Group Theory Methods in Physics
Prof. P. Ramadevi
Department of Physics
Indian Institute of Technology, Bombay

Lecture – 22
Tensor Product of Representation

So far I have kind of given you some flavor of how to see irreducible representations and all your energy wave functions which you are going to find will belong to one of these irreps for a system with symmetry let us say if suppose you look at a system with symmetry C 2 v.

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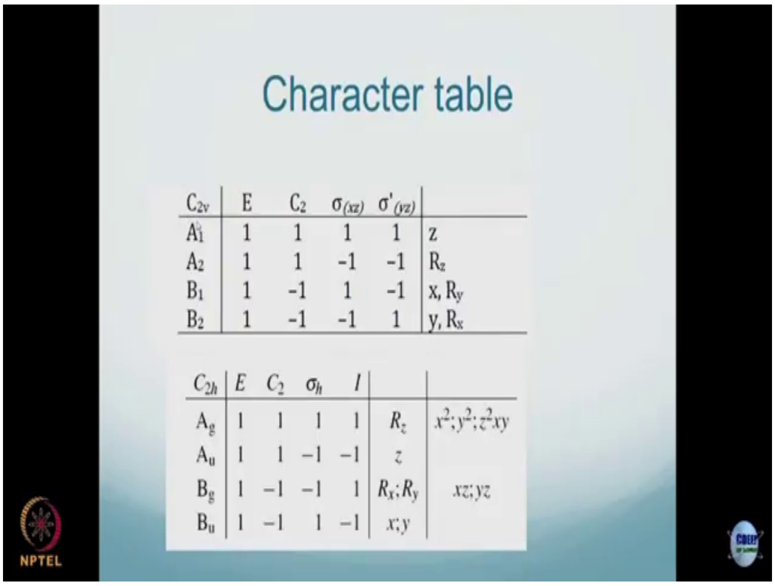

C_{2v}	E	C_2	$\sigma_{(xz)}$	$\sigma'_{(yz)}$	
A_1	1	1	1	1	z
A_2	1	1	-1	-1	R_z
B_1	1	-1	1	-1	x, R_y
B_2	1	-1	-1	1	y, R_x

C_{2h}	E	C_2	σ_h	I		
A_g	1	1	1	1	R_z	$x^2;y^2;z^2xy$
A_u	1	1	−1	−1	z	
B_g	1	−1	−1	1	$R_x;R_y$	$xz;yz$
B_u	1	−1	1	−1	$x;y$	

C 2 v will be a symmetry for a rectangular two-dimensional box right, you can start using them. And you can make sure that the wave functions are no longer having a 2 fold degeneracy as long as there is no accidental symmetry. For arbitrary length and breadth of the box you cannot find any degeneracy, there could be some accidental degeneracy if the ratios of the length and breadth are such that some numbers can go inside.

But that apart from this for arbitrary length and breadth a and b which is the length and breadth, you can show that it has to be C 2 v tells me everything has to be 1D irreps you cannot get degenerate wave functions without even doing the problem, do not even need to do the problem without solving them ok.

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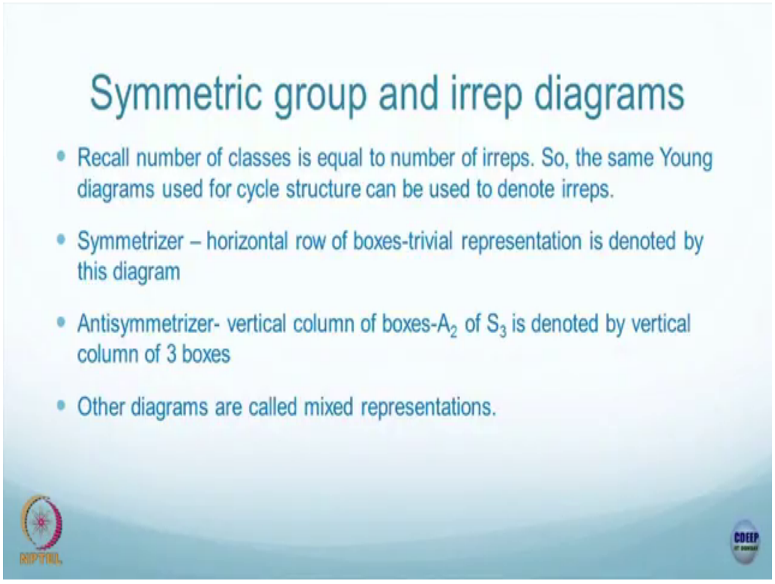

This also we went through the symmetrizer anyway I have recalled here for the group symmetry, group of degree 2 ok. And then only thing is in the symmetry group of degree 2, you do not get mixed representation because you have to only play around with 2 boxes. So, the mixed representations starts coming and you have to use the Hook diagram to write the dimension for symmetrizer and antisymmetrizer, it is 1-dimensional.

