Group Theory Methods in Physics
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Lecture – 25
Tensor Product and Projection Operator with the Example

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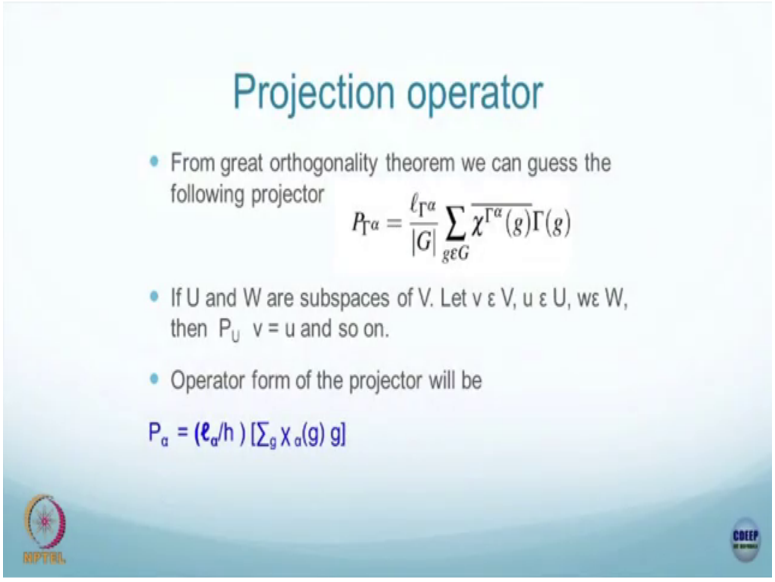

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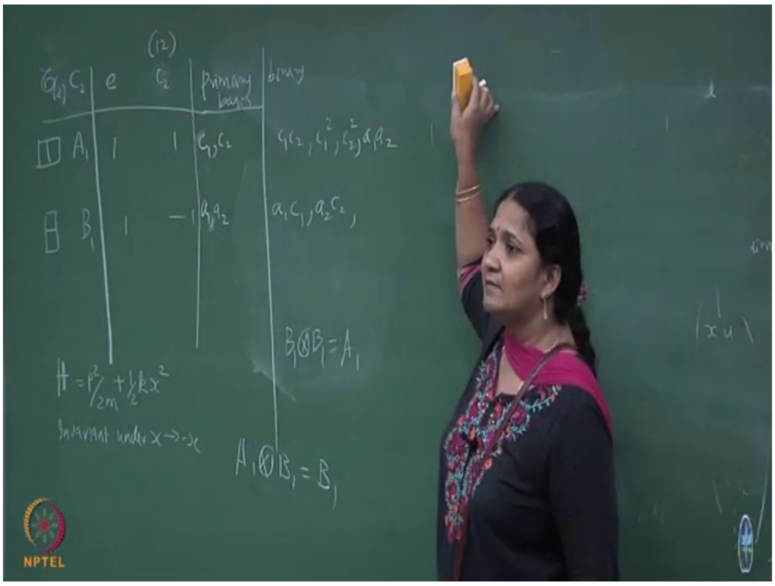

You can find tertiary basis by taking A 1 B 1 A 1 and so on. So, many that is the way you get your tensors which have which are you know 3 index object, moment of inertia has 2 indexed object, quadrupole moment has 2 index object. So, if you want to get a 2 index object it has to belong to a binary basis, if you want to get 3 indexed object it has to belong to a tertiary basis. And, how to construct the tertiary basis is to take tensor product of primary basis number of times, apply a projection operator and find out the basis ok. In this case it was trivial because they are all 1 dimensions, but we will do a non-trivial C 3 v now.

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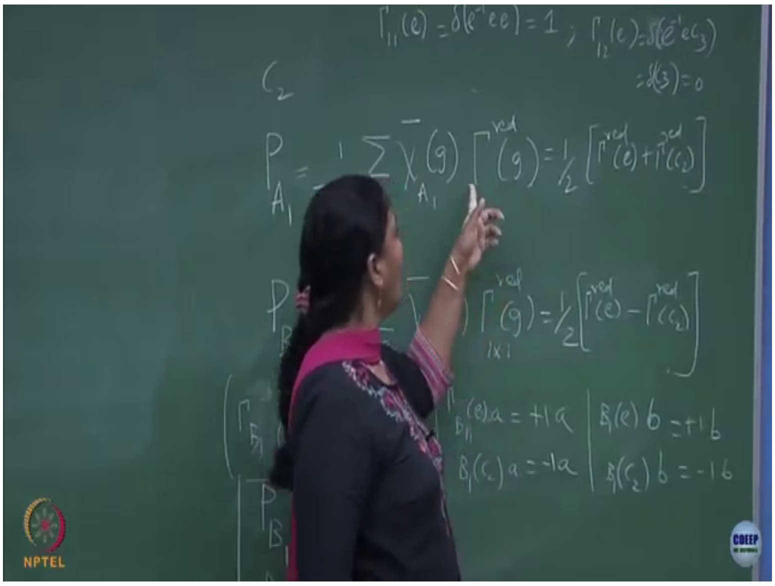

So, this was with the matrix representations which you can use, but you can also replace this by a formal operator g operate ok.

