has free particle respects translational symmetry, by any vector a x ok, any vector a x you do not get anything new ok. So, now let us do this operation.

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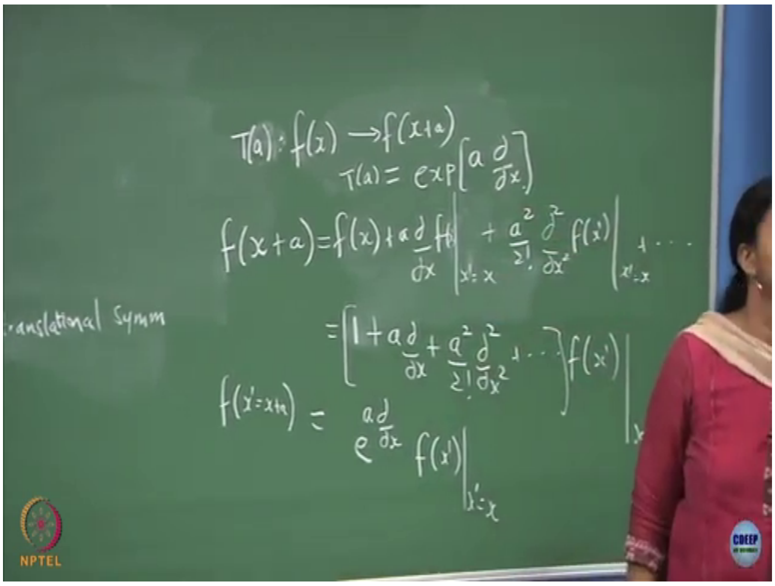

So, take a translation operator, let us do it on some arbitrary function of x, and let us take this translation operator, with unit a it is supposed to give you function of x plus a ok. Now, I want to find what is the explicit form of this T operator, which when acts on arbitrary functions shifts the x position to x plus a correct?

So, let us take x plus a, you are all familiar with Taylor series expansion. So, first will be f of x then a times del by del x f of x, you do the del by del f of f and then put it at you want you can call it x prime equal to x right you know what I am saying; a squared by 2 factorial del squared by del x squared and so on ok.

So, I would like to write this as 1 plus 8 times del by del x. So, prime is going to be ok, this is same as what I wrote in the previous step. Now, this expression which I have written here, can be write it in more compact form can put an exponential and write it out, exponential of del x times f of x prime ok.

Now, what is the T operator which does the translation by a, you know that? So, from these exercise you can show that T of a clear. A similar thing you should be able to do for rotations also ok. Before, we do rotation let us finish this and then you can try it out for rotation we can discuss in the next class.

So, what is the observation? Observation is that any operation which I do. Let us take an infinitesimal translation.

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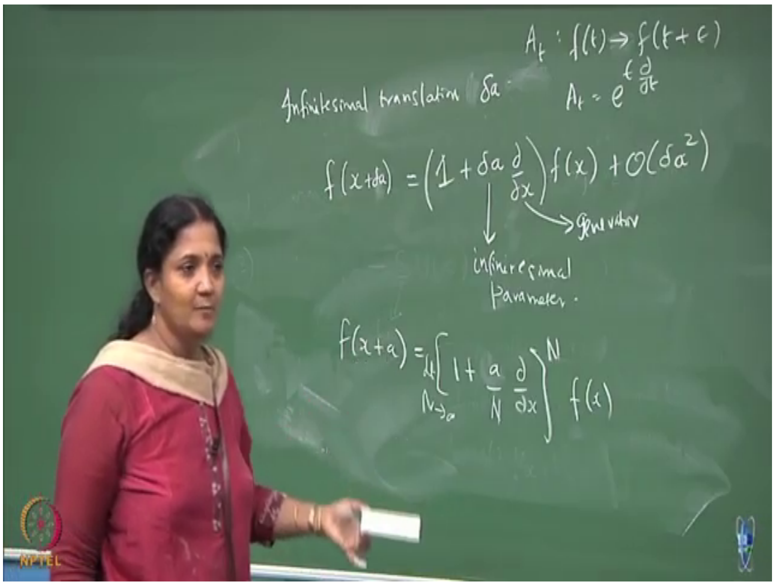

What do I mean by that? Let me call that a, to be some delta a delta a squared a small delta a cube is small. So, I can say that I can in principle write f of x plus a to be 1 plus delta a del by del x, f of x and ignore all the other terms ok. Anything powers of delta a squared and so on is negligible compared to this?

