Group Theory Methods in Physics
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Lecture – 03
Subgroups and Generators

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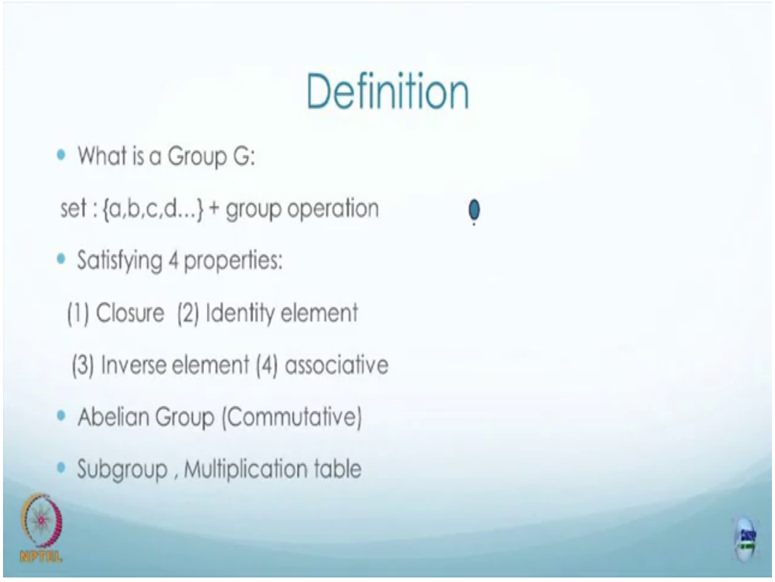

So, we discussed what is a Group and you need a set, not only a set in the group operation and it should satisfy the four properties closure, it should have an identity element. For every element in the set, there should be an inverse element and an associative property has to be satisfied.

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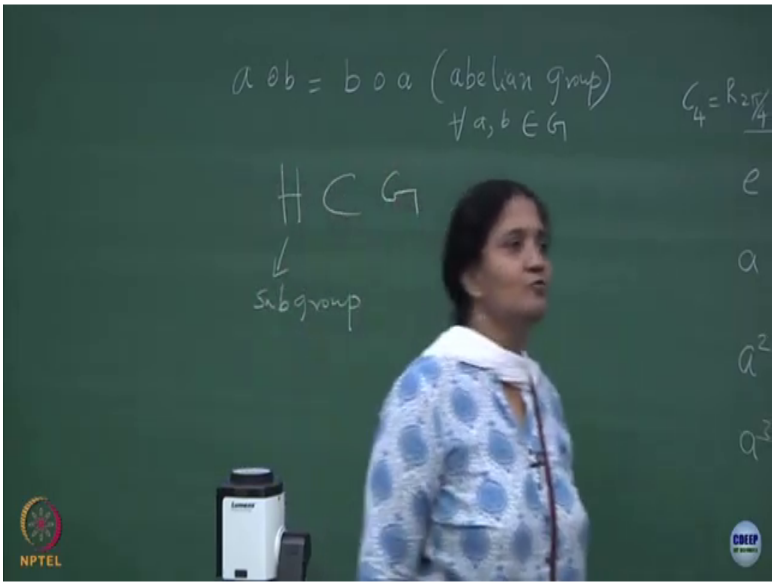

And if on top of it if you combine using the group operation, the order does not matter whether you write and a group operation b as b group operation a, then we call it abelian which is commutative property. And then we also went on to looking at subsets of this, above set; subsets of this above set, right which is called as a subgroup which is denoted by the subsets should also satisfy. This is called subgroup if it satisfies all the satisfying all the four properties, ok.

