Group Theory Methods in Physics
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Lecture – 18
Reducible and Irreducible Representation - II

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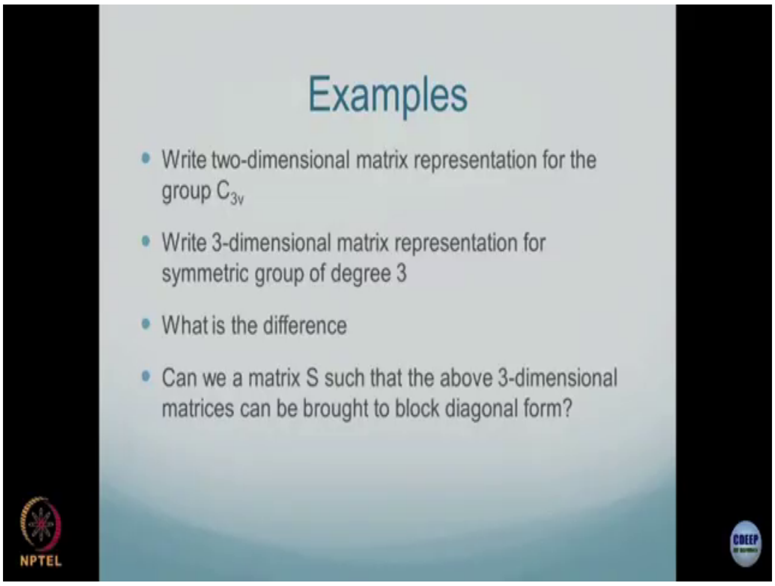

In fact, in the last class I gave you some matrix which was permutation on 1, 2, 3 objects right.

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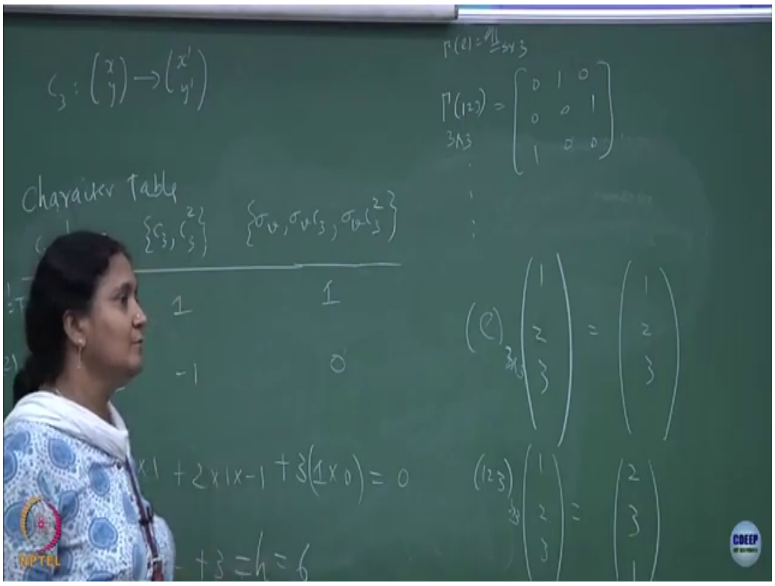

If you remember I also did permutation on 1, 2, 3 objects. The identity element will keep it as 1, 2, 3 and then when I do the 1, 2, 3 cycle on 1, 2, 3 it goes to 2, 3, 1. This is another abstract basis in which I can work where a permute the objects and that is also matrix representation. This is 3 cross 3. This is 3 cross 3 right.

We wrote this matrix. Can anybody recall for me what was that matrix? Am I right. Is this reducible or not. That is the next question you can ask right. You could write the similar 3 cross 3 matrices for all the elements not written it.

But if you see it as a symmetry group for degree 3 wound 3 objects you can get something non trivial like this which is not block diagonal. It is not a 2 by if you have done it on xyz you get it in block diagonal form. So, you know that you know you got a 2 cross 2 and a 1 cross 1. Can we further reduce the 2 cross 2. I said it is not possible, but here it is a non trivial 3 cross right. So, all elements you can write identity element which is an identity operator 3 cross 3 and so on. Question is that is this reducible or not reducible? How will you figure this out ok?