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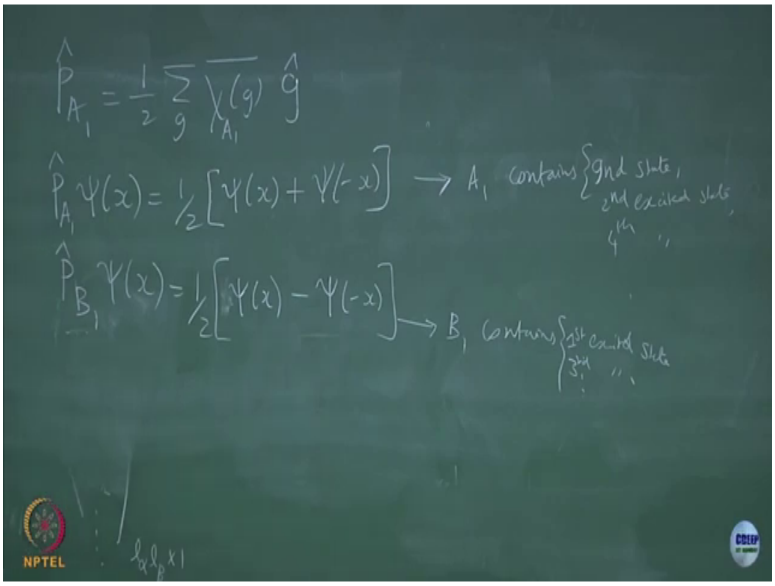

So, you can write P A 1 the projection for the irreducible representation as the character for the group elements in the representation A 1. And, then you can also put the g operator and not worry about the matrix representations for it, because the matrix representations depends on the dimensionality of your vector space; this is also another way of doing it. So, this is where you can apply P A 1 if you remember in the harmonic oscillator I said that this Hamiltonian is invariant under x going to minus x.

It has a group symmetry which is isomorphic to permutations of 2 objects or the C 2 group, this is something which I said. And, you can see that if we operate it on psi of x the projection operator for an irrep A 1 on psi of x using it this is a operator form of it which I am writing ok. So, operators are typically denoted by; so, if you do this then you see that it will give you the plus sign for the C 2 operation, this one anyway is plus 1 identity operator.

And so, this will be your projection operator or any way function it will project you to a subspace which is an even function. Similarly, if you do projection operator of B 1 on psi of x what will happen? This has to be replaced by B 1, B 1 character table is 1 and minus 1. So, what will the thing be? First element of course, is for the identity element is plus 1 ok, the g operators and identity operator so, that will remain a psi of x. The second element character for the B 1 irrep is.

Student: Minus 1.

Minus 1. So, all your which wave functions, ground state, 2nd excited state, 4th excited state and all belongs to which irrep? In the harmonic oscillator. This projector projects it onto a even wave function. So, even wave functions are A 1 is contains ground state, 2nd excited state all even. And, similarly this means B 1 contains 1st excited state, 3rd excited state, all odd excited states.

So, you do see that the just the group symmetry which is a simplest group symmetry C 2 has helped you to say that the wave functions which you are going to find or the solutions with you are going to find has to be to 1 dimensional representations. By 1 dimensions you mean that it is either going to be the A 1 is a 1 dimensional irrep, the projector will give you only a 1 dimensional state. So, it is either even if it is projecting on to an irrep A 1, if it is projecting to a irrep B 1 it should be an odd wave function ok.

So, now let me come to one non-trivial example, I did this simple 1 cross 1 matrix which is not that important, but it is a warm up. So, let us take this, we have been doing on the C 3 v the 2 cross 2 matrix representations which are nothing, but your rotation matrices right. Rotation matrices sigma v we have written it many times and I also told you that you can do a tensor product. So, this tensor thing is not coming, but this will give you a reducible representation ok. So, I think I have a matrix example here. So, E of the this E refers to the 2 dimensional irreducible representation of C 3 v for the elements C 3 ok.