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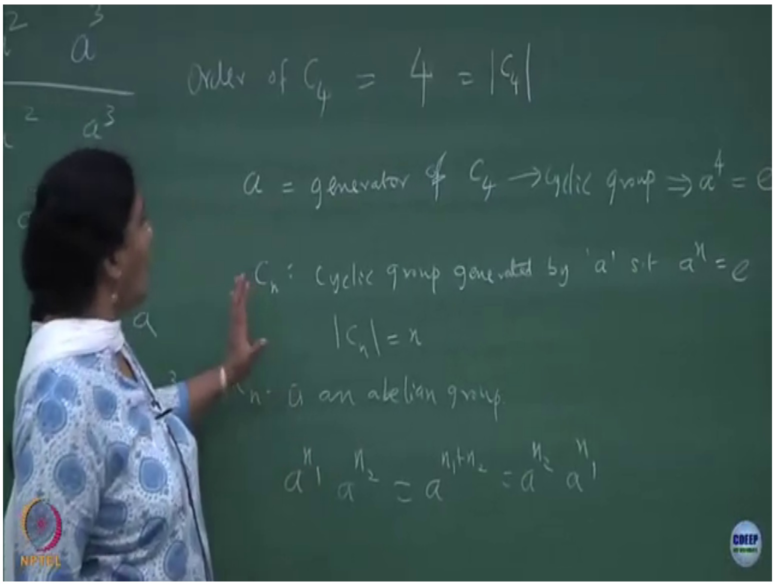
And then I also said that you can draw a multiplication table involving the elements, right. So, simple thing which we took was e, then we did this rotation by 90 degrees which is which let me call it as a rotation by 180 degrees can be called as a squared, rotation by 270 can be called as a cube ok.

So, this is sometimes denoted as c 4 which is nothing, but rotation by 2 pi by 4, ok. So, you can write a multiplication table for this and you can see that this multiplication table you just multiply e with e, a with e and so on, you can write those elements. And whatever entries you are writing here will all be some kind of a permutation, ok. So, some kind of a permutation and this is what we call it as a multiplication table.

So, in this particular case order of the group, order of c 4, c 4 is what I call it as this group of rotation by pi by 2 and this order means the number of elements. The number of elements is 4 here; 1 2 3 4 which is belonging to the column or the number of elements on the row of the multiplication table is what decides for you the order of the group. And we also went on to talk about we did some few examples. I specifically said if you take 2 cross 2 matrices with complex entries and if you put the group operation as multiplication, then this commutative property may not be satisfied.

If it is matrix multiplication 2 cross 2 matrices for each one of them, then that multiplication a times b for all elements in the set will not satisfy b times a. Some elements can satisfy, but that is not the criteria. It should be satisfied for all a and b belonging to g, ok. If that is satisfied, then it is an abelian group. So, a set of 2 cross 2 matrices with complex entries with the group operation which is multiplication, matrix multiplication, it is going to not satisfy that property. So, it is a non-abelian group, ok. (Refer Slide Time: 04:46)



So order of the l group is what I said is denoted by mode of that group. So, this we denoted by this is the short hand notation which we will keep following in future and then sub subgroup as I said is a subset which satisfies all the four axioms of the group and then, we went on to generators in this particular case which is rotation by pi by 2. You do see that with a simple rotation by pi by 2 you can generate other elements which belongs to the set, and so, we call this a is a generator of c 4, specially the c denotes the cyclic group, and this 4 should be interpreted as rotation by 2 pi by 4, ok.

So, c n will be, so then here you require that the order of the generator turns out to be identity element, ok. So, c n is a cyclic group generated by a, such that a power n is identity, ok. So, order of c n will be n is that ok and you can see that cyclic group is generated by one generator, and it is also an abelian group, c n is an abelian group. Because the definition of a generator is that all the elements in that group, the set has to be powers of a generator, ok.

So, any element in c n will be some a to the power of m and you can write it in whichever order you want. So, suppose I write a n 1 is one element, a n 2 is another element, write this does not really matter because it is a n 1 plus n 2 which is same as writing a n 2 a n 1. So, with a single generator there is only cyclic group has only one generator. With a single generator you cannot achieve any other group other than the abelian group, ok. You can write all possible powers of that generator for that group, ok.

