Group Theory Methods in Physics
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Lecture – 24
Tensor Product and Projection Operator - II

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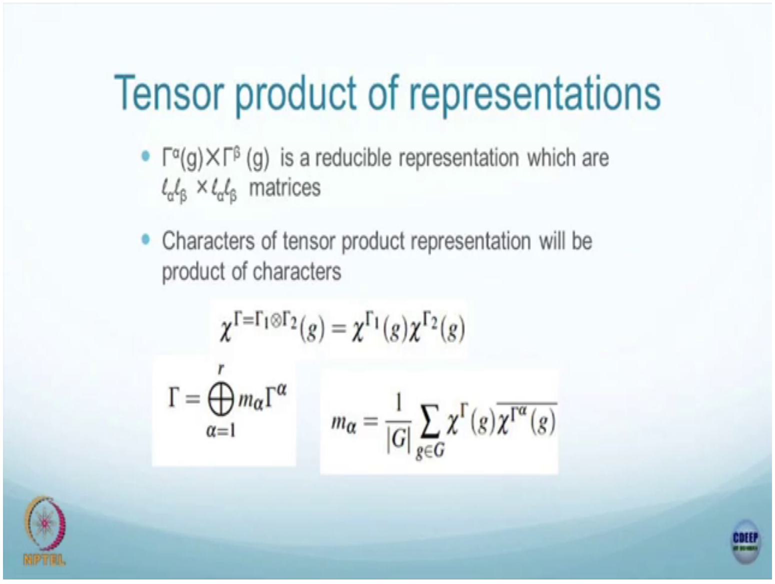

We also when through this tensor product of representations, ok. If you take two irreducible representation and take a tensor product sorry, this notation I have to put the circle on it which I did not put. So, the tensor product I explained in the last class right, I took a 2 cross 2 matrix, another 2 cross 2 matrix and then we did the tensor product. What was the dimension of that matrix, if you had a matrix l alpha cross l alpha and do a tensor product with the matrix l beta cross l beta, this will give you a new matrix with dimension l beta and it will be always a reducible representation ok, will always be a reducible representation.

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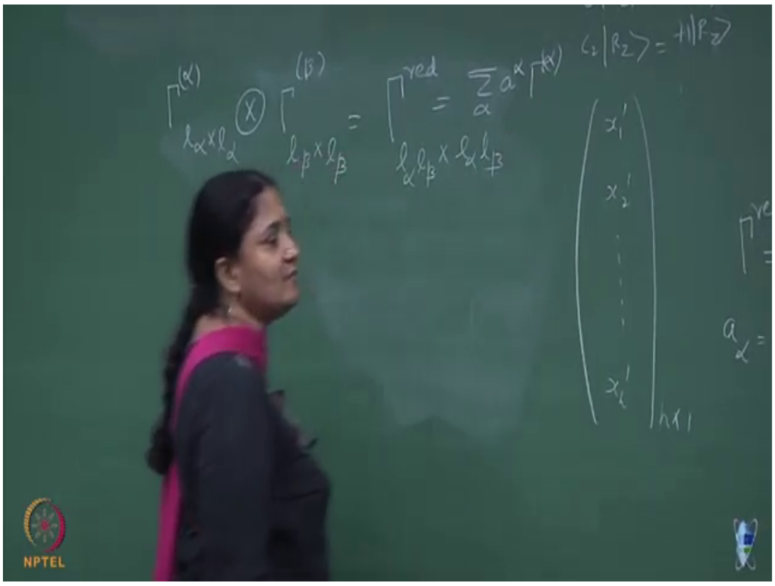

Another way of using this tensor product is that vectors of the fundamental objects in nature, observables fundamental observable, of course scalar is also one observable, vectors are the basic observables. So, if you want to look at these tensor products I said this in the context of saying that you look at xy coordinates of the particle 1 and xy coordinate of the particle 2 that is what I was saying last time, right.

When I do a tensor product of two particle system, I take 2 cross 2 matrix acting on the xy coordinate x 1 y 1 the other one acting on x 2 y 2. So, you can start looking at tensor products and look at what will happen to those reducible representations how to break this, these are the basis states, right ok.

