Group Theory Methods in Physics
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Lecture – 19
Great Orthogonality Theorem and Character Table-I

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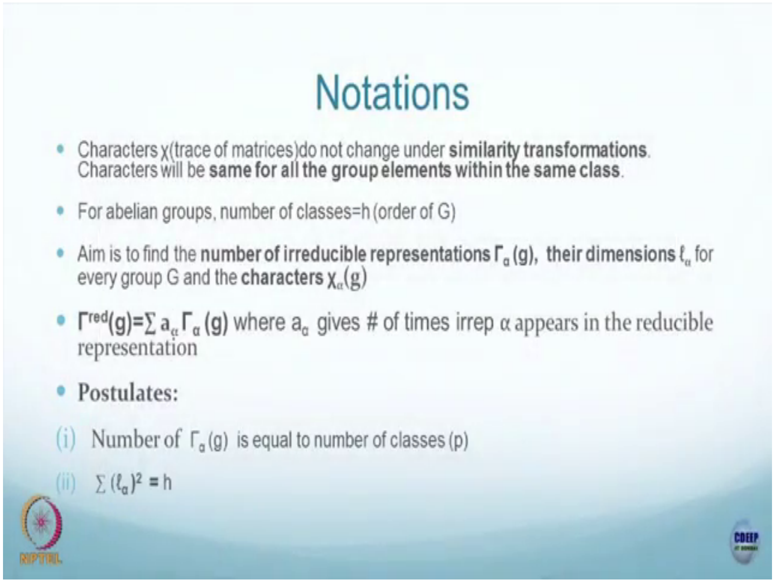

So, let me begin with the notations which we started yesterday. What were the notations, we are going to follow? We are going to take h is the notation for order of group.

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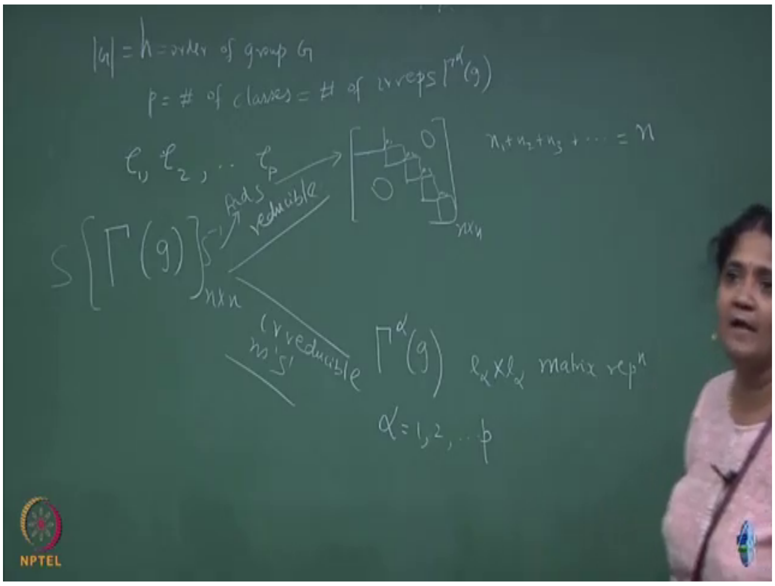

Typically, we write order of group g by notation, this is, I am using h just for clarity mod g is what we would have written, but we are writing this as h, p is the number of classes ok, p is the notation which we are going to use for the number of classes and classes we were denote it by C 1 C 2. So, do not confuse this a c with your group structure. So, you can, if you want you can put a some kind of curly C 1.

Cp, p classes are there that is one and then I also said that we will use gamma of g as a representation typically, it could be an n cross n matrix representation and this will be of two types; it could be reducible right and it could be irreducible. So, reducible in general, irreducible we are going to denote it by gamma alpha of g and these are l alpha cross l alpha matrix representation ok. I am just using the short hand notation that.

So, the matrices will have dimensions l alpha by l alpha, number of rows and columns will be l alpha and l alpha that is the meaning of this. And how many alphas are there for this specific group? Alpha will take values 1 2 someone number of classes, number of irreducible representations for the group will be equal to the number of classes is one of the postulates, irreps is the short hand notation I use for irreducible representations ok, this is clear? Notations are getting clear and then what I am saying reducible means I basically can find an S and bring it two block diagonal form ok.

So, this n cross n can break it up into pieces and so on. The requirement is that suppose this is n 1 cross n 1, this is n 2, this is n 3, this is n 4 and so on, what is the requirement; n 1 plus n 2 plus n 3, other are all 0. Reducible means you can find S to bring it to this form by similarity transmission, you can do an S, S inverse, you can find an S to bring to this form. Here, no S. Here, you do not have any S; you cannot break it that is the meaning of this hm. So, now, this number should add up to.

Student: (Refer Time: 04:31).

It has to be n right. The total matrix is n cross n, the breaking up into blocks. The total block dimension should add up to be the dimensions to start ok. So, that is one other thing, I said is that in principle I can rewrite this as summation over all the irreps.