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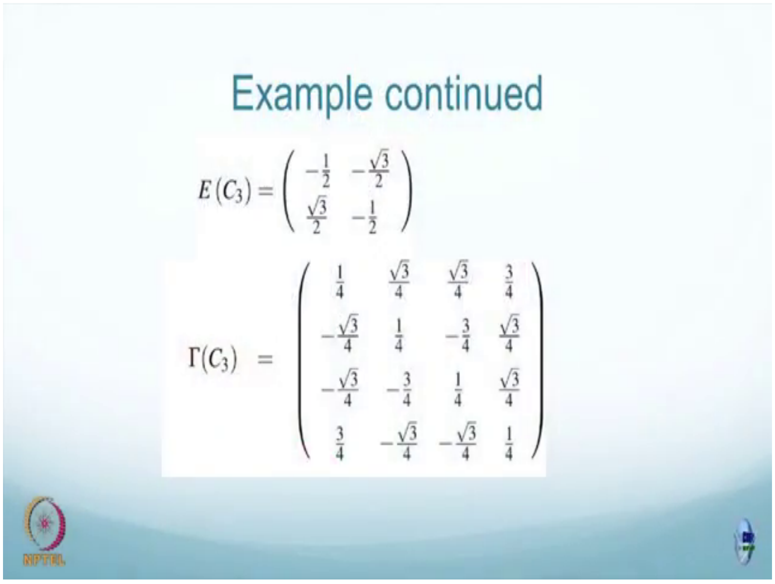

So, that is what that is just a rotation by 120 degrees which you have gone through many times. What is the corresponding vector space? It's just 2 dimensional vector space and we can take the basis to be the x y basis. Is that clear? Are you all with me? So, now you do the tensor product of E cross E, let us do it gamma E I am going to use the shorthand notation as E for g.

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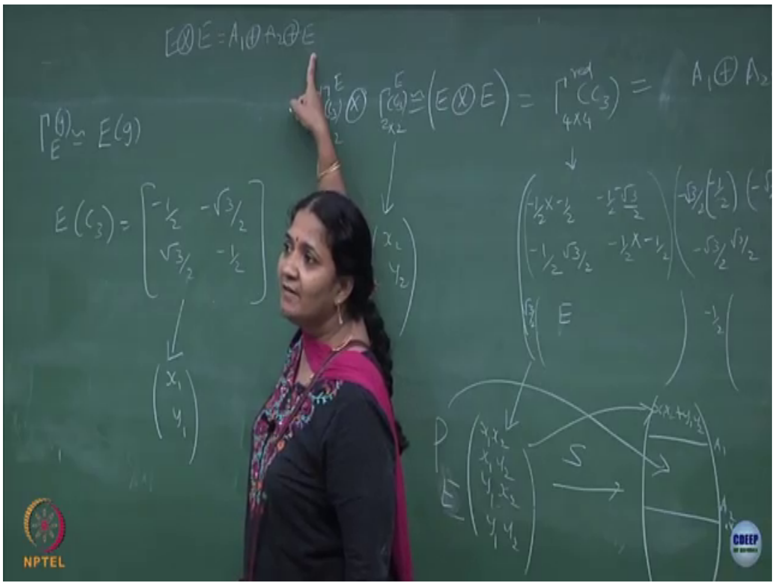

I am going to work it out for C 3 minus or plus ok, it will act on the basis the corresponding basis; let me take it as x 1 y 1 and take a tensor product.

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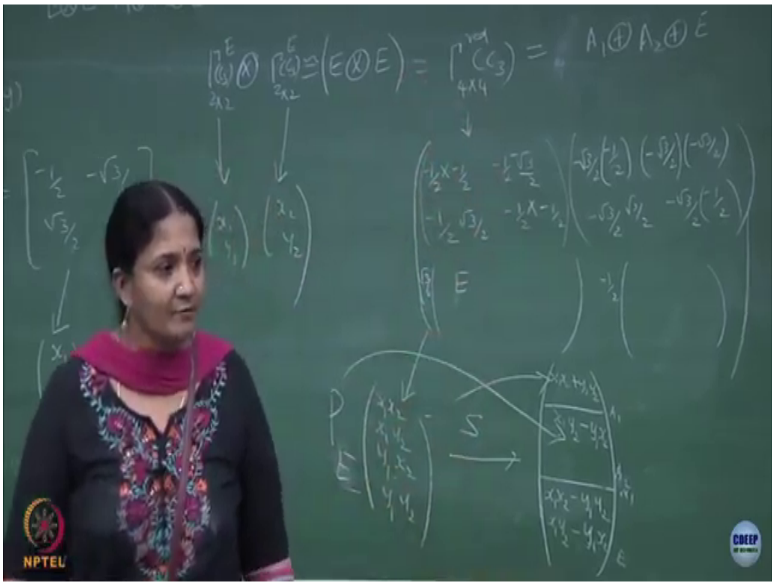

So, I want to do a tensor product. So, let me write it here which I am going to denote it as E cross E. This is 2 cross 2, this is 2 cross 2, this will be multiplied is going to give me a 4 cross 4 which is reducible and I am confining myself to C 3. You can do it on every element of C 3 v, just for an exercise I am doing it for you here. So, what do you have to do? You have to take this 2 cross 2 matrix, take the first element minus half and multiply with all the 4 elements right.