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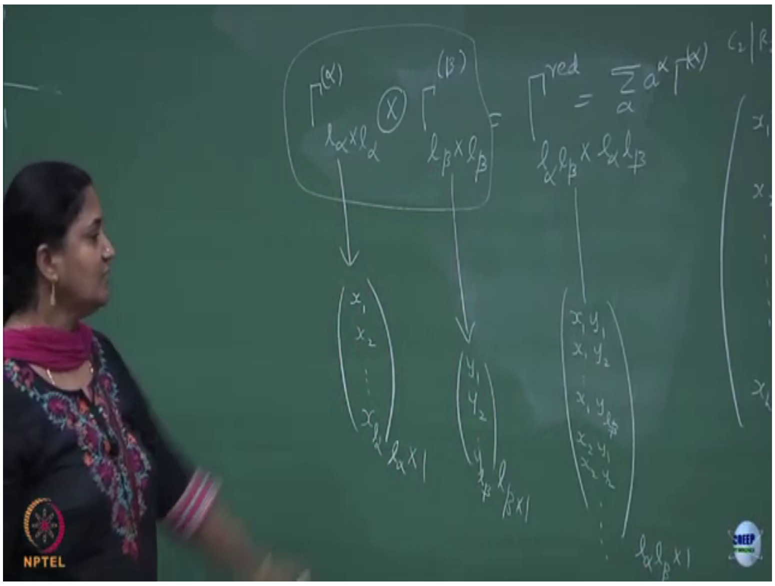

So, this is what is the and once I do this to start with this one, this one will operate on a l alpha cross 1 basis, this one will operate on a l beta cross 1 basis, this one will act on a, ok. So, let me say here it is x 1, x 2 up to will x l alpha l alpha; y 1, y 2 up to y l beta right. What will be this, I said that when you do a tensor product take the first one and multiply everything. So, what will this P l beta and then x 2 y 1, x 2 y 2 and so on, ok, that is why it will become. So, this is the vector space on which it is l alpha l beta dimensional vector space on which these reducible matrices will act.

So, here what was we doing we said that there is an s matrix which can it is a reducible representation means that there is an s matrix, which will bring it to a block diagonal form. So, this notation summation or alpha technically I should have written it as a summation over, direct sum over.

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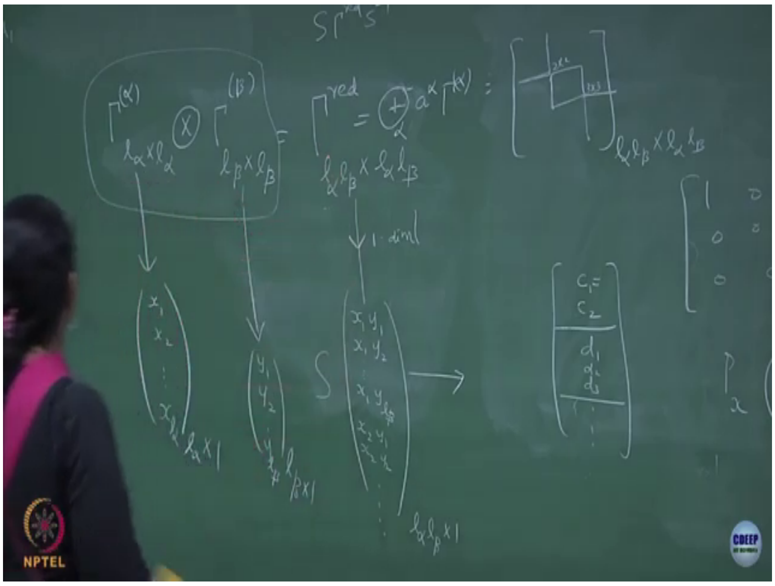

The vector space what this means is that you all know by now, it will break it up into block diagonal fashion and whose dimension totals up to l beta. Some of them will be a sub space depending on what are the allow irrep of your groups symmetry, is that right. So, it is like you can take two particle systems and you will do that explicitly. And then when you do that the vector space gets bigger and it is a reducible vector space.

Finally, I have to because these are like this block diagonal, this should also break up into pieces such that each block diagonal. Suppose this was let us say 2 cross 2 matrix, so this should have two basis ok; so let me call it as some c 1 and c 2. Suppose this was 3 cross 3 suppose, then there will be a d 1, d 2, d 3 and so on.