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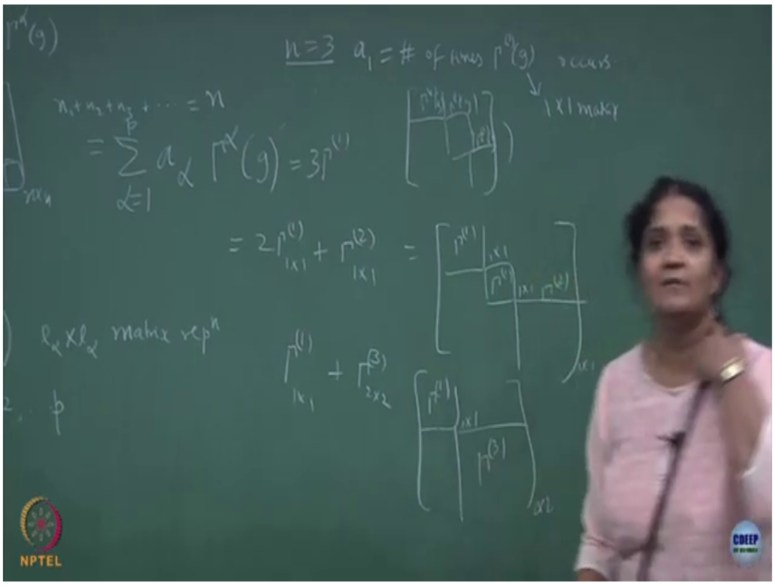

I am denoting the irreps by index alpha which goes from 1 to p, a alpha gamma alpha ok, for every element and someone was asking, what is the meaning of this? The meaning of this is that if gamma 1 appears twice, you will suppose, I say that let us look at n equal to 3 ok. For example: if you take n equal to 3 and let us say that a 1 is the number of times gamma 1 of g, this is an irrep notation occurs and how do I denote it in the matrix form? This will be a gamma 1 of g, this will be a gamma 1 of g and this will be a gamma 1 of g are you all fine.

The dimensions 1 plus if suppose, this is 1 cross 1 matrices ok, let us take this to be 1 cross 1 matrix, then the dimensions adds up to 3 ok. This is one way in which you can write this as a 1 times gamma 1 alone 3 times gamma 1 is one possibility, for this example you could also have 2 times gamma 1 plus some other gamma 2 and the only thing is that it should also be a 1 cross 1 matrix, it should also be a 1 cross 1 matrix.

These two entries to be different that is why I put gamma 1 and gamma 2, this entry occurs twice this entry occurs ones. How will I show this? It will be again gamma 1, gamma 1, gamma 2. This will also be 1 cross 1 1 cross 1 1 cross 1 for an example. So, this is the meaning of writing, it has the sum of a alpha times gamma alpha, where gamma alphas are the fundamental piece by which you can construct any higher dimensional matrix representation, which can be broken up into block diagonal form.

Some of the reps can irreps can repeat. Suppose, the irreps repeats, we say that a 1 is 3 here, the first irrep repeats so, a 1 is 2 and a 2 will be 1. Is that clear? It could also be that I get only a gamma 1 and a gamma 3, which is a 2 cross 2 matrix that is also possible, that depends on what is this S operation and how it block diagonalizes it ok.

So, if this is 1 cross 1 the irrep 1 occurs ones, the irrep 3 occurs ones. So, the block will become a 1 cross 1, under 2 cross, this is 2 cross 2 and a 1 cross 1. You understand what I am saying; this notation yeah.

Student: Direction, what do you mean the direct sum.

Direct sum, I mean technically the direct sum yeah, I agree ok; is just that you people do not understand that notation as of now, so I am using this.

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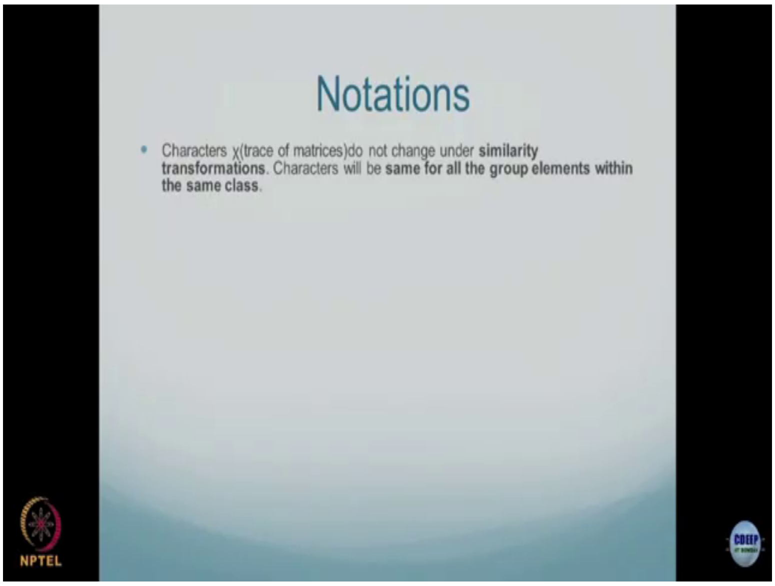

The meaning of this is the block diagonal fashion in which we write it is not that some of the irreps can be repetitive ok, that is the meaning of this. Is this clear? Is the reducible irreducible also clear? Ok. So, remember these two expressions, which I have written now, one is this, let me write here.