So, let us write that here that is the first block, second block will be minus root 3 by 2 with minus half minus root 3 by 2; you understand what I am doing right. Are you all with me? So, I have written that 2 by 2 block, right this is one block, this is another block. I need to do the same thing here. What will be the multiplication? It will be this element multiplying with all the 4 elements right. Let me write this as the E ok, you understand what I am saying; the last one will be a minus half with the same thing; this is what I have simplified here and put it on the screen for you so, that you can check it out ok.

These are the 4 cross 4 matrix representation. What is the basis vector? This one is x 1 y 1, let us take for this one x 1 y 1, this one as x 2 y 2 ok. So, the basis vector for this will be x 1 x 2 x 1 y 2 then y 1 x 2 y 1 y 2, is that right? Am I right? Ok. This is a reducible vector space, I want to use the projector to find out what is the irreducible subspace ok, is that fine. Here mechanically we took B 1 times B 1 and said it is A 1 by looking at it. But, technically you should be able to take the reducible representation and break it up and see how many components are there right; so, you can do that here.

Student: Ma'am.

Yeah.

Student: Why do we want to take x 1 is through (Refer Time: 11:49) because these are the same representation (Refer Time: 11:51).

Which same no no its a same representation, but one is acting on particular 1 another one is acting on particle 2 or one is acting on the position space, one is acting on the momentum space. So, you can take it like that. You can also do it on the same particle twice that is a different thing, then you will have quadratic powers that is what I wrote here like you can do C 1 C 2 C 1 squared is like on the same basis, but this is on a two different spaces that is why that. So, all those finer details can be worked depending upon whether both are same space or different spaces.

Right now, I am taking it as like let us take a this is something which you have been doing mechanically. I am sure your all been doing two spin half particles, you combine and then you find the irreducible sub space. Now, I am trying to tell you that this is purely from this projection, purely from multiplicity when you take the tensor product what are the irreps, how many times it occurs. And, then the projection operator will tell you what are the sub spaces ok.

This one which I have written is a reducible space, what I am going to show is that that S matrix will break it up into a piece which is x 1 x 2 plus y 1 y 2 that is one piece. Then you will have a another piece which belongs to A 2 and then another piece which belongs to E which are all binary basis. The binary basis is nothing, but the dot product of 2 vectors belongs to your trivial representation. This is like taking dot product of x 1 y 1 with x 2 y 2 and you will get it naturally. How will you get this? If you do a projection operator A 1, it has to give you this.

If you do a projection operator A 2, it has to give me this which is also 1 cross 1, if you do a projection operator E, if you do a projection operator E; the projection operator should be; what should be the rank of this projection operator? It is a 2 dimensional projection operator or it will give you two independent basis which means its rank is 2, it is a rank 2 projection operator. And, the rank 2 prediction operator will give you these two basis. And why have I said A 1 A 2 and E? I have not really done it here, but if you try to work this out you will find it is a direct sum of A 1 plus A 2 plus E ok.

I have not really worked it out, but this breaking is using that multiplicity argument; you can check this out, I have not done it here. So, basically I am trying to say that E cross E in the shorthand notation which I am using will turn out to be A 1 plus A 2 plus E ok. So, this is what? This is 2 dimension, this is 2 dimension, it is 4 dimension; this one is 1 dimension, this one is 1 dimension, this is 2 dimension; it breaks up into irreducible subspaces which is 1 cross 1, 1 cross 1 and a 2 cross 1.

The corresponding matrices will be block diagonal ok; the corresponding matrices will be block diagonal with first block diagonal as 1 cross 1, the next one is 1 cross 1, but it belongs to the irrep which is different. If it is 1 cross 1 you can look at the character table and see that it is A 2 characters entries and then the last one will be the conventional 2 dimensional irrep ok. So, this is what I am trying to show on the screen here.