So, you break it up into pieces, but what will c 1 be, c 1 will involve, it is an s matrix acting on this, because the diagonalization is done by, ok. So, which means s when it acts on this;

what will it do, it will give you some linear combination c 1 should be some linear combination which belongs to this block diagonal and so on right, this is what you will eventually achieve.

And what does these blocks mean; I can work with these blocks, I do not need to work with a big matrix only thing is I need to know them, clear. So, now we will do an example where this will become clear, ok. So, gamma alpha cross gamma beta is a reducible representation and characters of course when you do a tensor product of two representations, it will be the product of the characters, this also be verified last time. And remember these you know these identities or from the great orthogonality theorem where we have derive, this will give you the multiplicity of an irrep alpha, gamma alpha how many times it occurs in the reducible representation, ok.

So, this I will skip for the time being I will come back to this, let me get to an example, ok. So, I have already given you a fact that if you have a reducible representation, the dimensionality of these matrices will also determine the dimensionality vector space on which these matrices acts, ok.

Besides finding the multiplicity of each irrep can be find the invariant subspaces, by invariant subspaces I need the subspace, this subspace and so on and determine these basis vectors. This basis vector as trivial, but I want to find out which linear combination belongs to which irrep; each one is an irrep, I want to find which linear combination will be an irrep.

Clearly these involves product of two basis two primary basis. Two primary basis are involved right tensor product; this has one primary basis, this has one primary basis and this skate involves binary basis, ok. Some linear combination will give me the correct binary basis on with the same irreducible representations of these characters will act, ok.

So, we will get to this and we want to understand what linear combination gives you c 1, which belongs to that vector space is the vector space on which this irreducible representation acts that is what we want to understand. It is not new to you; you have done projection operators in your x, y, z three-dimensional space, right. Suppose I say that the vector space is three-dimensional its take x, y, z, ok.

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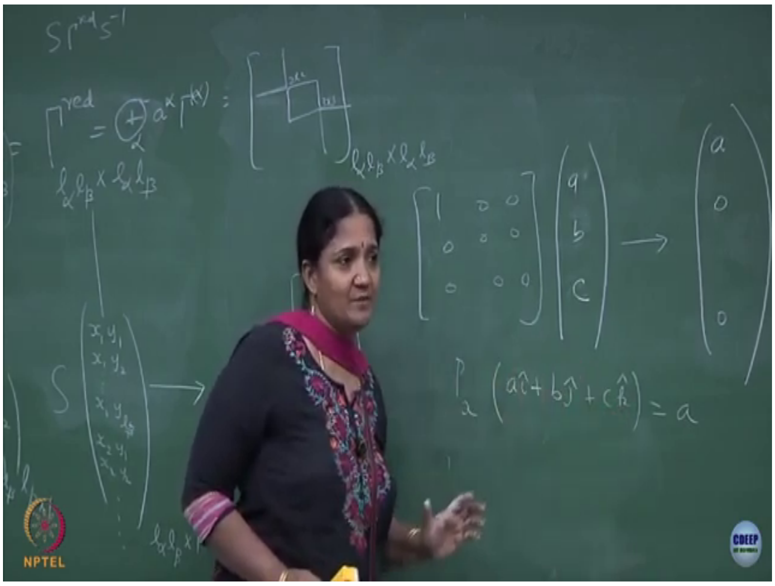

I want to project this to some a times ok; so let me call it as a, b, c what is the meaning of this, it has a component a along i hat, b along j hat and c along k hat, this is a meaning of writing it in this column. And if I want to find what is the projection operator along x what is the answer you will get, you have to get a right, this you know projection operator is basically take a dot product with i, i dot j will be 0, i dot k is 0; so you will get the projection operator along x direction will give you the competent a.

How do I do it in the matrix, I just put a matrix this is the matrix representation for the projection operator in your conventional three-dimensional vector space to project you on to the x component. What is the rank of this matrix, its 1; so, the corresponding non-trivial basis which you can find depends on the rank of the matrix, so that will be one-dimensional vector space, this projector will give you a one-dimensional vector space, is that clear.