What do we see here? We see that the translation takes you away from it is initial position identity position. So, non-trivial generator, which is associated with the parameter involved in such a transformation ok. And, that is the one which gives you the change or transformed function due to such an operation ok.

So, what I am trying to say is that this is the infinitesimal general, infinitesimal parameter ok. This is the operator which takes you away from it is identity state, this is what we call it as a generator ok. And, once I have these 2 I am able to try and do this for arbitrary translation also. How do I do arbitrary translation?

If I want to do suppose I say n times delta a is a arbitrary transformation, then delta a can be written as a by n am I right. So, if you want to do phi of x plus a I can do this as 1 plus delta a del by del x, how many times?

Student: N times.

N times you can do this and replace this by a by N take limit of N tending to infinity expression also you know.

Student: (Refer Time: 10:37) a by N tends to 0.

It is the same thing right a is finite. So, it is the same delta a tends to 0 is equivalent to N tending to infinity, this is all familiar to you, what is this exponential function right you are all familiar with this. What am I trying to show you is that any finite translation, can be broken up into infinitesimal translation, it is all it is not discrete like the way you did in the earlier part. Discrete means if I say that the object has a c 4 symmetry, I can only do it in the multiples of c 4 symmetry, but here the translation or in fact, rotations also.

I can do an infinitesimal rotation and then I can make the finite rotation by taking you know N operations of infinitesimal rotation. So, that is the advantage of getting any finite transformation, you can achieve it from the infinitesimal transformation ok. So, in that sense the only important information is to confine to an infinitesimal translation or transformation, all other information, which I want is already contained any finite transformation can be broken up into infinitesimal transformation.

So, I have to only confined to an infinitesimal transformation, they are all continuously connected ok, is that clear? This is the first difference between what you did in discrete groups and continuous groups. Whatever I have said for translation the same thing I can say for rotation the parameter will be delta theta and then you can do an infinitesimal rotation. If, you do an infinitesimal rotation, you can find that there will be a deviation from identity and the corresponding operator is what differential operators, what we have to figure it out, but right now we will do it we do it slowly ok.

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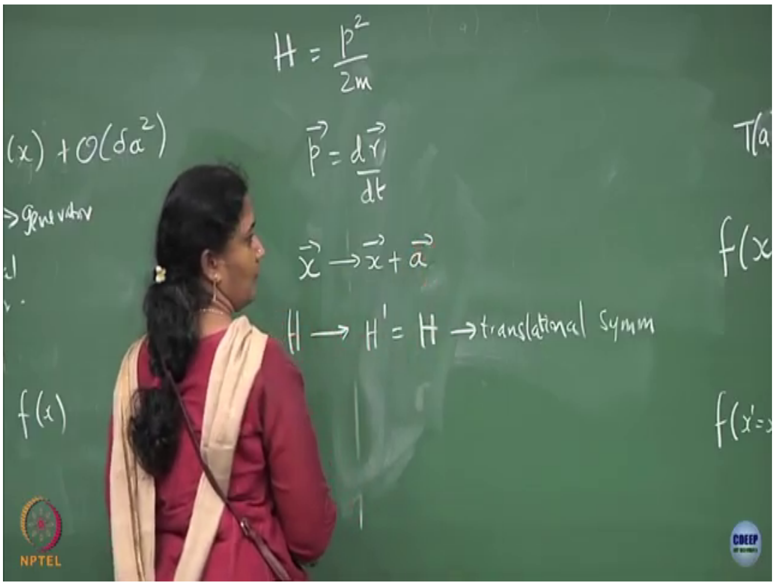

Student: Ma’am.

Yeah.

Student: Why you should write as f of x plus delta?

Yes here it is fx, thank you only when you do the powers it is thank you. Yeah is that clear. What happens if you go to 3 d, it is also translationally invariant, what happens there?