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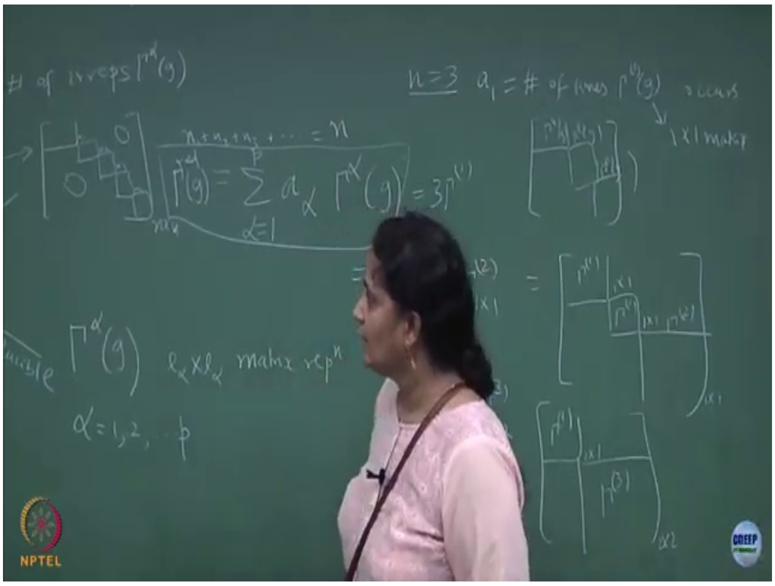

Gamma of g reducible will be rewritable in terms of this ok, where these are the irreps and then what did we do and we had set of a properties which follows for the.

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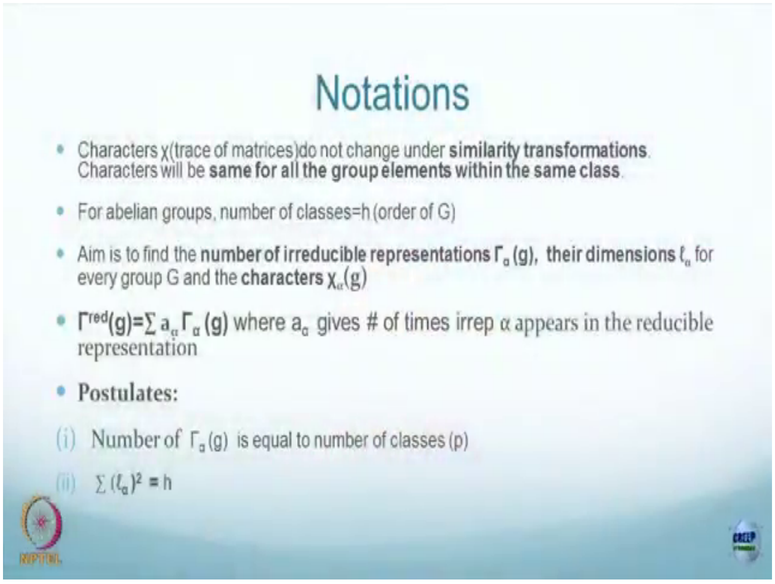

So, I showed these things for a abelian groups, number of classes is also order. So, you will get number of irreps to also be equal to order of group and then you need to find out the reducible representation also now, I put in here that you understand the notation.

A alpha is a number of times an irrep alpha appears in the reducible representation ok. Is this clear? Postulates were the first postulate, I have already written here that the number of classes equal to number of irreps and then I have one more property which is summation over alpha l alpha is squared equal to order of the group ok.

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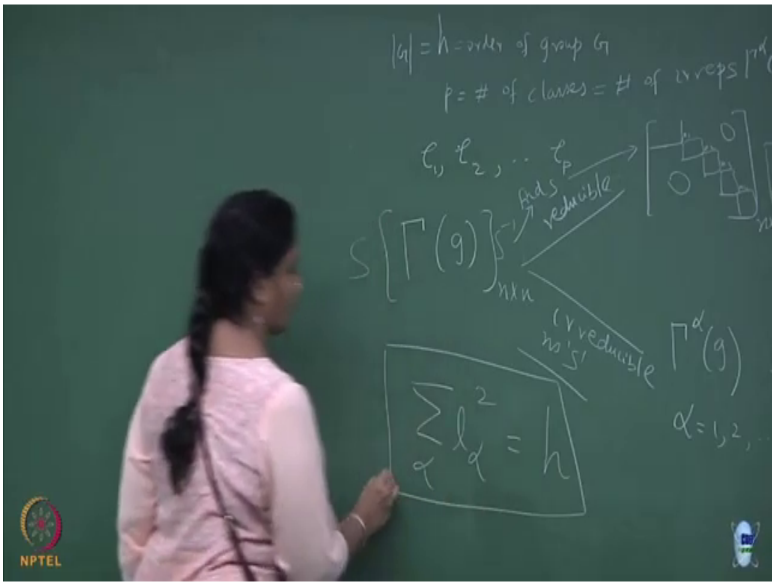

So, l alphas are integers and squares of those, if I have positive integers squares of them is constrained by the number of elements in that group and it is a very easy problem to find what are those integers, there will not be more than one solution ok. There will be only strictly one solution and then I got to the postulate of the great orthogonality theorem, I am not here to. In fact, you can use this great orthogonality theorem to prove all this things, I am not proving it.

I am saying as a consequence, you get these as a product, you can do the proofs if you want, but as of now from the great orthogonality theorem you can prove this l alpha squared equal to h and you can also show that if you take the trace of those matrices.