So, similarly you can write just like the way I have explained P x, you can do it for P y on the projection operator which will also be having a rank 1, but it will give you the y component; P x multiplied with P y has to be has to be 0, right. One is projecting onto the x component another one is projecting one to the y component. So, projection along x once you have done if you try to do projection along y on that it will be 0, because there is no y component. So, P x, P y is 0; P y, P x squared will also be P x ok, so those are the properties of your projection operators.

Why am I doing this, I want to find a projection operator instead of writing it like this, I want to find a projection operator which when operates on this basis. The projection operator is going to be for different irreps I want to write it, so that I can pull out the c 1, c 2 for a specific irrep projection operator; just like your x, y, z are the three projection operators which I am going to write P x, P y, P z.

Typically for any group I will have irreps, I want projection operators for each irrep ok. So, I am just proposing a projection operator for a specific irrep gamma alpha where this depends on the dimensions of that irrep. And as I said many of these characters could be complex sometimes, so in general the great orthogonality theorem it is better to use one of them to be a complex conjugate ok; so this is the definition of your projection operator. So, take it as a definition.

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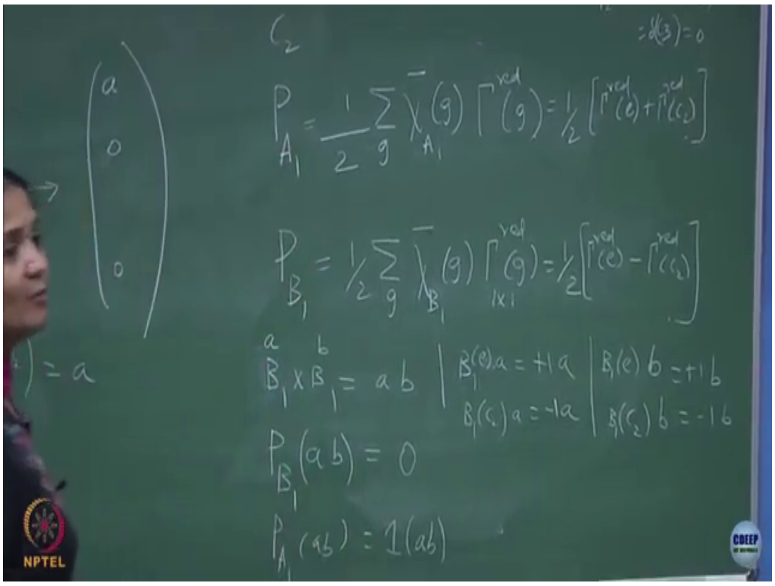

So, projection for the suppose I want to write what is the projection for P A 1 using this, it is get to the so let us do this for the conventional permutation of two objects. So, using this expression A 1 is a one-dimensional irrep, so you will have 1; order of the group is 2, let us do it for the c 2 group ok, order of the group is 2 and characters are A 1 has character 1 and 1, it is a unit representation so let us write that, ok.

So, I am not going to write what this gamma reduce this, but let me write this explicitly what is just going to be half of ok, yeah. What will be the projection operator for A 2 no B 1; why B 1, because the principal axis has negative eigenvalue. So, this will be so the character is 1 and minus 1 ok, so these are the two projection operators which I could write for a reducible representations. And however I constructed the reducible representation, I have taken some l alpha cross l alpha matrix multiplied with l beta cross l beta matrix.

So, let us take as a simple exercise. So, let us take B 1 cross B 1; so, if I take this B 1 cross B 1, then I will have so let us say this is a one-dimensional basis I will have. So, I will have it to be a b and then I want to do a b, ok. What does it a definition of B 1 the basis is a on it what is the meaning of it, what is the meaning B 1 on a, the corresponding element; the irrep corresponding element which is c 2 on a will be minus of a, that is the meaning of this, right.

So, now you can try and do these projection operators on this and see what you get, ok. So, it will operate on a b the reducible one and the reducible one is also 1 cross 1 matrix only would not change, right. So, you will find it to be which is a b and this will be a minus and then you will see which one will survive, what one which one will survive, is that right. What I have written it for a, the same thing happens for the b also right; c 2 element on b is minus 1 times b. So, use this fact on this projection operator right, if you do that. Then what you see is that the, this tensor reducible representation of c 2 will cancel with the does it cancel or does not cancel.