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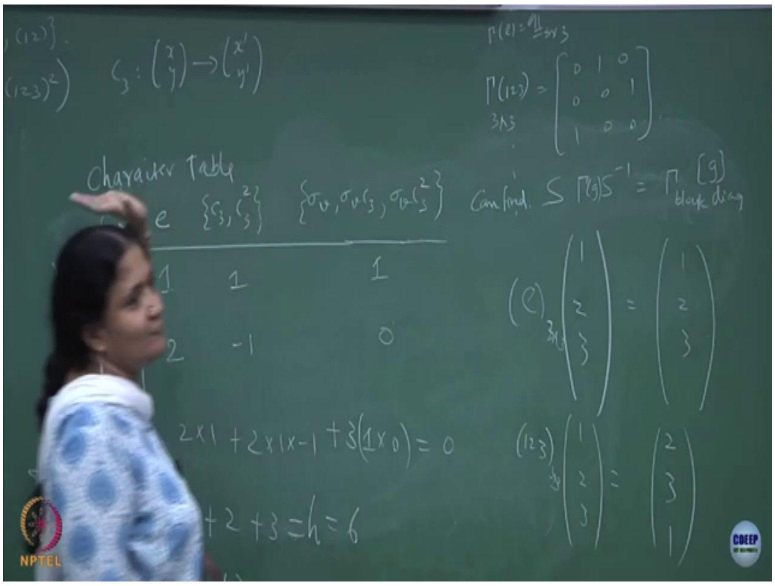

So, I am saying that you can find an S; you can find an S which will diagonalise this into some block diagonal form. I am not saying what it is. You can find some S with just this. What does that mean? This set which I write is a 3 cross 3 representation for the symmetric group of 3 object which is isomorphic to C3v, but it is reducible, but it is reducible ok. This representation is reducible when I write the character table I write only the irreducible representations irreducible characters all the other reducible characters.

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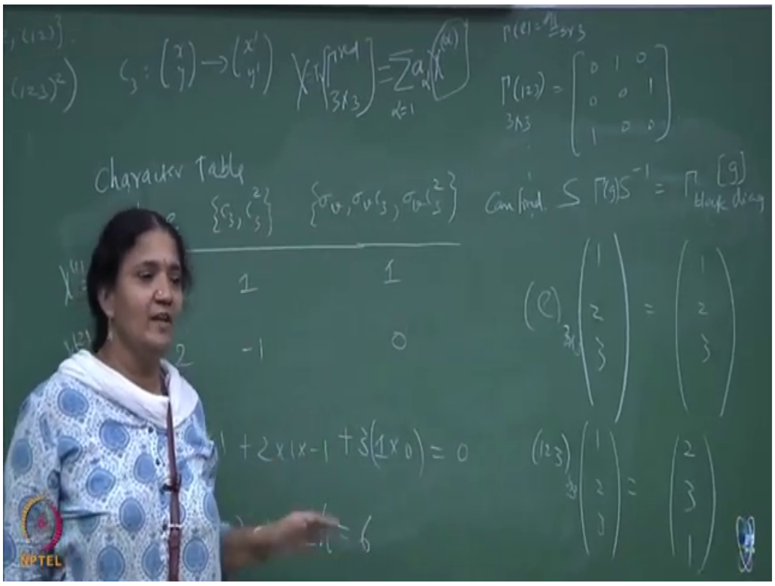

Suppose I take this I call this as reducible character of it which is trace of these ok. I can write it as something like a alpha character of alpha x. This rotations a alpha is some integer and if it is a 3 cross 3 matrix character it should involve at least a 1 cross 1 3 times or it should involve one 2 cross 2 and one (Refer time: 04:12) 1 cross 1 and that will define for you that a 1 will be 1 or a 1 will be 3 and so on.

So, this alpha in this particular case will be going from 1 to some number. This list I have not fixed it and a alpha is the number of times it will happen in that representation and the question is that how to find these things is mainly dependent on the irreducible characters. If you know the irreducible characters just like an alphabets 26 alphabets you can make words. Irreducible characters has all the information about the discrete group and any other representation like this if I give you which permutes this you can try and break it up using the irreducible characters ok.