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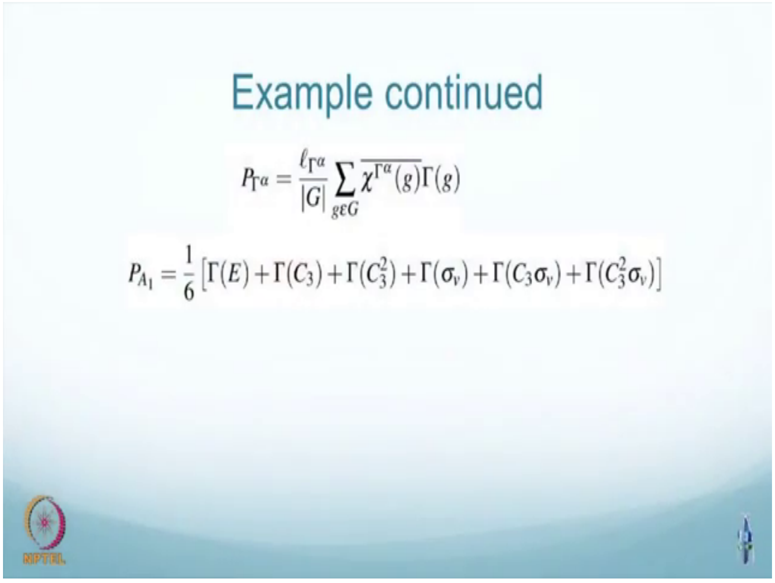

So, I have explicitly tried to P A 1 in the case of C 3 v group all the characters are plus 1 using this formula ok, but now you replace it by all the elements of C 3 v and do not worry about this.

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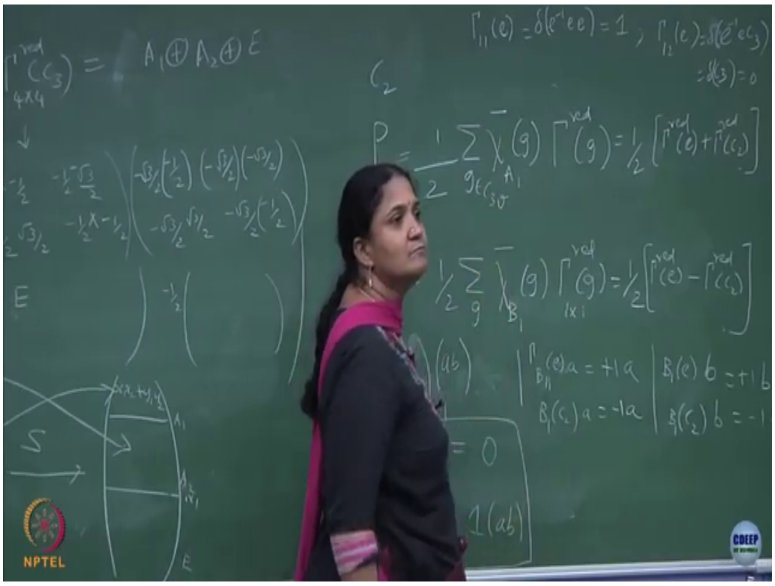

So, this is what I have written it on the screen here, order of the group is 6. The dimension of the matrix dimension of the irrep A 1 is 1 cross 1, 1 dimensional irrep and then the all the characters of this A 1 which is a unit representation is all 1. So, you add up all the matrices. I wrote only 1 matrix, you have to write the matrices for all the elements and add it up. What you get is what do you call it as a projection operator for A 1. So, I have shown that here P A 1 turns out to be this, sorry this is not right is that right yeah it is right. So, this is a 4 cross 4 matrix, write the 4 cross 4 matrix sum it up.

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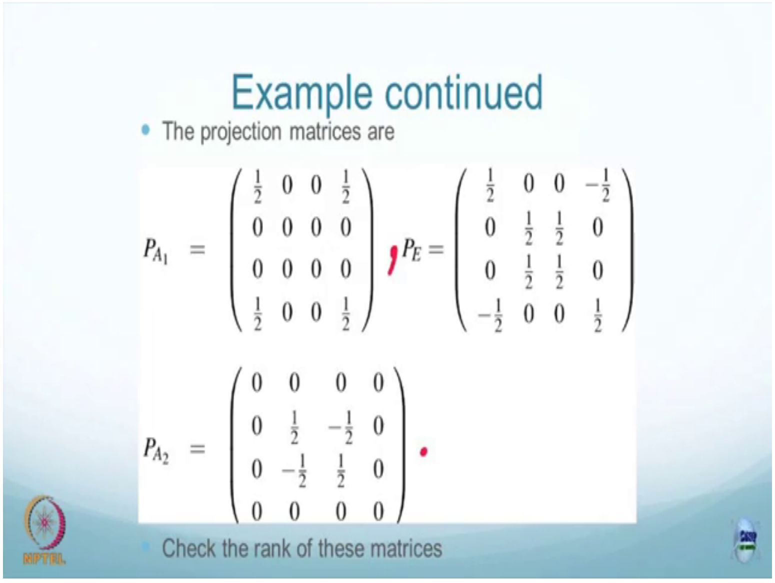

And what you get are the projection operator for A 1, again do it for using the characters of the irrep E; please check this and the last one. And, you can check that the rank of these matrices are these two are 1 and 1 and this one will have a rank 2 ok.