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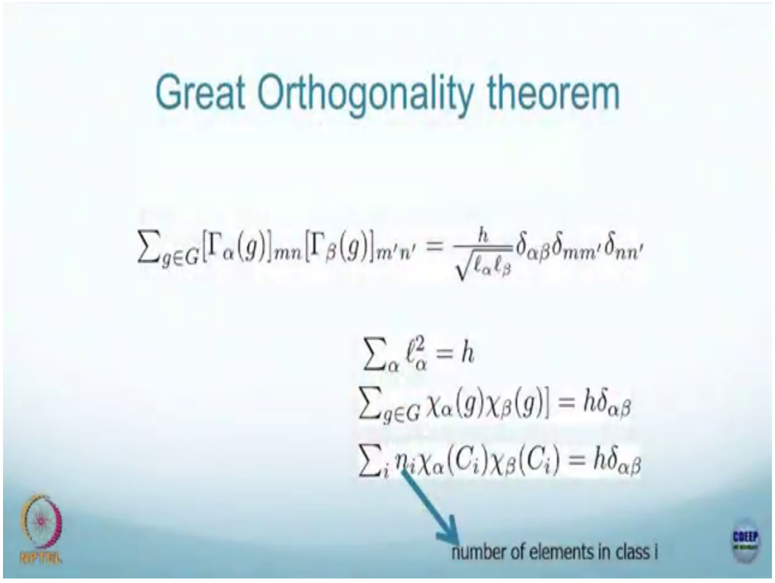

So, these are matrices with m n element the specific element this m n means a specific element of that matrix multiplied with some other specific element of another irrep matrix and if you take the sum over all the group elements of them, it will satisfy this relation that it will be ortho orthogonal, alpha and beta they are not same you get it 0. If alpha and beta are same you also require the same elements m should be m prime n should be n prime and then you have a factor which is h divided by the dimensions of those irreps, l alpha is a dimension for the irrep alpha, l beta is the matrix dimension for irrep beta is this clear.

So, this is something which I do not want you to memorize, you can put it in your formula sheet. Everything you can put it in the formula sheet ok, you do not need to memorize and from this you can start playing around taking the trace. So, we define character character alpha of g as trace of gamma alpha of g right. This is the notation we have. So, you can play around on this great orthogonality theorem by taking a trace and then you can derive summation over g chi alpha of g chi beta of g to be equal to h times delta alpha beta. Is this clear? Are you all following?

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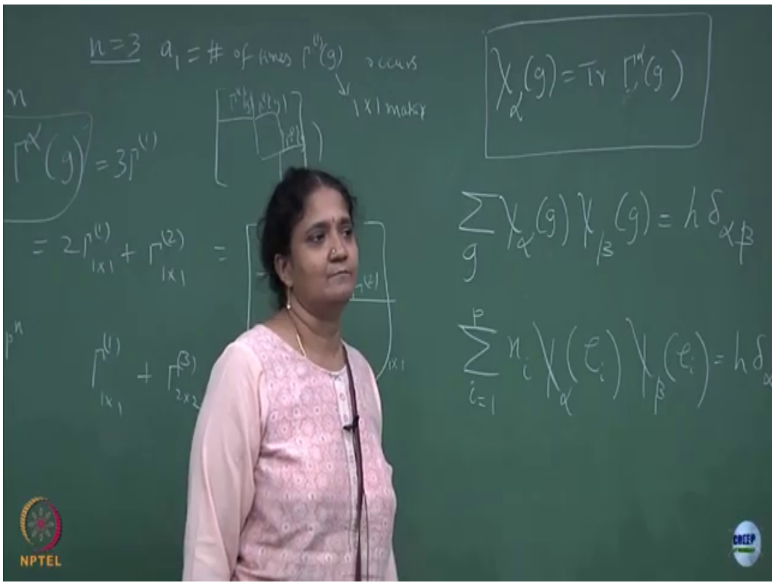

Student: Yes ma’am.

So, the characters are orthogonal, they are different irreps and you have to sum up over all this elements and then do that. Equivalently, since the character within the class is same ok, you could replace this by summation over classes 1 to p number of elements in the class i and then you have a character for the irrep alpha for a candidate in that class which you can take as formally as C i and chi beta for the candidate in the same ith class ok.

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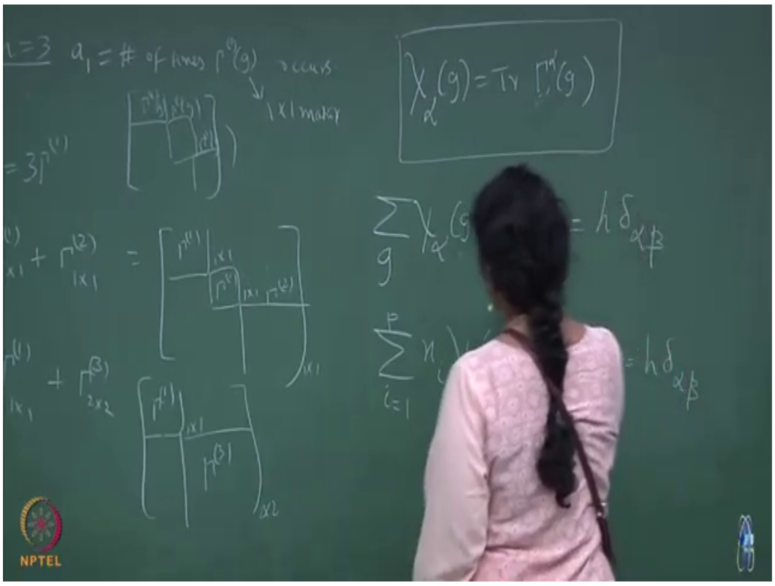

And this will also be exactly h times delta. So, these two are equivalent statement. Instead of doing it over a summation, over group elements, you can do it by taking one element in the conjugacy class, but put the number of elements in the conjugacy class as a multiplicative factor, is that clear. So, that is what I have written in the next equation.