Only when you start going beyond degree 2, then you start seeing mixed representation and I gave you a formula for writing the dimensions. The ways of deriving it, but right now just take it as a working principle formula ok.

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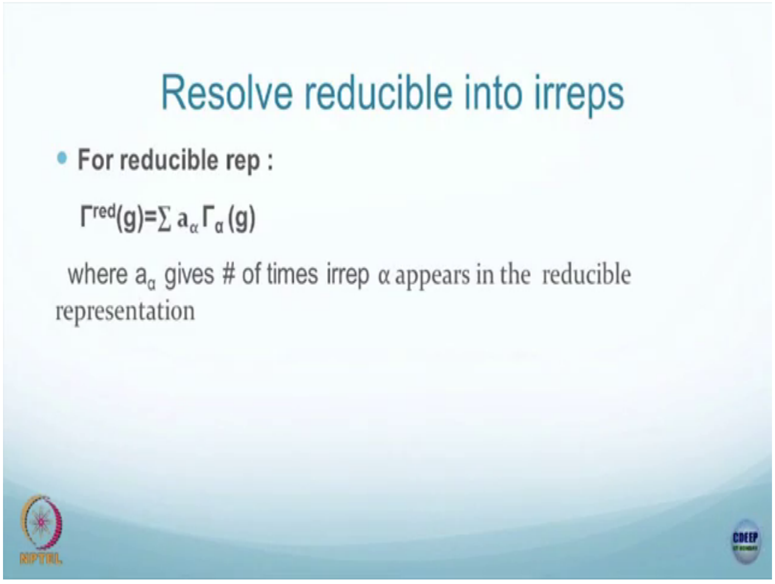

And this is where I left you last time saying that reducible representation if you have I took one simple example I think I made a mistake in writing the matrix elements.

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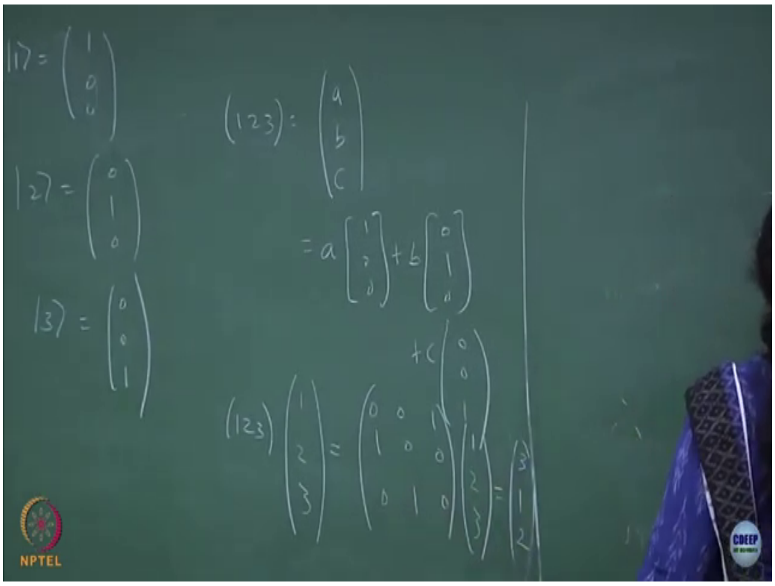

So, it defines states like 1 which is 1 0 0, 2 which is 0 1 0, and 3 which is 0 0 1. And if you take the 1 2 3 element ok, the property of this is that if it operates on let us say a b c ok, this can be seen as a times 1 0 0 plus b times 0 1 0 plus c times 0 0 1 right this is one way of seeing it.

So, you can write 1 2 3, on 1 2 and 3 will become what did we find 0 0 1 ok. This will be the this will be the matrix operating of 1 2 3 goes to should get sliding it, thank you. So, write the matrix representation which does the job and we call these matrices as reducible matrices, and we found what is the reducible nature of the matrices right.