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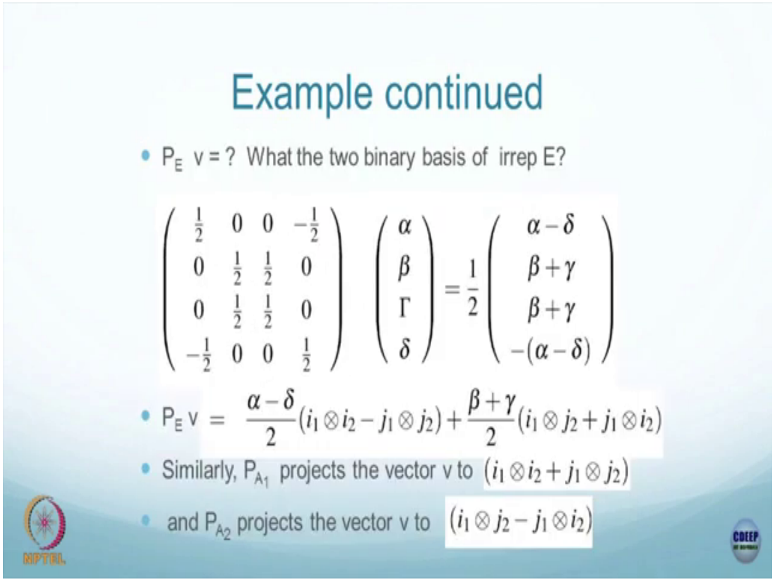

The immediate step after this as you take an arbitrary vector just like I took x y z or a b c take a arbitrary 4 dimensional vector, you know what is the basis state right. I am using here the i cap j cap notation, but it is like x y coordinate if you want to take it. So, i 1 cap i 2 cap j 1 cap j 2 cap ok. So, what is happening is that if you do the projection operator for each of the irreps on this arbitrary vector, you get this answer especially for P E I get this answer ok.

What does it mean? I get a linear combination for arbitrary alpha beta gamma delta so, this me that there are; so, this one will be according to that. It will be x 1 x 2 minus y 1 y 2 and this one is x 1 y 2 minus or plus y 1 x 2. This is just basis for the E projection and for the A 1 anyway I have written and A 2 will be the mistake I did, this is minus I have used i 1, i 2, but it is the same as the i component and the j component.

The thing which I want you to do is I will put this on Moodle, please check this algebra; it is its just to I have given you the concept today. But, unless you work it out all the 4 cross 4 matrices and substitute and check it out, you will not understand what I am saying. So, I have given you this data, this also you should check. You should also check that the projection operator P A 1 when operates on an arbitrary alpha beta gamma delta, it gives you only this subspace whose basis is going to be given by x 1 x 2 plus y 1 or I call it as i 1 i 2 plus j 1 j 2 ok.

And, you can check P A 2 will give you this and P E will give you this. So what does that tell me, that I can start writing binary basis z squared is a trivial binary basis which will come from A 1 cross A 1. If you do A 1 times A 1 you will get a z squared. How were you getting x squared plus y squared? It is this one when x 1 and x 2 are taken to be the same x, you get x squared plus y squared that comes from the tensor product of E cross E and it belongs to the A 1 irrep ok.

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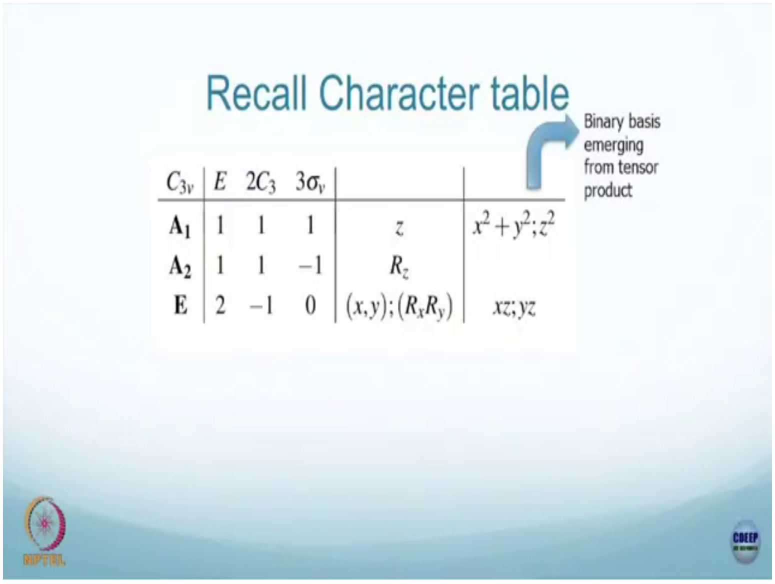

C_{3v}	E	$2C_3$	$3\sigma_v$		
A_1	1	1	1	z	$x^2+y^2; z^2$
A_2	1	1	−1	R_z	
E	2	−1	0	$(x,y); (R_x R_y)$	$xz; yz$

And similarly you can work out all the tensor products ok.