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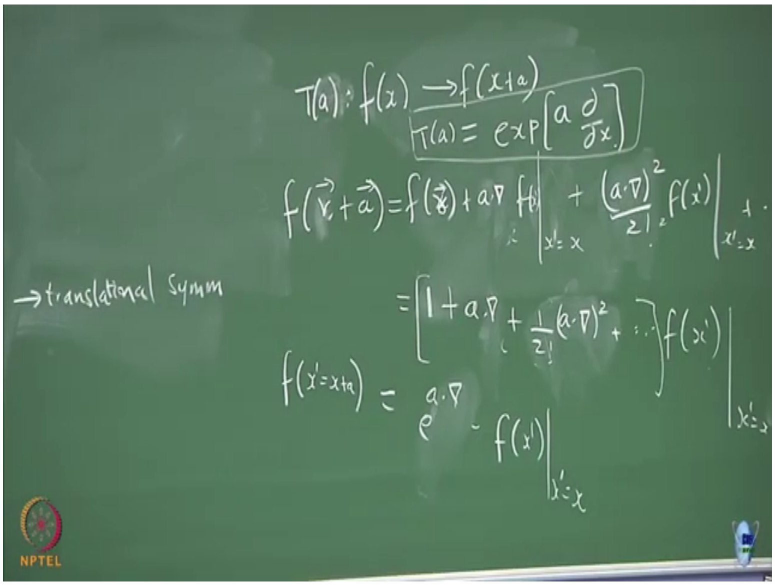

Somebody a dot del and so on. So, this will get replaced as clear. So, what do we see you will have translations generators will be del by del x del by del y del by del z. So, let me write that also ok.

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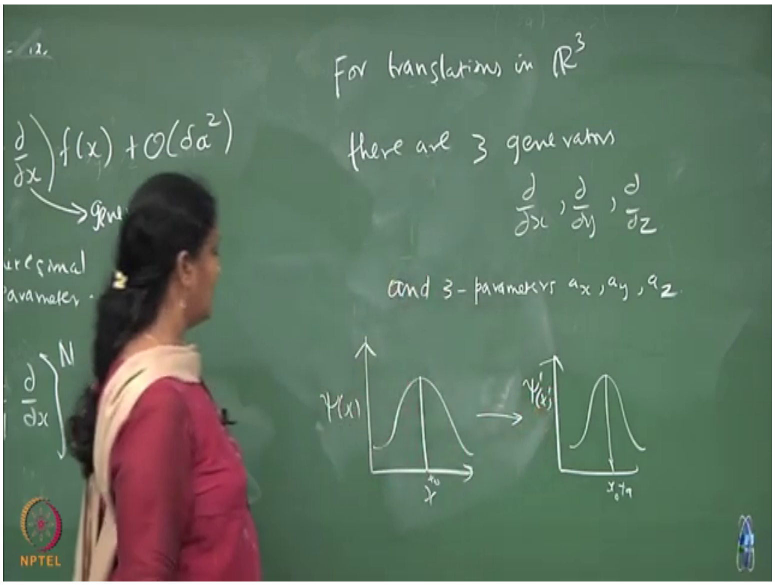

And, so, always comes as a dot del, what does it mean? The number of parameters must also be equal to the number of generators and 3 parameters ok. So, number of parameters is equal to the number of generators it is the next observation and you also have this parameter having values between minus infinity to plus infinity right.

So, in some sense the parameter spaces there is no bound, it is a complete 3 dimensional space with no boundary like it is unbounded ok. So, this is the observation for translation ok. Now, if you go to quantum mechanics and look at the wave function psi of x for a free particle and if you say that it has translation symmetry, what does it mean? You will have a wave function; let me do it in one dimension.

Suppose, it is like this which has a peak let us say at x naught or something, if you do a translation the wave function changes to psi prime and the peak is going to get shifted right x naught plus a, but whatever I am doing here is same as whatever I am doing here ok. So, this is a function of x prime.