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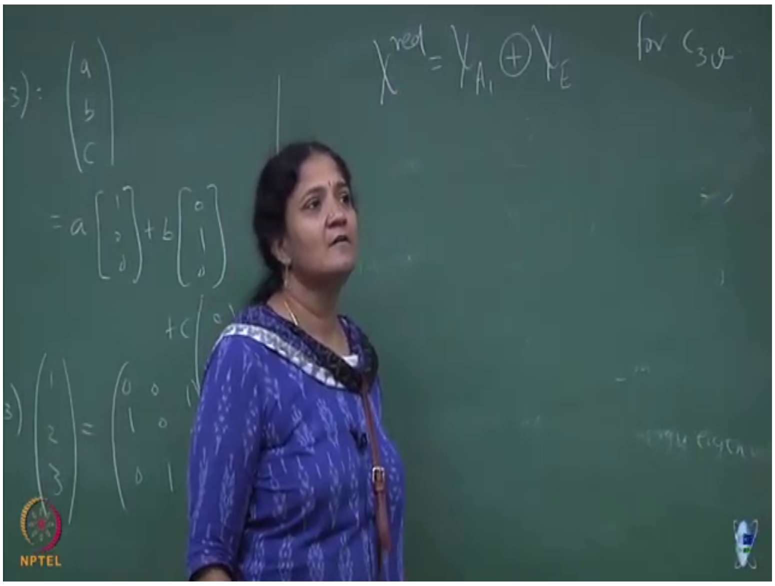

What did we find character for reducible for these kinds of reducible matrices turned out to be, someone, is that right? This is for the C 3 v group right. Last lecture I am sure you are all there, none of these things will get altered because the characters do not change by this similarity transformation, but if you take the matrix representation find the traces and we did this.

So, a alpha is the number of times an irreducible representation alpha occurs in a reducible representation. It turned out that the irreducible representation A 1 occurred once, and irreducible representation E occurred once, and then that linear combination gave you the character for the fact this is not the right way to write.

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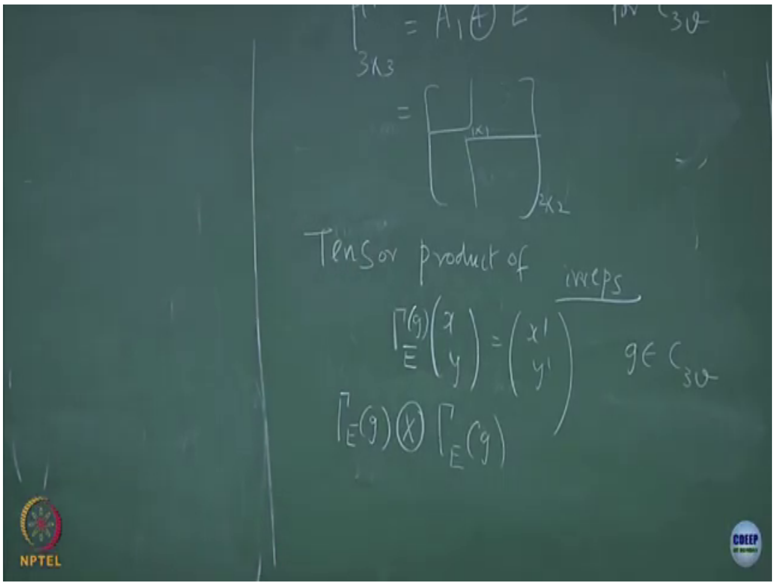

I should say that you have to say that the reducible representation gamma reducible not the characters, characters are just numbers is A 1 plus E this is the right method of writing it. Characters are just numbers they do not have some direct sums. So, any 3 cross 3 matrix which I had here can be brought into a block diagonal form which is 1 cross 1 and a 2 cross 2 and these are irreps of the C 3 in the character table.

Student: (Refer Time: 06:08).

So, I am just saying that when you do this is the cycle structure.

Student: (Refer Time: 06:15).

So, I have just written that if you call this as the two corresponds to 0 1 0.

Student: What is that mean?