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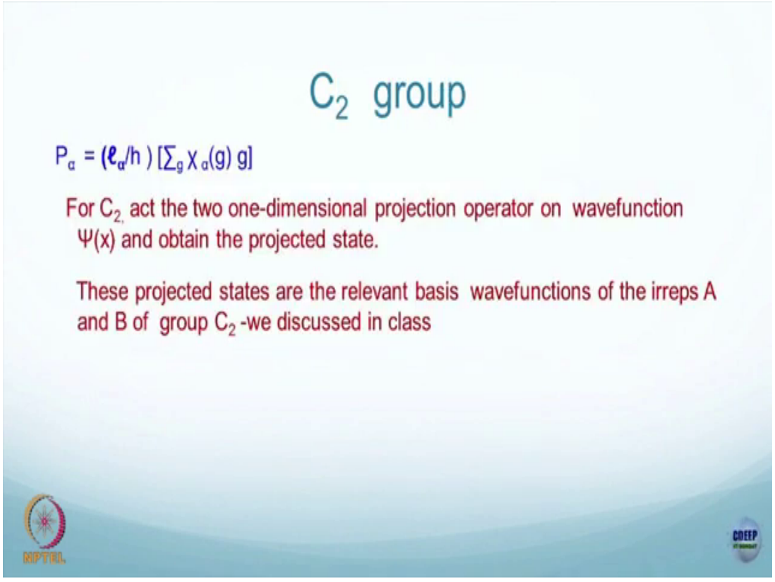

So, we will anyway this I have already discussed with you.

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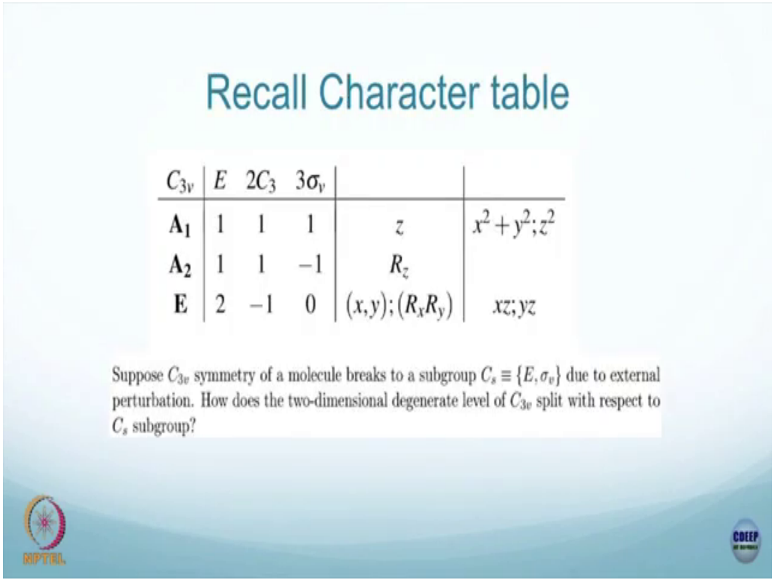

This also I was telling you that typically a system will have a bigger group and then you break it up into a smaller group ok. This you can try it out and there was one more assignment problem which you have supposed to do it and take a look at this. Essentially you will have a system, sudden perturbation will break it up into a lower symmetry system. Once it does whatever irrep in which the system was there it becomes the reducible representation and it will break it up into irrep of the lower group representation ok. So, this is what will happen in nature ok.

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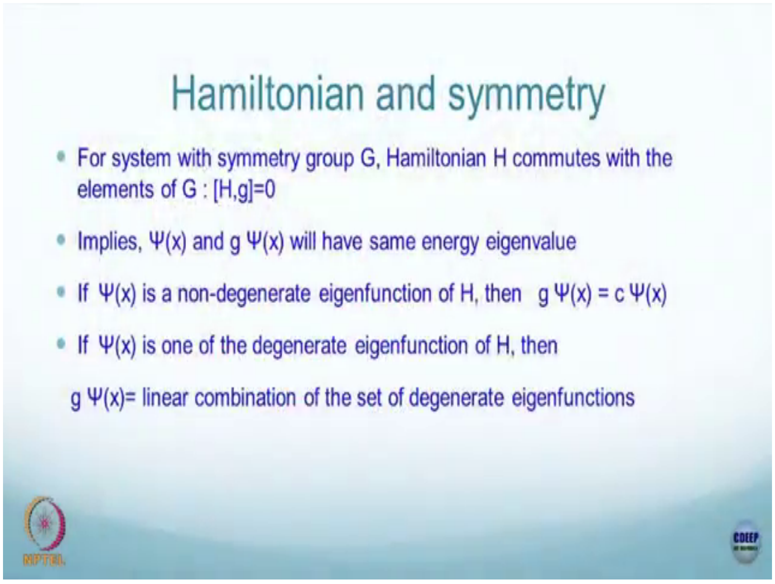