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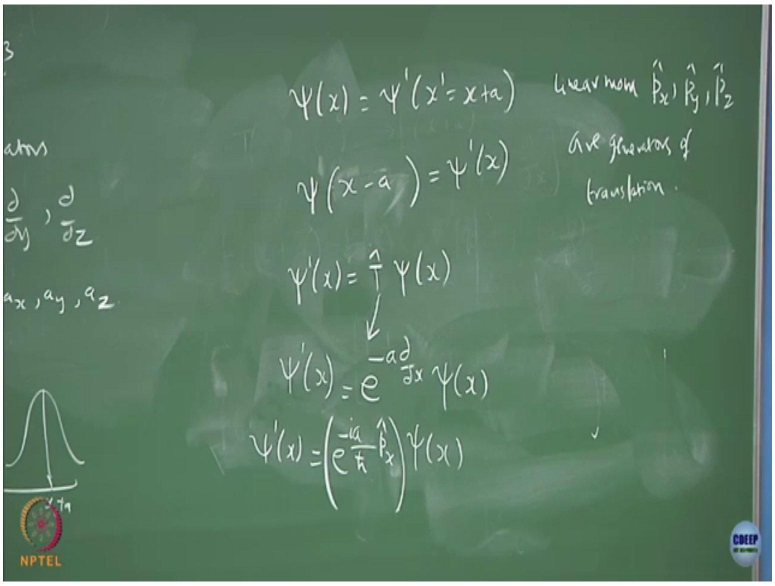

So, psi of x and psi of x prime are not going to be different ok. So, suppose I want to write psi prime or psi of x prime x plus a, or equivalently I could use this property and write a slightly different thing, we shift the x to x minus a. So, this will become psi prime of x ok. And, then do the same Taylor series expansion is some T operator on psi of x and you need to figure out what this operator is what will that be e to the power of ok.

Now, in quantum mechanics it is nicer to write it in terms of Hermitian operator's del by del x is not Hermitian. So, you introduce your Hermitian operators which are px operator. So, that psi prime of x is plus or minus minus a ih by ih cross h cross will be yeah I will be up I think del by del x ok.

Student: px.

P x yeah sorry del by del x is already replacement ok. So, this is what you will have, such a transformation it is not going to affect your physics, it is going to give you the same energy of the particle right. So, now, what do we see in quantum mechanics, because of making this operated to be Hermitian you end up getting this, I you still have a parameter multiplying a generator clear.

So, whatever I have written here for 1 dimension you can continue for 3 dimensions. So, we see that there are generators, which are 3 and the generators for translation in quantum mechanics is momentum generators linear momentum generators ok. So, let me write that. So, linear momentum p x, p y, p z are generators of translation.

Student: There is.

Student: There is no formula (Refer Time: 22:18) therefore, just translation could be (Refer Time: 22:21).

So, non (Refer Time: 22:21) there is no bound that is why that is what is the meaning of it I will come to slowly those juggles I do not want ok. There is some more observation you can see, once I write them to be the generators you have to look at, what is the algebra amongst the generators? Ok.

What I mean by that is that if I do a translation along x direction, if I follow it by translation along y direction or reverse it will your result be different or same will it be same or different if you do a translation by a x.

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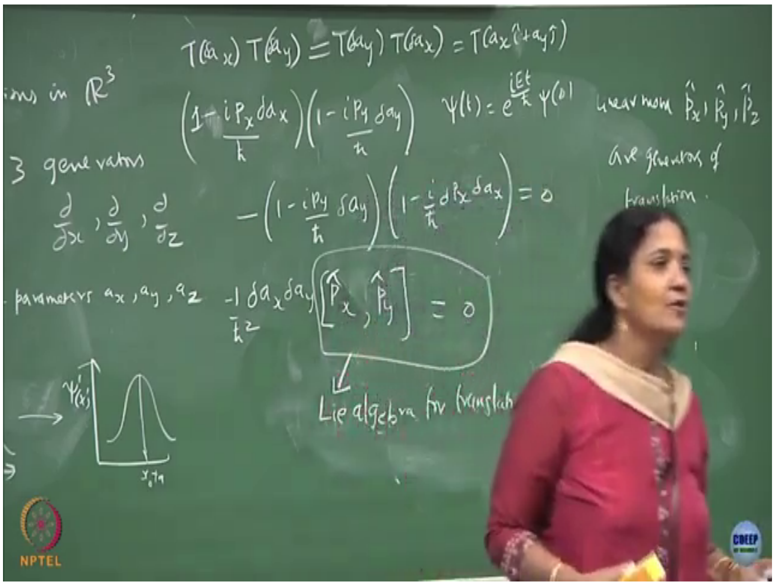

If I do a translation by a y followed by that, question mark will it be same as what is the answer?

Student: (Refer Time: 23:30).