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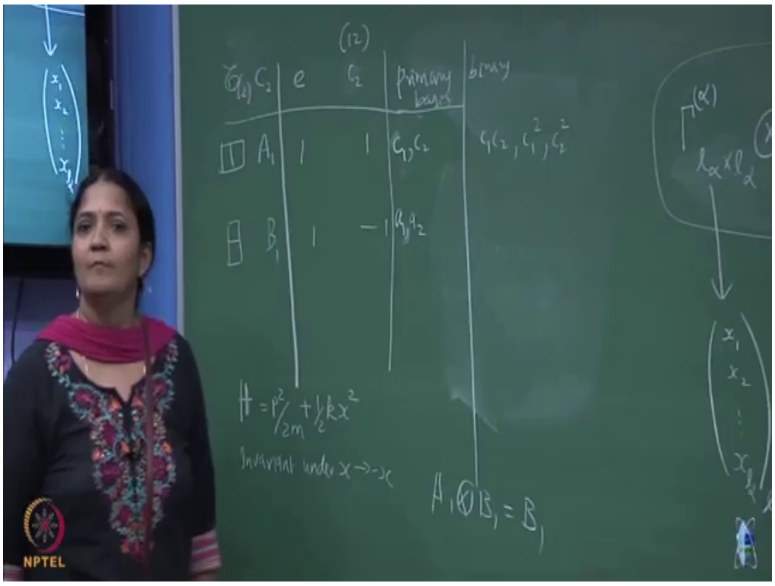

Or equivalently what I am saying is if I take B 1 times B 1, it will become characters is also going to be multiplicative right, character of B 1 times B 1 is what is that, ok. So, I am just trying to see from this that B 1 times B 1 is A 1, so you cannot get a projection of a b on the B 1 representation, but you can get it on the A 1 representation. Yeah, someone was saying something minus sign; its fine, yeah.

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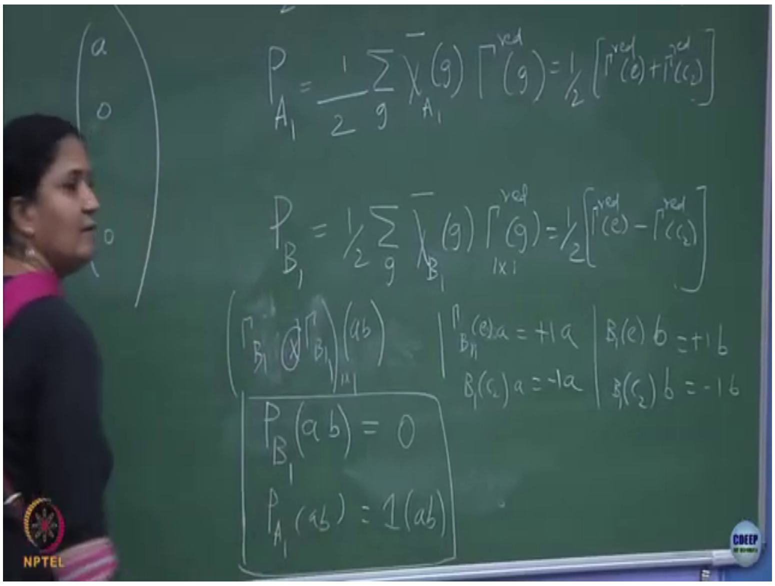

I am just showing it in a shorter notation, technically it is gamma the irrep alpha which one, here also it should be a gamma, gamma B 1 cross gamma B 1. This is what I am writing here right; take one irrep multiply with another irrep that is all I meant as shorthand notations.

Student: It is a tensor products.

It is a tensor product yeah did I not put a tensor product, it is tensor product, yeah. So, the tensor product will act on a basis which is a b that is all I am write this. This is a 1 cross 1 matrix, it will act on a basis which is a binary, it is not even a basis I should say it is a reducible vector space, because just like here it is x 1 and y 1; if it is B 1, it is a one-dimensional vector space, one-dimensional vector space I just take that and the tensor product matrix should operate on that binary component not on the. And I am only seeing, whether the binary component does it belong to B 1 or does it belong to A 1, you just get only a one-dimensional vector space again.