So, this is cyclic group. Is this clear? So, this is for cyclic group. Now, we look at for general groups, ok.

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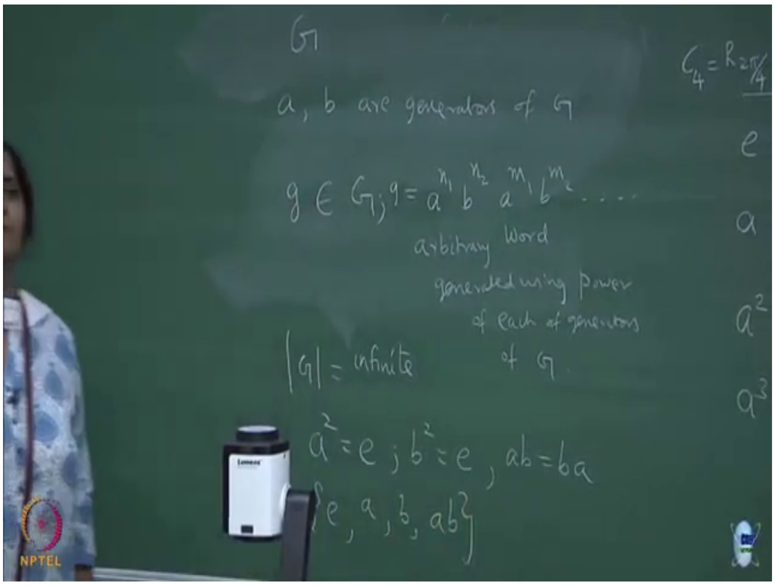

So, suppose you take some abstract group G and say that a and b are generators of G, ok. So, what we mean is that any arbitrary element of G should be rewritable as a to the n 1, b to the n 2, a to the m 1, b to the m 2 you know dot dot. So, this is what I call it as a arbitrary word generated using powers of each of the generators of G, ok.

This is what you mean by saying it is a generator just like I said that 26 alphabets is a generator of all words in the group. You can create arbitrary words using those alphabets, some could repeat itself the group operation. There is concatenation, but it is definitely not a group because we do not find inverse of them just as an analogy all possible words. So, we can take powers of one generator, multiply with another power of another generator. It does not stop here, you can keep continuing like this any number of times, ok.

So, if you do that that is what I call it as an arbitrary word which you generate using powers of each of the generators, then the those are the elements of this group, ok. So, this is what I call it as G for example. Is this ok? So, it is going to be order of that finite group can be infinite type, clear.

So, order of G may be very large, ok, so let me call it infinite. It is countable, but infinite. If suppose I put an additional constraint ok, so constraints like a squared is identity, b squared is identity and I also add a b equal to b a, then many of these words may not be independent, ok. It will give back the same elements this. So, points suppose I have do a to the power of 4, a to the power of 4 is a squared squared a square is identity. So, it is not going to give you anything more.

In fact, that will put in restrictions of giving you independent elements of the group to be. If I put this condition, then the group will involve elements in the set as e, then a b and a b. What else can you have? Anything else which I missed out? If I write a squared it is a identity, you will get back b; if I put b squared, you will get back identity again.

So, this is the maximal set which you can get for arbitrary words subjected to this condition. So, this is given, if you give this condition, then it is not going to be infinite. The order of this group is going to be 4. Is that clear? So, let us look at some more examples and then it will be clear this reverse couple of questions on generators. So, I thought let me stress on this.