So, to sum up what I am trying to tell you, I just wanted to flash this for you since I have done so much. For any system with symmetry group G Hamiltonian H will commute with the elements of G that is the definition of saying that, if suppose the Hamiltonian has a symmetry, Hamiltonian will commute with every element of this C 2 group. The non-trivial element is only the C 2 element, Hamiltonian will commute with that. Once it commutes whenever there is a commutation relation like this in quantum mechanics you allow for you allow for state psi of x and g psi of x to have the same energy; you all know that right.

If you have operator A and operator B commuting, you can have a simultaneous eigen states of both operator, the two operators which commute. Here the first operator is Hamiltonian; so, I am saying the energy eigen states have to be degenerate in general. When will it not be degenerate? If g on psi of x is proportional to psi of x that is what happens in this harmonic oscillate, g on psi of x is proportional to psi of x then you will get non-degenerate eigen state. How do I see it in this irrep language? How do I see it in this irrep language?

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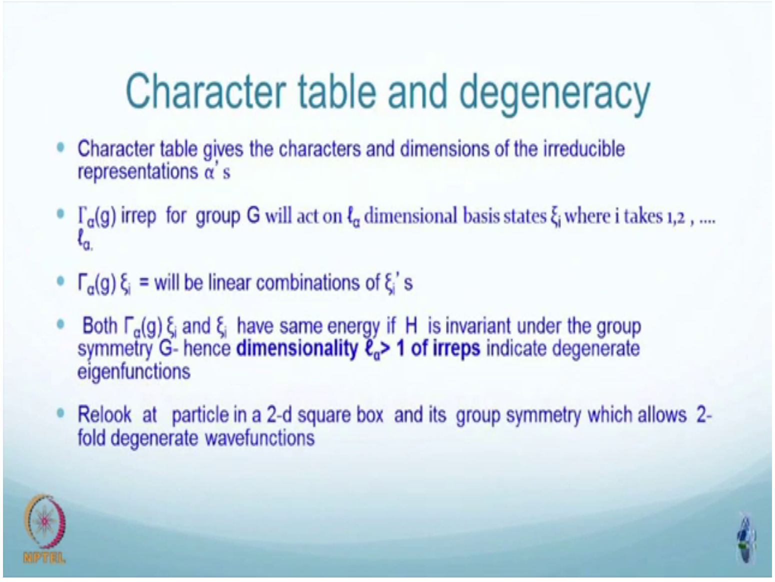

If it was degenerate then you will have g of psi of x will be a linear combination of the set of degenerate eigen function, this is known in quantum mechanics. Now, in the irrep language look at the character table. So, when you take the character table C 3 v has two 1 dimensional irrep, one 2 dimensional irrep, C 2 has two 1 dimensional irrep. Once you look at this character table, the corresponding matrix representation will have to act on a vector space which will have basis states. So, xi i is what I am calling it as a basis states, where i will run from 1 to l alpha.

And, what is the meaning whenever there is a group symmetry in the system when I operate this on xi i, it will be a linear combination of xi i. If it is 1 dimensional vector space, if l alpha is 1 what is the linear combination available? You do not have other linear combination ok. So, gamma g of psi i and psi i will have to have same energy, if the Hamiltonian had the group symmetry. And, gamma g of psi i will give you a linear combinations of basis states of that irrep, if there is only a 1 dimensional irrep, what is the linear combination?

So, it is non-degenerate, if you are looking at 1 dimensional irrep xi i and you do gamma on xi i it better be only proportional to xi; you cannot get anything linear combination. The linear combination will happen only when you go to C 3 v or other groups with 2 dimensional irreps, where the basis will be 2 dimensional basis or 3 dimensional basis depending on what group symmetry we are looking at ok, is this clear?

So, now with this information you take a relook at the particle in a 2 dimensional box ok, that is why you start getting degeneracies when you go to 2 dimensional box is because of the C 4 v symmetry. Because, C 4 v symmetry will allow for a 2 dimensional irrep and then gamma g of psi i will be a linear combination that is why you start getting degenerate states for 2D harmonic oscillatory or 2D particle in a box, but not for a 1D harmonic oscillator. 1D harmonic oscillator has to be non-degenerate because the irreps of the character table are only 1 dimensional, you cannot get degenerate energy eigen states, is this clear? Ok. So, let me stop here.