Yes or no will be same right, translation what is this in your discrete group called as abelian, you call it as abelian right, it is the order of operation is not mattering then it is an abelian group. So, here also if you do a translation arbitrary directions, you can take an arbitrary direction by a vector and a b vector, or you do a b vector, and an a vector the final answer will be this one can be written as T of a x plus arbitrary vector right you all agree. So, this translation operator has this property.

What is this property imply on the generators of translation? What does it imply? Generators are p x and p y, if you try to write this explicitly do that. Let us do it for infinitesimal transformation, let us take it to be delta ok. So, what will this p one plus minus right.

Let me take the difference and see whether it is 0 or nonzero ok. What does this give us, we already argued that this is equal to this right, it is an abelian, which means this difference has to be equal to redo ok. So, this one has to be equal to 0. So, what are we getting the generators of an abelian group, the generators have to satisfy commutators to be equal to 0 ok. The commutators of the translation generators will be 0, because the group is abelian and the generators have to commute ok.

So, I have slowly introduced for you instead of looking at group elements, I would like to look at the commutative brackets of the group generators, that itself will have all the information, which I need, if I want the group elements I am going to just exponentiate, it with the parameter multiplied with the generator and I will get all the group elements. So, it is not really very much needed to work with group elements, but it is nicer to work with group generators.

And, the algebra of the group generators is essentially looking at the commutator bracket of the group generators will help us to look at, what whether the group is going to be abelian or there is a sub algebra which will give you abelian on all these informations and that is where this whole thing this is what I would call it as a algebra lie algebra for translations ok.

So, now we can redo the same thing what I did today? What will happen if you do time translation alone can also have a situation where the Hamiltonian is explicitly time independent right. Most of our time independent hamiltonians has to have time translation invariance ok. What will be the generator of time translation by the same argument, which I did f of t to be f of t plus epsilon let us say sorry goes to this under time translation, I do not know what I should call it as.

So, let me call it as some operator a with time translation and you can find what that operator is that right, it is like I did with a and del by del x a is replaced by epsilon del by del x by del by del t. And, you all know in quantum mechanics I H cross del by del t can be interpreted as Hamiltonian operator. So, the generator of time translation is the Hamiltonian. Just like linear momentum or generators of space translations Hamiltonian is a generator of time translations is that clear is the jargon getting clear to you ok.

So, what I am trying to tell you is that the physical conventional translation can be seen as an exponential operator with the parameter multiplying a generator. And, what is the generator is there a physical meaning to those generators del by del x is a differential operator, if I go to quantum mechanics, I do see that del by del x can be interpreted up to proportionality constant as linear momentum.

So, linear momentum is the generator of space translation there are 3 linear momentum components and correspondingly there are 3 parameters, by which you can do a translation. Similarly, if you do time translation del by del t will be the generator, if you want to map it to quantum mechanics del by del t multiplied with I H cross is Hamiltonian and you can call Hamiltonian is a generator of time translation ok so, yeah.

Student: What do you mean by (Refer Time: 32:02).

Take t to t plus epsilon.

Student: Like if I (Refer Time: 32:06) time period like (Refer Time: 32:09).

Yeah, if the Hamiltonian is independent of time it should not really matter right. The Hamiltonian is independent of time, you do the evolution mechanically, you know you do the psi of t how do you write it? You remember how do you do? You do write psi of t, how did you do this?

The evolution at different time is due to this time evolution operator and this is exactly like my time translation operator. This was kind of you know from the differential equation postulate you did I am saying from the group theory point of view, you can interpret the corresponding operator as the generator of the transformation.

So, if you doing space translation linear momentum is the generator, if you are doing time translation Hamiltonian will be the generator. If, you do rotations now you can understand what will happen?

Student: (Refer Time: 33:22).

Student: Angular momentum.

Angular momentum, but you I want you to derive it by taking a rotation about z axis and see what you get? Ok. Do it as an exercise and Monday we will discuss this part.

Student: Ma'am.

Yeah.

Student: Why did define (Refer Time: 33:39).

So, I have to define commutator brackets of the generators involved in the system. Suppose, I am looking at a system with translation symmetry, the generators are px py pz, I have to look at the commutators amongst the generators, that is what is the lie algebra?

In the case of time translation you do not have, if you look at space time translation then you have to see what happens with the Hamiltonian also, but right now let me not get into it, but Lorentz transformation for example, we will do them systematically and see what the algebra comes into it.