It is just a representation. It is just equivalently the quantum state which I am writing in the direct bracket notation can also be written as a column vector whenever I have to do matrix operation I need to remember that I need to define a basis vector space vector space. So, these are the three basis vector space, and then what I am going to do is 1 2 3, takes 1 to 2 to 2 to 3, 3 to 1, and which matrix that is it is what I was trying to figure it out.

And it will be a three-dimensional matrix 3 cross 3 matrix because the vector space is 3-dimensional right. Whenever you have a xy component, the corresponding matrix with which you have to multiply on the xy vectors have to be x y a 2-dimensional vector space is that it has to be a 2 cross 2 matrix, this is all I am saying ok.

Student: Maam.

Yeah.

Student: Can we (Refer Time: 07:21).

Very good, good, yes you can have 100 by 100, 1000 by 1000, how do you construct it, it is a good question ok, so good points, so that is going to be just a tensor product irreps ok. So, we know the 2-dimensional representation right

Let us take the 2-dimensional representation, all the elements, we know for the C 3 v just a rotation by 120 degrees sigma v in the x y plane and so on right, the basis is x y, so 2-dimensional ok. So, you know these matrices, all the matrices for the group elements, where g is a C 3 v. We have constructed this, and you can write the 2 cross 2 matrix.

What is tensor product means you take this gamma E of g let us take a tensor product with gamma E of g ok. When do you have to do this is a question you can ask. If suppose x y is the coordinate of one particle, and I want to look at a system of two particles, let us say x 1 y 1 is the coordinate of one particle, x 2 y 2 is the coordinate for the second particle ok, and then essentially it becomes a how many dimension vector space? 4 right, it is going to become a 4-dimensional vector space.

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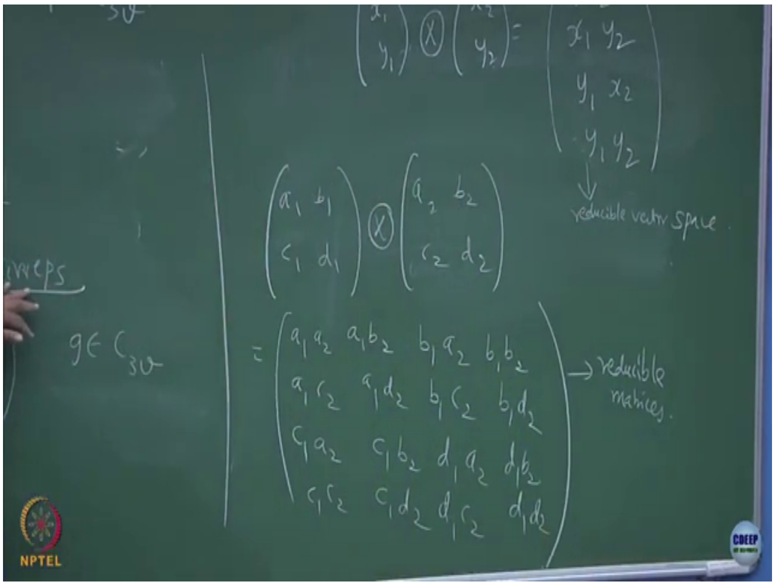

So, what happens, so let us write that. So, x 1 y 1, x 2 y 2 will give you x 1 x 2 ok, take the x 1, take the x 1 and multiply with x 2 y 2 that is what I have done x 1 x 2 x 1 y 2 y 1 x 2 y 1 x 2. So, the vector space of a two particle system becomes 4-dimensional. And the corresponding matrices are given by again this tensor product, how do you see this tensor product if you had a 1 b 1 c 1 d 1, and if you take a tensor product with a 2 b 2 c 2 d 2, you have to take a 1 and multiply with this, then go to the next block b 1 multiply with this.

So, let us write it out explicitly somebody can help me out a 1 a 2 a 1 b 2 a 1 c 2 a 1 d 2, then b 1 a 2 b 1 b 2 b 1 c 2 b 1 d 2, the last one c 1 a 2 c 1 b 2 c 1 c 2 c 1 d 2 and d 1 a 2 d 1 b 2 d 1 c 2 d 1 d 2. So, what can you say about this matrix?

Student: Why you are not taking the great sum, why you are taking great sum there, you can take great sum?

What is great sum?

Student: Direct sum, direct sum (Refer Time: 11:41).