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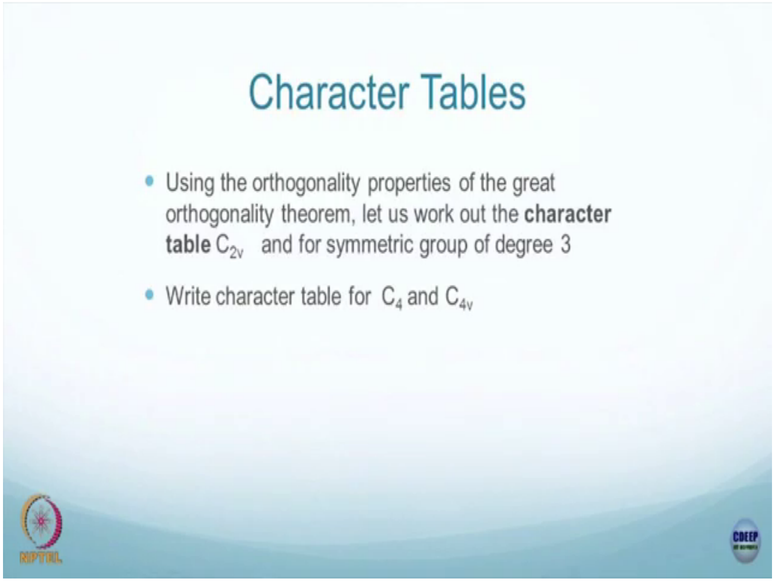

So, now comes the writing the character table using these properties ok. So, C 2 v was the first example, we took and here, you had e then you add C 2 then sigma x z, which I will call it as sigma v and then I call sigma v C 2 ok.

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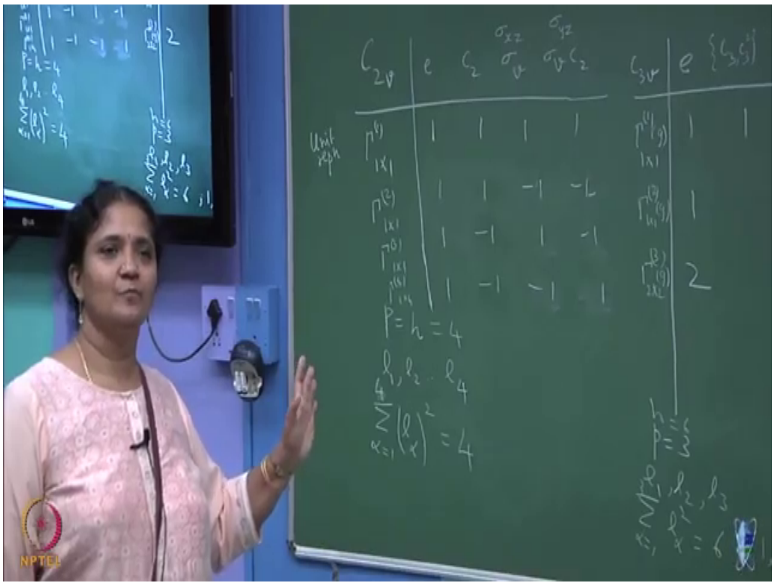

So, this is nothing, but sigma xz. I think this one sigma yz both are equivalent right. So, first of all we need to determine how many alphas are there, p for this cases it is an abelian group number of classes is also equal to the order of the group p is h.

So, which means l 1 l 2 up to lh correct and we need to satisfy this condition. So, summation over alpha 1 to h l alpha squared has to be equal to 4 right. In this case, it is 4 ok. This is an abelian group maybe, I should just keep to 4 ok. I am doing it for this abelian group. So, what is the solution for this? What is the possibilities for this? Can you have a two dimensional matrix? Everything should be non zero, there should be four non trivial at least 1 cross 1.

So, the solution is that you can have a gamma 1 which is 1 cross 1, gamma 2 which is 1 cross 1, gamma 3 which is 1 cross 1 and gamma 4, which is 1 cross 1 ok. The 4 irreps have to be each of the irreps, have to be only one dimension ok.