I am just trying to argue, the characters also multiply if you just multiply the characters what do you get characters if you multiply, do you get A 1, A 1; B 1 times B 1 will give you A 1. And then you can also show that the projection operator P A 1 on a b will give you plus a b, but P A P v 1 one on a b will be 0 this I leave it you to check, please check it should happen. So, please check this in case you have not checked it.

So, what does it is tell me that I took the two irreducible representations which are one-dimensional. I took the tensor product of those two irreducible representation, it will give me a binary basis and that binary basis does it belong to A 1 or does it belong to B 1, it is not clear to me. I introduce a projection operator and just blindly apply the projection operator on that binary basis; if it gives me 0 that means, it does not belong to it sub space; if it gives me non-zero and it is a one-dimensional so it gives you just a b. So, this means that this reducible representation turns out to be it is a very simple exercise when it is l alpha is 1 and l beta is 1; it is again one-dimensional, ok.

But, I do not know whether there is one-dimensional is going to be B 1 or A 1 for this two element group, there are two possibilities. And I am using that projection operator to say that that product element or the product basis which I am going to get, it belongs to B 1 or not sorry, belongs to A 1 and not B 1, ok.

So, what I have shown here is that the shorter notation which typically people write is B 1 cross B 1 will give you A 1, ok. So, the character so I am not writing gamma subscript B 1, many times they write only the Mulliken symbols. So, if you take product of these two, you will get in A 1 basis. So, the basis of B 1, basis of B 1 if you just put it together, it belongs to A 1 or not to B 1.

And another simple way you can see in one dimension is that the product of these characters will be this. So, now you can blindly write A 1 cross B 1 will be what, A 1 cross B 1 is B 1 is that right. So, these are things which A 1 cross A 1 is trivial that is going to be again A 1, ok.

So, if you had a binary basis sorry, let us say that the basis was here let me call it as c and here I will say a as the basis, ok. So, let me call c 1 comma c 2, a 1 comma a 2 are the basis; so, this is the primary basis, ok.

Binary, you can take combinations of c 1 with itself, c 2 and c 1 if you multiply; c 1 multiplied with c 2 will be what, it belongs to A 1 cross A 1; but you know A 1 cross A 1 is A 1. So, the A 1 will have c 1 c 2, c 1 squared these are the possible basis for the binary basis. Similarly, if you come here what will be the basis binary basis possible, you have already seen that A 1 with B 1 will give you B 1, right.

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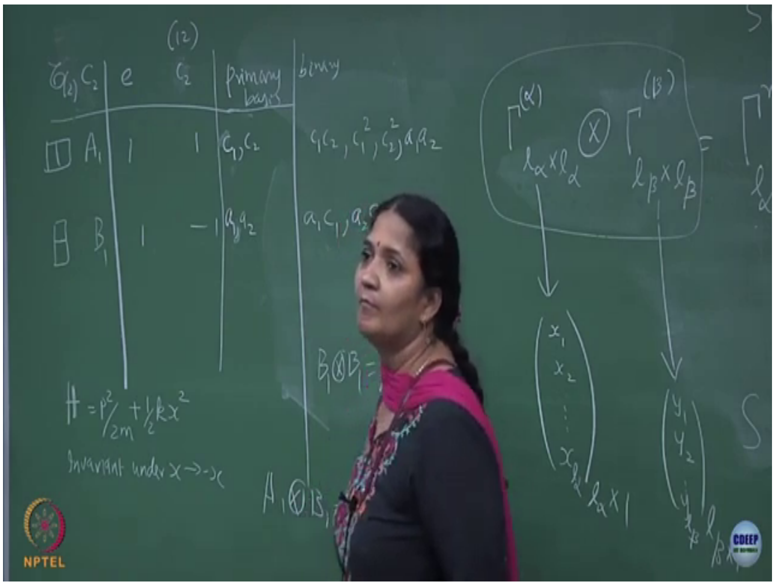

If you see that A 1 with B 1 is B 1, right. So, can you tell me what will be the basis you will have a 1 c 1, a 2 c 2 all possibilities; but you cannot have a 1 a 2 you agree; a 1 a 2 can go here. Why it can go there, because B 1 times B 1 is A 1 clear, taking the simplest example to explain it to you.