No, this is going to be a tensor product by definition. I am not doing direct sum. Direct sum is each block is irreducible that we have already done. We need we will do this after we do the tensor product. Tensor product gives you a higher dimension vector space, and these matrices are reducible, reducible matrices or this one is a reducible vector space ok.

So, I am just defining for you what is a tensor product, it is not a direct sum vector space I am looking at I am looking at a tensor product of irreps. Typically, you will face this when you have to do a system of three particles, system of two particles, you have to first combine them becomes a higher dimensional vector space corresponding matrices operating on it is also higher dimensional matrix, but they typically are reducible, and then you try and find out how many irreps are there in this combination ok.

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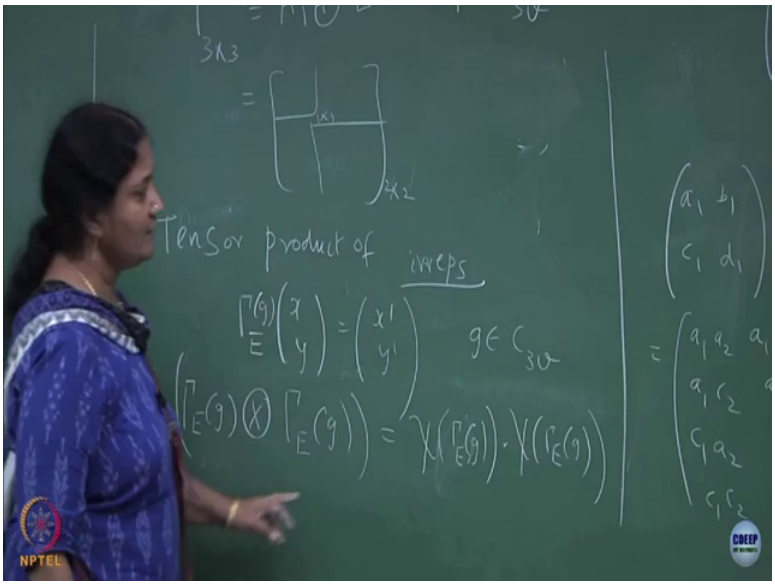

What happens to the characters, can somebody tell me what happens to the characters of them? Characters will also be a product of the characters check it out ok. So, the character here is a 1 plus d 1, character here is a 2 plus d 2. Can you check what happens? It will become, so one of them this will be just a multiply ordinary multiplication. Is that right? Can you check, what is the character here?

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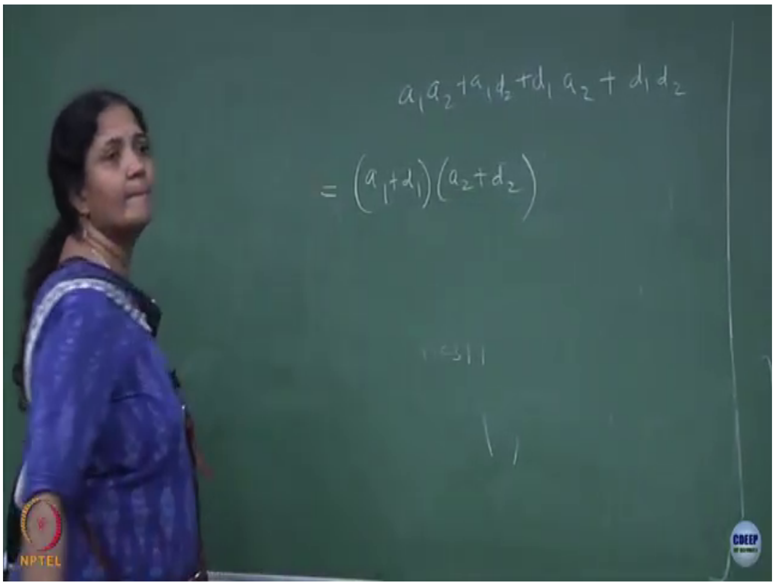

Character here is that right that is the character for the 4 cross 4 matrix, which is this left hand side for a particular element. Right hand side is what, this is the right hand side. Is that same, are the same? So, the characters of the tensor product will be a product of the characters of the irreps, but this tensor product representation will always be reducible someone was asking can we not make 4 dimensional representation, yes, by taking tensor product any number of times you can get reducible representations.

And then you can also break it up into how many components are there and how many times each of those irreducible components are occurring by using this formula ok, is that clear.