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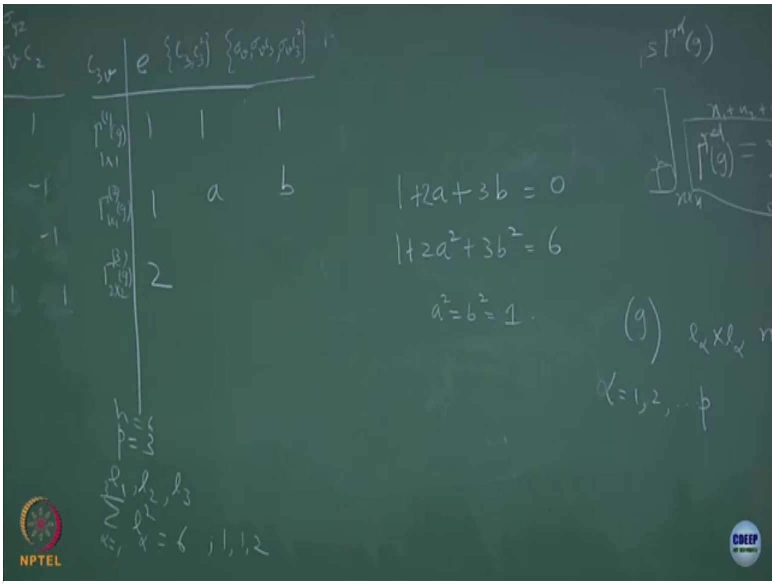

So, let us do the C 3 v also here, identity one conjugacy class as C 3 C 3 squared the second conjugacy class as sigma v sigma v c 3 sigma v c 3 squared. So, in this case what is the classes number of classes 1 2 and 3 right. So, you have p equal to 3. So, you will have l 1 l 2 l 3. There will be 3 irreps. What are the dimensions summation over l alpha squared from alpha equal to 1 to 3 should add up to give you.

Student: (Refer Time: 18:54).

Order of the group is.

Student: 6.

6. So, now, tell me what are the representations possible here? 1 1 and 2, anything else possible? Nothing right, these are the only possibility. So, you will have a one dimensional representation another one dimensional representation and a two dimensional representation and then we write the characters of these representation elements ok.

I said that we are going to take this to be the unit representation, which is trivial representation, I am going to blindly put it as 1 1 1 1 ok. Here, also we will write the characters. So, now, I am writing the characters here. So, entries which I am writing are characters, characters of which irrep. To remember these irreps, I have denoted it here.

So, this one will also be 1 1 and 1 and now, I need to fill the complete table. Last time, I tried to write the irreps by looking at a representation. Now, I am going to use these identities to fix the table ok. So, using these identities part of the properties, we have seen that there are 4 irreps for C 2 v and there are 3 irreps for C 3 v and one of the irreps in C 3 v is 2 cross 2 matrix.

Student: (Refer Time: 20:55).

Student: Is it always possible to (Refer Time: 21:01).

1 1 1 1 is like a trivial see if you can put any group to have identity element. Every element to be identity element, any multiplication table will be satisfied right. So, I will take a trivial representation as a matrix I can put 1 1 1 1 and make the whole multiplication table to be satisfied.

Student: That is why do we know (Refer Time: 21:22). So, if you take a.

I agreed.

Student: (Refer Time: 21:24).

I agree. So, that is another way of saying, this is, this at least I wrote last time, if you do a rotation about z axis. The matrix representation is just 1 1 1 1, that is another way of set ok.

So, that is a trivial representation or unit representation as we call it. Next thing is I need to fix these things, the characters. So, inside whatever I am writing, I have to write the character for representation two, do not know what notation I used here. So, I have used character for alpha 2 for identity element character for the irrep for the C 2 element and so on ok.

I need to fill this; I am not done it yeah. How do we fill it? We just use property the product and summit up over all elements should be orthogonal, because one irrep is a unit representation, other irrep is a non trivial representation. So, let us fill it only thing what you have to remember is for identity element is nothing you can do, whatever is the character it is has to be, if it is 1 cross 1 matrix, it is 1 if it is 2 cross 2 matrix, it is 2.

Student: (Refer Time: 22:44).

So, the character for the identity element you can write blindly. It is just dimension of that matrix right, trace which will give you the dimension. So, this is actually l 1 is 1 l 2 is 1 l 3 is 2. So, similarly here, you will have. So, this part is 1 1 1, no change. Now, I can play around with this C 2. The thing as that property can be done for alpha refers to a row, beta refers to another row, if alpha is not equal to beta then those two rows have to be orthogonal.

So, you have various conditions, what are the conditions you will have; a plus b plus c plus 1 has to be 0 and you can try and play around what all you can do, because the products also we have to take it is a 1 cross 1 matrix entries should be either 1 or minus 1 and you can play around and try and fix this. So, so you can try and see that suppose, I put this as 1, these two has to be minus 1.

So, it is a simple way to check it. Here, also I put minus 1 and make of 1 of them to be minus 1, any other possibility I have missed up, these are the only non trivial possibilities I can have right. So, basically what you are seeing is that this multiplied with this should also be 0. We did 3 of them last, in the last class by looking at this as a x axis, y axis and z axis right.

We also have a fourth one, I do not know which one is the fourth one, but this fourth one which I have is kind of very different, but still it is orthogonal to this as well as this and this is that right are you all with me I have just used that property and I am just playing around with the numbers. What about here? What about here? Someone help me, this should be a and b. It is not just a we have to multiply by a 2 and then 3 times b has to be 0 I am just multiplying this with this, alpha is this one and beta is this one.

I am using this, this one this relation. So, n 1 is 1 n 2 is 2 n 3 is 3. There is also one more requirement, what is the other requirement? You could just take this particular row alpha and beta to be same, then you will have.