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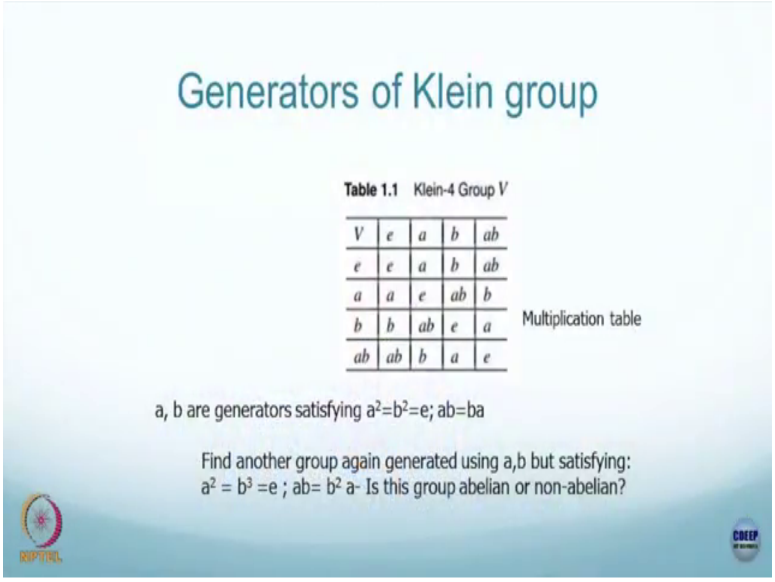

So, the Klein group, it is called in the group theory literature as Klein-4 Group which is denoted by V, so it is made of V e a b a b. So, this is the multiplication table a and b are generators satisfying a squared equal to b squared equal to e and a b equal to b a, ok. So, this is what is the multiplication table.

The next thing I want to give is this is one group I have try to give with some condition for two generators. I can keep varying this condition and start generating different groups, ok. So, the next one which I want you to actually try it out which is really useful.

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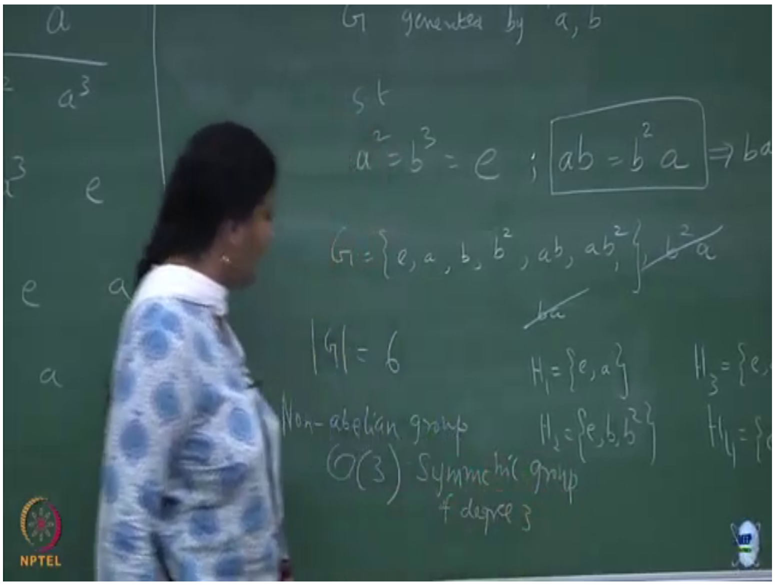

So, group G generated again by a by two generators such that a squared equal to b cube equal to identity. So, this is different from the Klein-4 Group which you have been seen, ok. And I also want to have a b equal to b squared a.

Now. you start cooking up writing all possible words and see how many will be independent elements. I leave it you to try it out, ok. So, the set G will have e a definitely a squared is identity, it does not mean anything. You can have b b squared and then, you can play around socio properties a with b, a with b squared, right. What else?

Student: b squared.

b squared a this is independent.

Student: Right.

I have given you a condition that b squared a is a b. So, it is already there in the set. So, this is not there. Anything else?

Student: a squared b.

a square b.

Student: a square (Refer Time: 14:57). What do you want?

Student: (Refer Time: 15:02).

So, it will truncate if you give conditions and you can determine the order of the group. What is the order of the group? Anything else?

Student: b a.

b a let us write, b a. Is b a independent?