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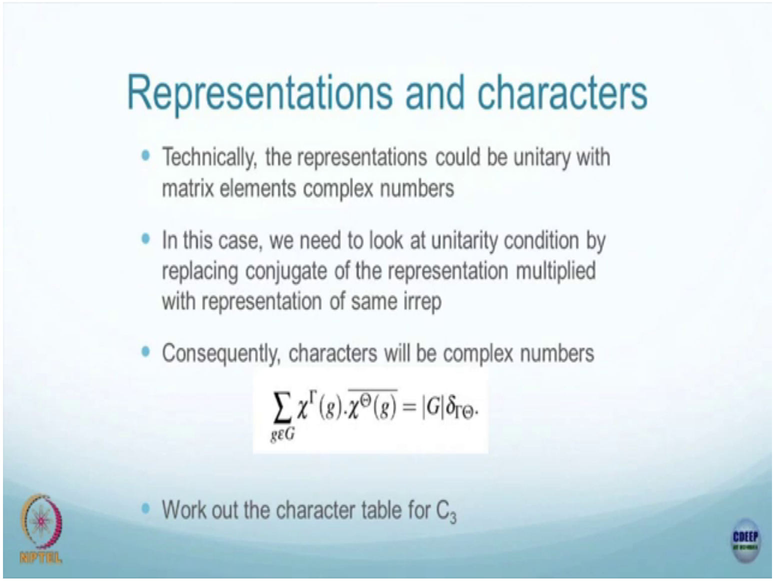

A alpha is actually the multiplicities. By multiplicity what I mean is that one particular irrep can come many number of times, and then you find out what is that a alpha which we elaborately worked it out for this particular case, and I have given enough problems in the assignments. So, please go back and do it, and find the multiplicity a alpha how it is going to be evaluative ok, ok.

So, that is the a small small technicality which I thought I should mention it because most of the books if you browse through I just used it as if it is orthogonal between rows whenever I wrote a column character table, I said look at the orthogonality. Essentially when I write the matrix representation you know for a group, it could in general be a unitary matrices. Unitary matrices could be having complex entries, some of the matrix elements could be complex, characters could become complex.

So, I should remember that there should be a unitarity condition are not orthogonality condition in general, most of the cases will pass with real numbers in the character table. There will be some situations where you will be forced to use unitarity conditions, some of the characters can become complex ok. We will we will just go over C 3 character table to understand this and that is why I have put in a bar here, bar to remember it is the complex conjugate of the character. Gamma is one irrep, theta is another irrep formally ok.

So, if I take these two irreps, sum up our all elements, earlier I did not put this bar assuming it is real, but in general for any arbitrary group it is nicer to put a unitarity kind of condition and this condition is important ok. So, just remember this, and let us do the C 3. C 3 is an abelian or a non abelian group, non-abelian? Abelian.

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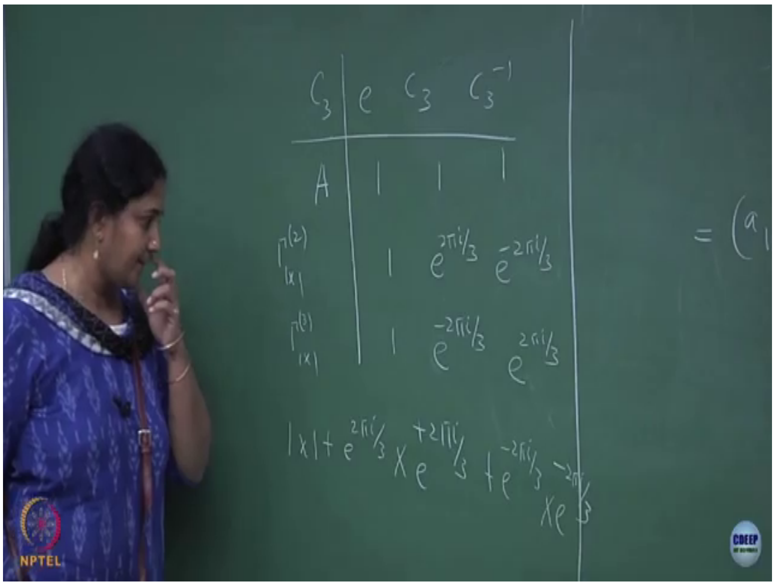

So, character table for C 3 somebody can tell me, each one is the class by itself, you will have 1D representation trivial representation and then non-trivial representation should also be 1 cross 1 matrices, it could be either A or B will figure it out. So, this is 1 1 and 1 right. Now, what we do for the next 1 cross 1 matrix, we will write the Mulliken symbol afterwards if we know it ok.