Student: (Refer Time: 26:55).

It has to be 6. So, you have two conditions and then figure it out what is the best a and b have to be, some integers see, which is the best one, which will fit to this ok. So, which one will fit a squared equal to b squared has to be 1 is that right.

Student: (Refer Time: 27:26).

So, we will have a 1.

Student: (Refer Time: 27:30).

Very good. Now, let us come to this.

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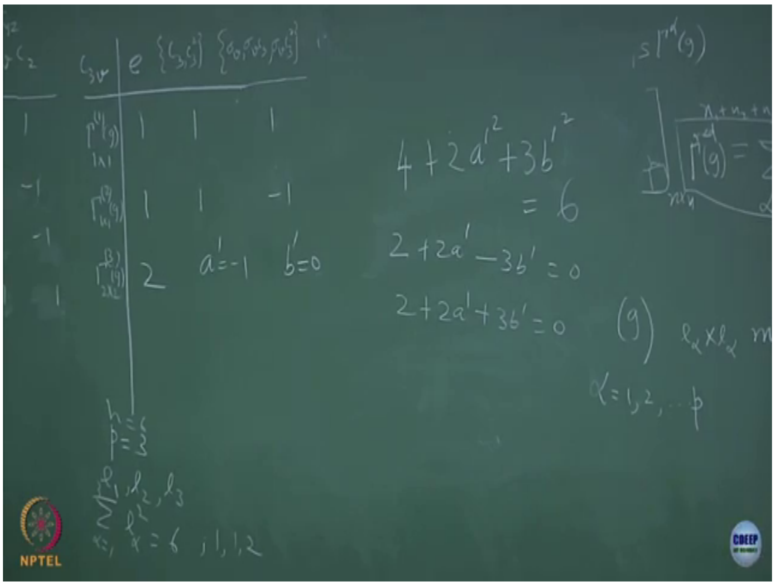

Student: (Refer Time: 27:40).

So, just taking the square of this alpha and beta to be same, we will give you the square. So, that will be 4 plus 2 times a prime square plus 3 times b prime squared, it has to be equal to 6 and what is the next condition? I could just play around with this 2 plus 2 a prime minus 3 b prime has to be 0.

Student: (Refer Time: 28:31).

One more you can write 2 plus 2 a prime plus 3 b prime has to be 0 by multiplying this with this. Clearly, these two equation will tell you what is the solution, b prime has to be 0. So, b prime has to be 0 and is it satisfying that other condition yes, a prime could be plus or minus 1, but then this condition will tell you minus 1 ok. There will not be a ambiguity even though I do not know how to prove it, but case by case when I take the discrete groups, I will find that the character table is unique ok.

So, whatever I have written today is a character table for C 2 v 1 is an abelian group, C 3 v is another group which is not abelian and you do see that as you go to as you go from abelian abelian will always have only 1 cross 1. What is the reason? It has to satisfy this condition and there will be the same number of elements in the abelian group, the only solution is all the l alpha has to be 1 ok. You cannot put any l alpha more than 1 cross 1 matrices ok.

So, abelian group, the irreducible representation is always going to be one dimensional ok, but in the non-abelian you can have one dimensional, you can have two dimensional, we could also have more dimensional matrixes in this particular case with these orthogonality criteria. It turns out that you will have irreps where there are two one dimensional, two different one dimensional irrep; one is a trivial unit representation, another one is a non trivial representation, which is also 1 cross 1, but you also allow for a 2 cross 2 non trivial irreducible representation.

Is this clear? Ok. So, that is what I have done the same thing it could have done it for the symmetric group of degree 3, there is no change. How will it do it? For the symmetric group of degree 3, this is isomorphic.

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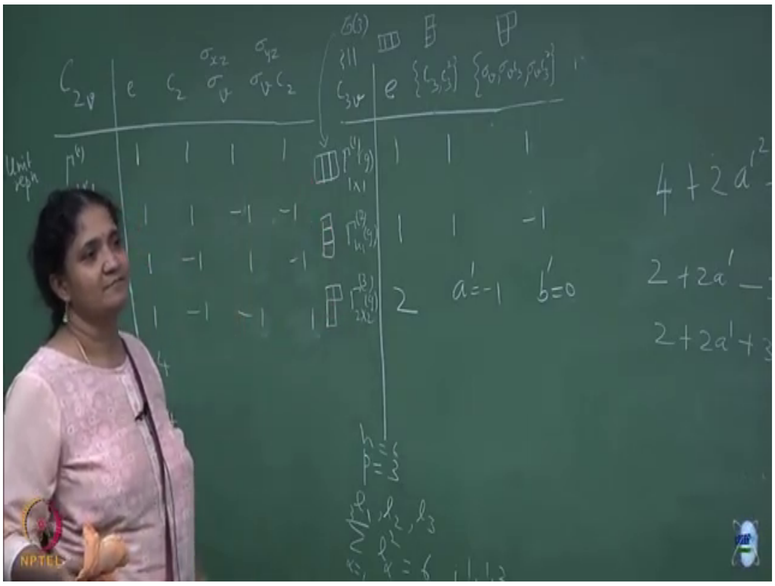

To symmetric group of degree 3 right. You can represent these classes by diagrams, this class as 3 1 cycles right, this class as somebody 1 3 cycle. Am I right? Am I right and this one is 1 1 cycle and 1 2 cycle. You are all with me?