Student: b a is independent.

Good. So, this implies b a is same as a b squared. Using these conditions you can show that b a is same as a b square. So, just check it out. So, this is also not.

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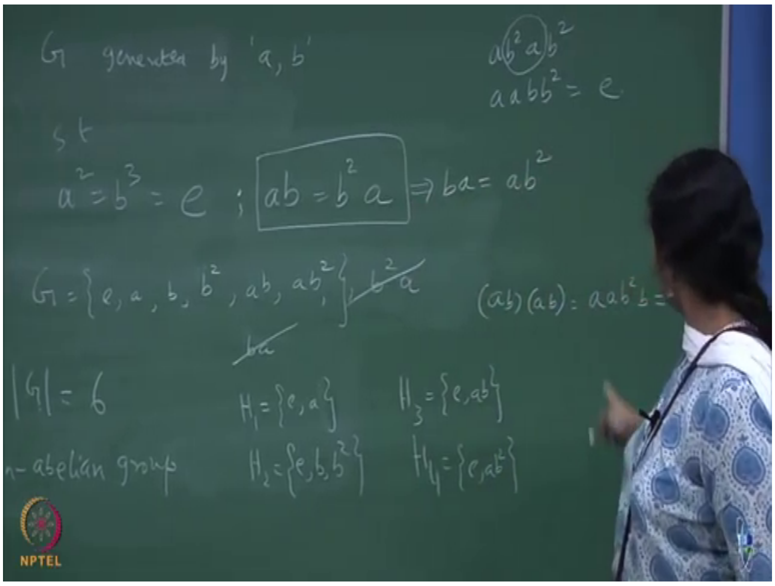

So, order of G is; order of G is 1 2 3 4 5 6 is 6. Is this abelian?

Student: No.

Trivially, it is condition which I have written clearly shows that it cannot be a abelian, right. So, G is this non-abelian group, ok; so non-abelian group. What else? What are its subgroups? Can we find out its subgroups? e and a forms the group subgroup. So, let us write let me call H 1 as e and a, H 2 is e b b square, ok. Is that right?

These two I can trivially write order of an element is the order of the subgroup. Order of a is 2 by this definition, right; a squared equal to identity means order of the generator a is 2, b cube equal to identity means order of the generator b is 3. So, using that I can write cyclic groups, one cyclic group which is its abelian generated by a, another cyclic group generated by b. Anything else?

Student: Group of ab.

a b.

Student: e a b.

E a b good is that a group.

Student: Yes.

So, if you take e a squared b squared a squared is identity, then you will get a, no a b a b, right. So, let us write it in c a b; a b. So, b a is nothing, but a b squared; b a is a b squared, ok. So, a b is also a order two element, ok. So, this is also a subgroup and anything else? That is it. What about a b squared? Is that ok? Order two is easy to check order two groups, right.

So, a b squared; a b squared b squared a is a b, right this one is a b. So, this is also identity. So, you have another order two subgroup which is e and a b squared. It is interesting to play around you know. I am not saying that given an instant for you with so many elements or so many generators with certain properties, you can actually sit down and see whether you can generate the group's subgroups, ok.

So, clearly you do see that there is H 1, H 2, H 3, H 4, the common intersection is only identity element, but then there are all these other elements which are there, ok. So, just to how completeness I just put the multiplication table which you can also rewrite instead of this 4 cross 4, it will become 6 cross 6, ok. So, you have to write using these properties which is what I have put it in here on the 6 cross 6 matrix. So, which is what I have put it here on the 6 cross 6 matrix.