So, this is the theme of the next 2 2 or 3 lectures and we would like to see whether a given representation is reducible or irreducible depending on whether you could find an S which is a non-trivial question, but we will go through the theorem which says how to find the irreducible components, how many irreducible components are there and what should be the dimensions of these irreducible that is the next question ok. So, this is the theme and we will go over it.

So, when you if you remember we wrote this 3 dimensional matrix representation can we find a matrix such that the above 3 dimension can be brought to block diagonal form. I am saying the answer is no. If we find an S which diagonalises or block diagonalises an n cross n matrix.

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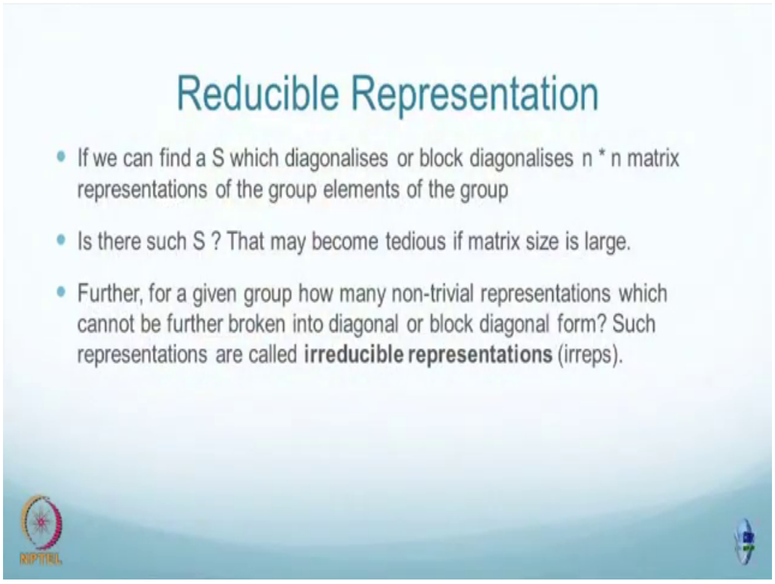

I am just repeating whatever I put it on the board. You call if you can find then you will see that it will be a kind of each of these irreducible each of these blocks will be called as a irreducible block. The matrix n cross n matrix you can call it to be a reducible matrix which is broken up into irreducible blocks. Is there such an S ok. Generally it will become tedious if your n cross n is even 4 cross 4 matrices.

For a given group how many non trivial representation you know how many you can find in such block diagonal form or can we pull out those block diagonals forms out of it which is what we call it as an irreps. Suppose you cannot break it up that is what I am saying if you had an n cross n matrix if you cannot break it up then that matrix is an irreducible representation. Is that clear?

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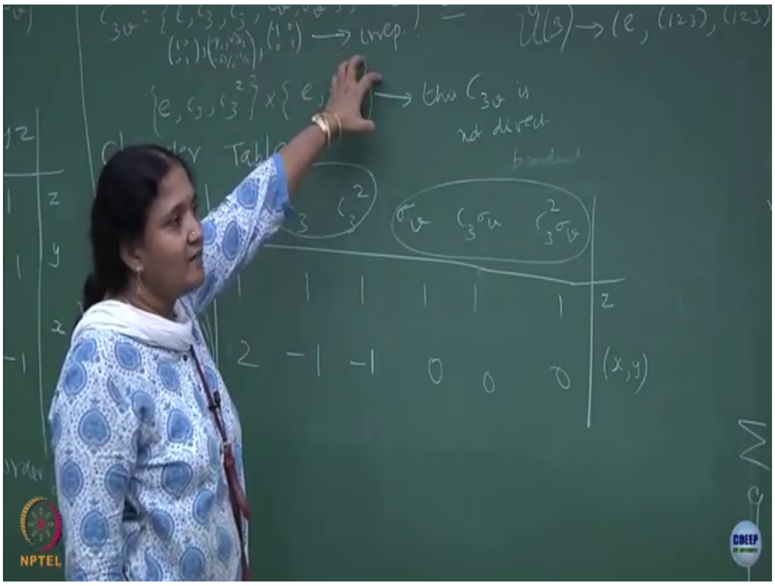