So, here you can have a 1 ok, can have a e to the 2 pi i by 3. This is where the complex entries are (Refer Time: 18:47), because you have to make sure that the orthogonality satisfied, sorry unitarity satisfied. And when you take the square of this, you have to take the this multiplied with the star of this by that condition.

If gamma is equal to theta right, you have to take mod square of it, is that clear. If you take the mod square of it, this is unit modulus 1 plus 1 plus 1 which is 3, and unitarity the orthogonal combination is going to be orthogonal because 1 plus this plus this is 0, you all know that right, cube root of unity satisfied 1 plus omega plus omega squared equal to 0. What will be the next one, somebody, it is work? Does this satisfied that condition between these two?

Student: No.

No or yes?

Student: No.

The bar of this will become e to the plus 4 pi i by 3 right. So, it is 1 into 1 plus e to the 2 pi I by 3 into e to the minus minus will become bar. So, it will become plus 2 pi i by 3 right 4 pi by 3, and then this one will be this is complex conjugate. Is it satisfying now, can you check, yes, no?

Student: Yes.

So, this is what will happen that you have to remember when you have to do orthogonality, you have to remember she already said no by blindly multiplying right. Please remember that if you have complex entries, one should be complex conjugate of the other when you multiply, so that is why I put this with the plus sign ok. Now, this will also give you clue for c 4 what you should play around there. So, this is one.

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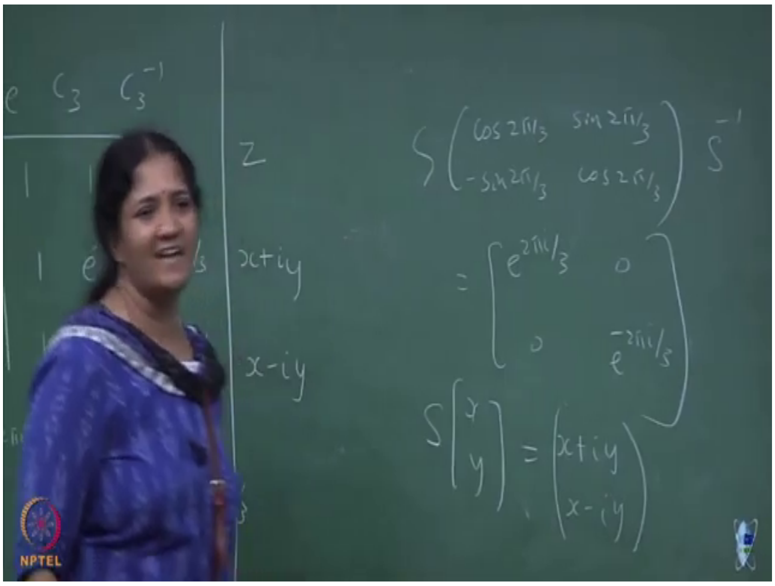

And what is the basis states here, z is a basis for this. So, this being complex entries the basis could become, I do not know which one is right, but you can check it probably one is x plus i y and the other one is x minus i y.

Student: (Refer Time: 22:36).

Student: How do we get this (Refer Time: 22:38).

How do I get this is I am just only playing around.

Student: (Refer Time: 22:42).

So, you can diagonalize the matrices because it is an abelian group ok. So, if you take the rotation matrix which you have cos 2 pi by 3 sin 2 pi by 3 sin 2 pi by 3 cos 2 pi by 3, you can diagonalize this by in S matrix ok, which will give you 0 0, I am sure these people would have done.

And then this matrix was in the xy vector space right, this S matrix if you do it on x y, the S which does this job find the S which does this job, if you find it you will find that you will get it to be what will that be x plus i y right, this is what will happen. I leave it to you check some of these things can be checked.