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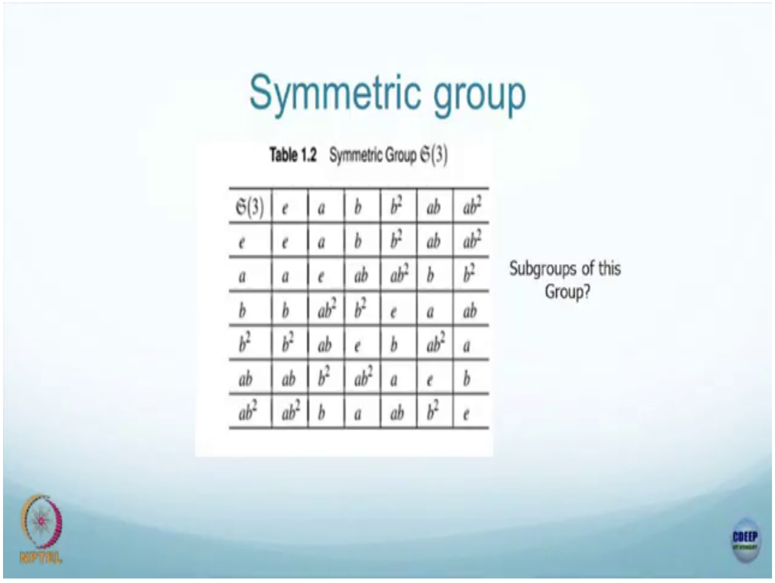

$\mathfrak{S}(3)$	e	a	b	b^2	ab	ab^2
e	e	a	b	b^2	ab	ab^2
a	a	e	ab	ab^2	b	b^2
b	b	ab^2	b^2	e	a	ab
b^2	b^2	ab	e	b	ab^2	a
ab	ab	b^2	ab^2	a	e	b
ab^2	ab^2	b	a	ab	b^2	e

So, e a b, b squared a b a b squared on the column and again on the row and then, I multiply it and write it down explicitly. Even here you can check whether e and a b and similarly e and a b, what happens? You can see that it is between e a b, a b e. So, you can see that there is a subgroup with e and a b, fine. You can figure it out by looking at the subgroups, by looking at the multiplication table, also you can see which one is the subgroups, ok.

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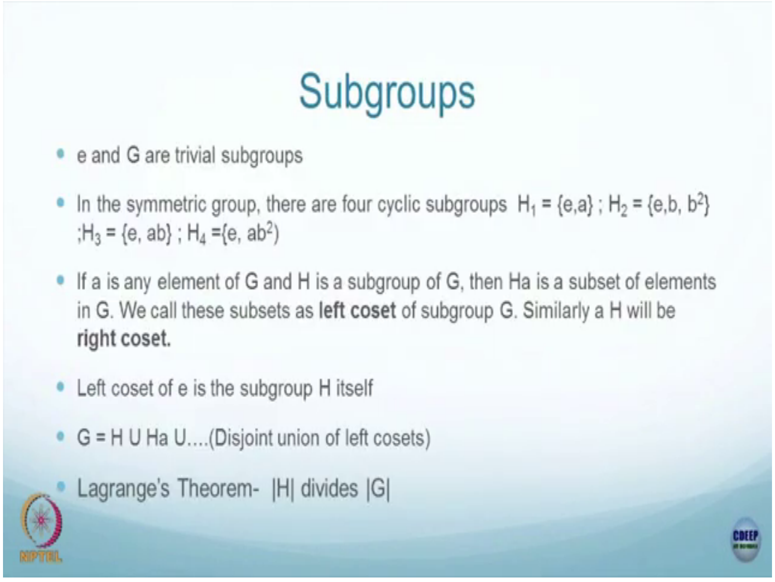

So, just to summarize keeping these examples. So, let me call this example as some kind of a non-abelian group. Let me call this with some notation as O3 will come to it what it is. This is this will be known as symmetric group of degree 3.

I will explain why it is called degree 3, why this 3 is coming and all. Right now you take it as an abstract group G generated using through two generators. One is order 2 generator; another one is order 3 generators satisfying this additional constraint and you have generated a six element group. And that group later on we will see we call it to be symmetric group of degree 3 we will see why we call it and it is clearly non-abelian because of this property. Is this clear?