This 2 cross 2 cannot be further broken down into 1 cross 1 simultaneously by a similarity transformation. These things are called irreps. Is this concept clear? Given a representation I am saying you can find an S to break it up. If you can find an S to break it up then you call this to be a reducible representation. If you cannot find an S to break it up then you call it to be a irreducible representation. I have given you 2 examples just to remember. Any questions?

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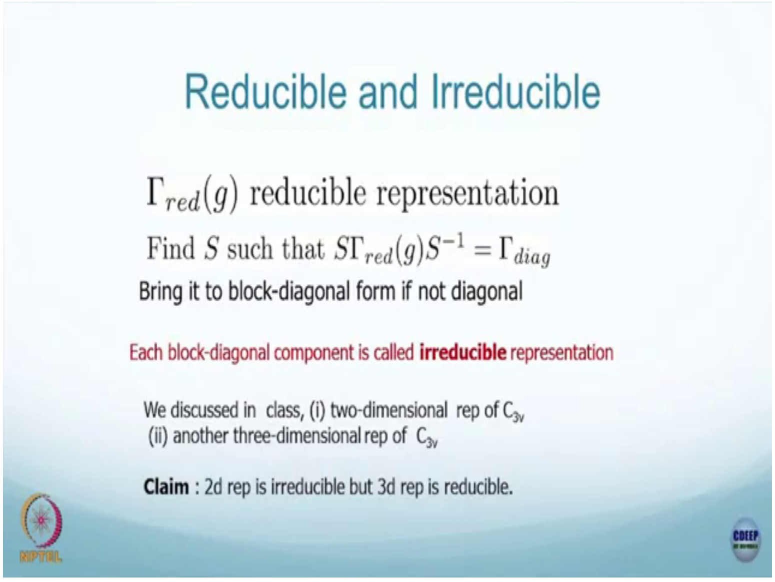

So, I am repeating again here gamma reducible by that I mean if you find an S for all g the same S which brings it to either diagonal form or it could also be block diagonal form then each by a diagonal component or a block diagonal component is called irreducible representation. So, we have discussed 2 dimensional representation for C 3v and another 3 dimensional representation for C 3v by that I mean these this is another representation for the same group C 3v.

Because 1, 2, 3 can be taken to be the C 3 element by isomorphisms right. C 3v is isomorphic to the symmetric group. So, this is another representation. This one is reducible whereas, this one is irreducible ok. So, 2d is reducible and 3d is 2d is irreducible two-dimensional representation is what I mean by that 2 cross 2 matrix representation I am writing a short form as 2d rep is irreducible you cannot further break it, but 3d representation the 3 cross 3 matrix representation for the same group it is reducible. You cannot find an S which brings into block diagonal ok.

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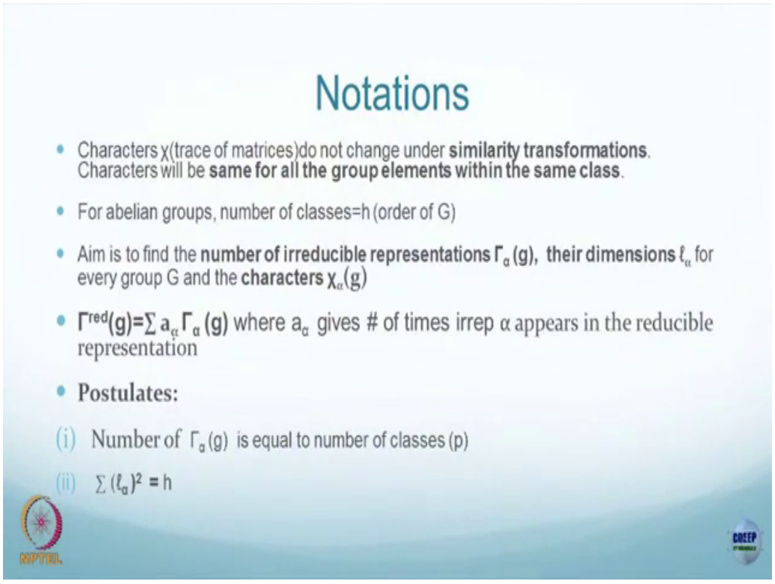