Student: Yes.

Ok. So, in the context of symmetry group, these irreps can also be represented by the same diagrams, with the different meaning. Here, if I ask you how many elements are there for this diagram, you know what to do right. You have done this. You look at how many cycles are there and we have a formula. Similarly, unit representations are always denoted by this diagonal for the symmetric group ok. Do not confuse it with any other group, not every group will have the symmetric group of degree n to do this.

Whatever I am doing it for symmetric group of degree 3, you could do it for 4 5 and all, but C 3 we have happens to be isomorphic. So, this diagram seems to match, but in general I cannot write any diagrams here ok. For any discrete group point group I cannot write a diagram, but at least for symmetric group I can write a diagram ok. So, there is another diagram which is this and there is a diagram, which is this.

Now, you can ask the question, why should I use this diagram structure which is how do I know that looking at this diagram that it is two dimensional matrices. That is not a obvious from here, I could have put this here, right. So, how do I know this is two dimension, this is one dimension, this is one dimension ok. So, let me just review on that.

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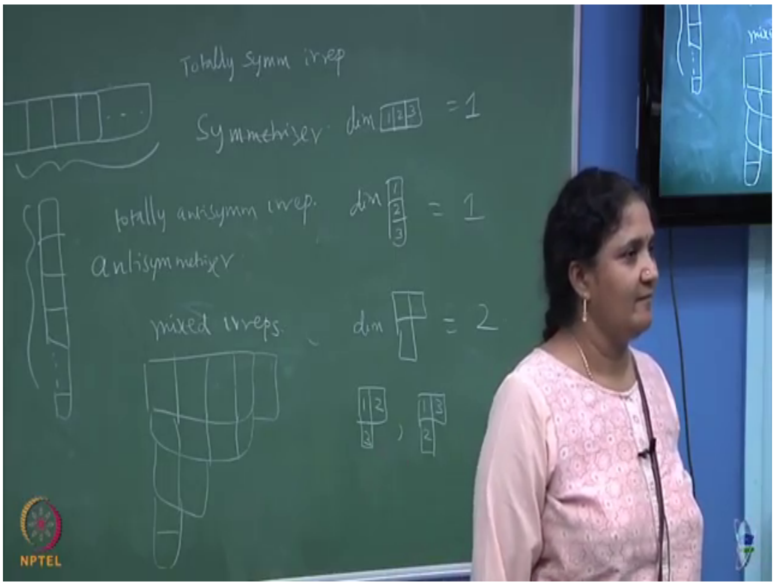

So, this diagram is called symmetrizer, this diagram is called anti symmetrizer, but you can also have mix diagrams, only requirement is that the number of boxes as it goes it to the second row should be almost equal to that, that is the convention we are going to follow.

And you know you can have this. So, this is what I will call it as mixed. Why mixed? It is not totally symmetric, it is not totally anti symmetric, but it as some of them are symmetric some of them are anti symmetric and you can mix and match ok. So, these are called mixed irreps ok. So, this is anti symmetric irrep, this procedure is called symmetrizer. So, I can say that this is totally symmetric irrep, this is totally anti symmetric irrep and this is some mixed irreps.

Now, my claim is that dimension of three box is one, which means that irrep is a 1 cross 1 representation, then I said that dimension of this one is 1, dimension of this one is 2 this is what I am claiming, one of the ways you can see is that symmetric group of objects 3, you are allow to put in every box one of the objects, you can put object 1 here, object 2 here and object 3 here that is it.

There is no other possibility left, that is why it is one dimension. Here, also you can put object 1 here, object 2 here, object 3 here. You will ask; can I not put object 3 here, object 1 and object 2, but it is totally symmetric or you can take it to the identical objects, you do in total symmetry.

Student: Ok.

Whether you call the first box, second box, third box or third box, first box, second box does not really matter, because it is totally symmetric. Similarly, the same thing, this is the only option when it comes here, what happens? You can allow 1 and 2 to the symmetric and 1 and 3 to be a anti symmetric right, this is one possibility.

But you can also have one more possibility, which is 1 and 3 and 1 and 2 is there any more possibilities you can ask, but I am trying to tell you that if you try to do that you can show it to be a linear combination of these two. It is not independent ok. So, these two are the two independent possibilities of putting the three objects in a mixed diagram ok. So, that is why it is 2, but now if I start doing this, this will be for some, let us say n objects, why you become tedious.

Again like conjugacy class I said, there is a formula which will tell you, if you had a breaking up of cycles, what is the number of elements in the conjugacy class you had a nice expression combinatorial experiment. Similarly, there is also an expression for how to determine number of or the dimensions of the irrep, for the permutation or symmetric groups of degree 3 or more degree n also there is a formula, which I am going to show you and then I will explain it ok.

But, this is the motivation and the irreps can be shown exactly like the diagrams which you have here, but the meaning there is 3 1 cycles, 1 3 cycle, 1 1 cycle and 1 2 cycle. Here, it is totally symmetric, totally anti symmetric and a mixed representation, mixed representation will be two dimensional. There will be a 2 cross 2 matrix and these 2 will be 1 dimensional, that is why it is a 1 cross 1 matrix. Is this clear?