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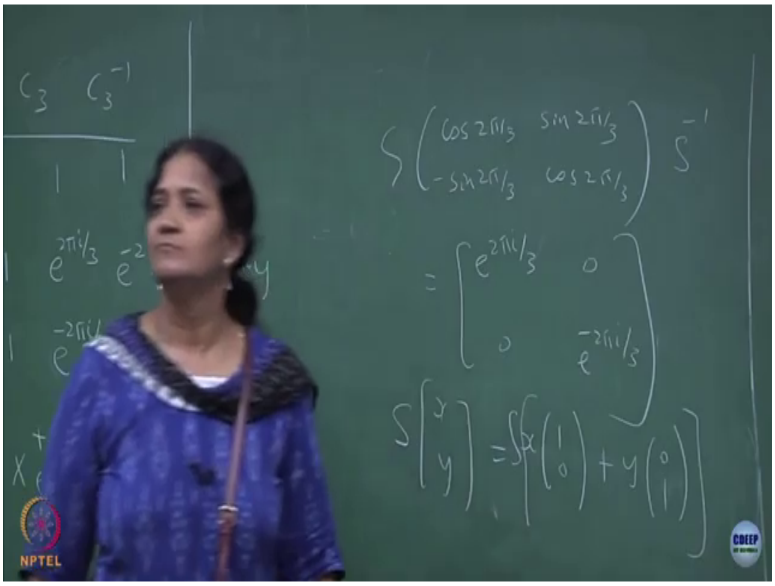

What I mean by this is that, so basically I am just saying that x can be written as 1 times 0 plus and then you can operate the S on this and y times 0 1 ok. What you can show is that this S matrix which is the 2 cross 2 matrix on 1 time 0 will give you only 1 time 0, but here when it operates on 0 times 1, this 1 0 1, it will give you an i factor ok.

So, you have would done it in classical mechanics right. You have not done diagonalization of matrices, you all done right? You have done basis vectors, after you diagonalize it, it will become some linear combinations of those original basis vectors. I am only saying that original basis vector which are x and y some linear combination is with this combination.

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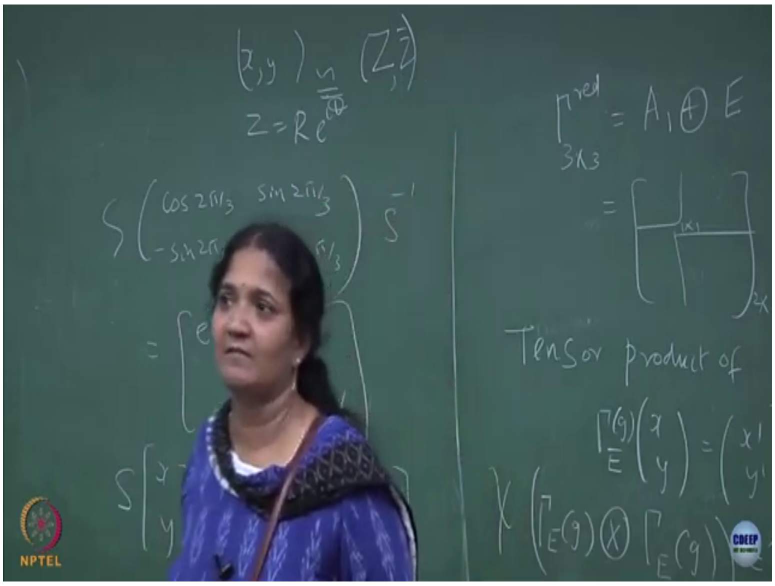

It is also not its fancy way of writing, one side say it is a rotation you know that either you can work in xy plane or you could work in the complex z plane, do not take it as z axis complex z plane, z z bar, both are equivalent. If I say under x and y gets rotated, the same rotation is about the z axis, z coordinate will not change right, it will go to e to the i theta z is R e i theta. Am I right?.

So, that is the way it is going to be. So, z is what z is x plus i y in z bar is x minus i 1 both are equivalent basis right instead of looking at a 2-dimensional real basis you can look at it as a 1-dimensional complex basis right, z bar is complex conjugate of z, so that is all I have return here the basis is say x plus i y and the other one can be seen as a complex y.

All known to you, it just that I am trying to write it in a group theory language, and I only added one more additional variant that sometimes the matrices can have complex entries and then your great orthogonality theorem should involve one of those with the complex conjugations, so that everything becomes real.

If you blindly multiply with e to the minus 2 pi i, you will start getting this which is not going to be real right. You need to make sure that if this multiplied with this you take this and conjugate this and multiply which is what I have done ok.

So, I will probably stop here.