So, now, let me put in the notations. Characters chi the notation chi denotes trace of matrices do not change under similarity transformation. Characters will be same for all the group elements within the same conjugacy class. For abelian groups number of classes is also number of elements calling using the letter h to denote the order of the group g ok. What is our aim? Aim is to find the number of irreducible representation.

This set this list is it only 2 irreducible representations or more I need to find that. I need to also say that this one was a 1 cross 1 matrix this one is a 2 cross 2 matrix representation and so on. That information also I have to give. I am going to use the notation l alpha; gamma alpha is for this superscript 1 I could call it as alpha ok.

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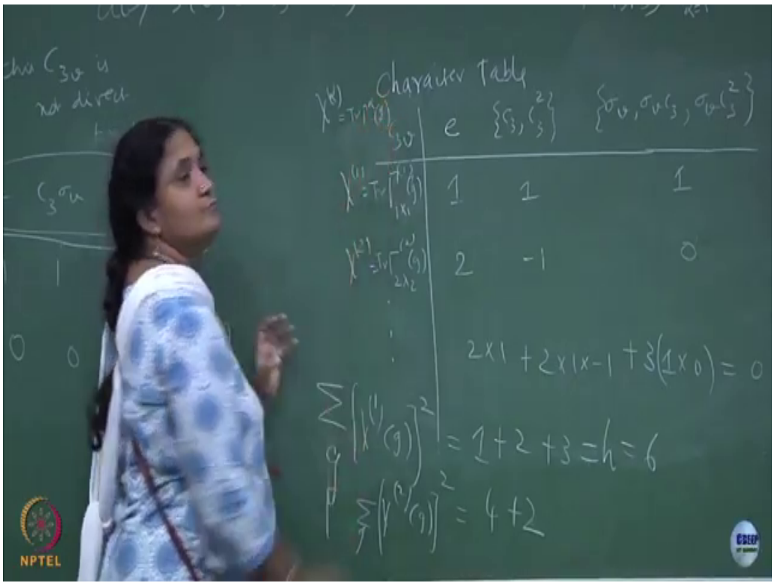

So, I can use an alpha here which is 1 and this one is 2 ok. So, this is the notation I am using.

Student: Take a value you will scale let us say chi of 1 chi of 2. These are the different scale factor 1, 2. So, if we get a scale factor let us say chi of 1 is irreducible then any value less than that which is n minus 1, n minus 2 they would also be irreducible because we would construct a matrix such that.

See thing is that is a good point. See suppose I give you a 3 cross 3 matrix. Suppose I given a 2 cross 2 matrix which is which can be brought into a completely diagonal form then I may say that this like this particular representation occurs twice or suppose I find another one dimensional representation I could say one of them another one of them. So, these kinds of questions can also happen in the lower dimensional matrix.

Student: What I see from this is in the lower dimensional.

Let me do some examples and then you come back to this question right. Now it is not you are thinking that if I have a 4 cross 4.

Student: Because what I see here is.

I am only saying the list is not only 2 cross 2, the list can continue. It can be a 3 cross 3 it can be a 4 cross 4 5 cross 5.

Student: Ok.

And given a 3 cross 3 I have already argued that it can come as a linear, it can be a reducible into blocks.

Student: What I see from the kinds of element I see the kind of elements we have constructed for C 3v for a 3 by 3 matrix they were correct, but what we have constructed for a 1 by 1 we have released relaxed a lot of constraints. So, we have got all one.

I agree.

Student: So, we have irreducible representation at order 2 then we could always have an irreducible (Refer time: 13:17) for less than that.

In fact, you will find 1 cross 1 there is non trivial representations also. I will come back to it. In this case you do see that you have 1 cross 1 non trivial representation. It is not that you do not have a non trivial representation. I am just trying to tell you what is the aim what is the notation and we will come back to this (Refer time: 13:36) back ok. So, aim is to find this list of a reducible representation this column and we also need to know the irreducible representation is 1 cross 1 which is what I call it as l alpha.

L1 is 1 here. l2 is 2 here. It is a 2 cross 2 matrix. We need to find what are the dimensions of those irreducible matrix representations and the corresponding characters ok. So, these are the things which we need to determine then only the character table will become complete. What I have done here is I took some simple examples acting on x y z basis or permutation of 3 objects and gave you some matrix representations and I said some matrix representations cannot be further broken into block diagonal or diagonal form, some matrices can be broken as a data you have not proved it for you.

And then I said you can start constructing a character table and the only thing is the character table here will be always we see that they are all 1 cross 1 non trivial ones on the both second column second row and third row first row is a trivial which I call it as a unit represent this is what you have seen and now I am going to get to the definition.

Student: Lot of elements have the same character value, but they belong to different conjugacy classes why is that.

Thats right. Which?

Student: C 2v table.

Different conjugacy. That that is just an accident getting a one identity element will satisfy any multiplication table.

Student: Yeah.

So, that is like a trivial one, but with 1 cross 1 matrices you can only play around with one and minus one. So, this is just by an accident ok.

Student: So, is this representation correct I mean if (Refer time: 15:34).

No no. It is correct it is what we have done explicitly right.

Student: Ok.

So, we wrote this in the xyz basis. You cannot say that the representation is wrong, but we will come to formal definition and see whether they all fit into those definitions. And I also tried to tell you that if you had a reducible representation you will find an S and break it up into block diagonal form and those block diagonal form some of the irreducible components can occur more than once and you call that as a multiplicity factor or a alpha. A alpha is a number of times an irreducible representation which I am calling it by the symbol alpha.

So, this one is chi 1, this is chi 2. Alpha is the notation I am going to use and you can write any reducible representation as some kind of a block diagonal pieces ok. So, this one can be written in block diagonal pieces and those block diagonal pieces is what I am writing it as in this former rotation. Postulates I am going to now postulate blindly ok. This can be proven, but I am not going to prove in this course ok.

Anybody is interested can take a look at (Refer time: 15:57) and you can prove it. Number of irreps number of irreps irreducible representations is always equal to number of classes ok. So, how many classes are here. It is 1, 2 and 3 ok. I will find only 3 irreps that is it ok.

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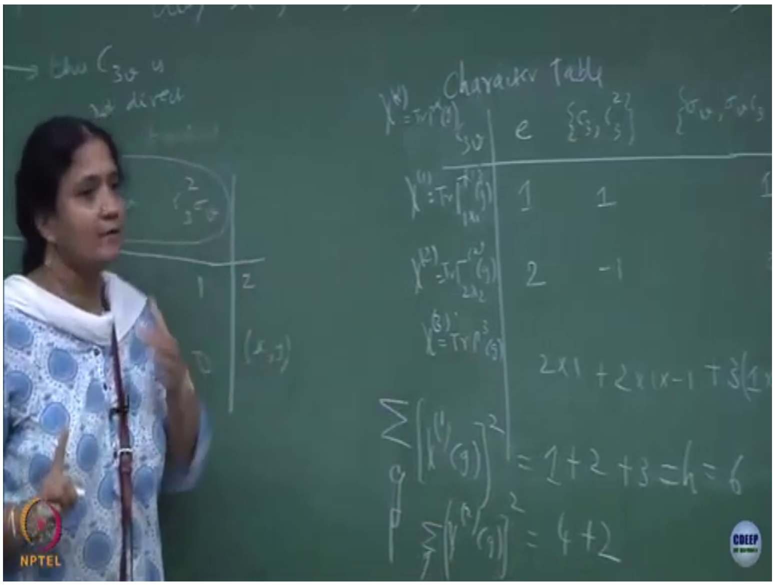

So, let me write it here. There will be a third irrep trace of gamma 3. I do not know what is the dimension here. We will find that out. There be a third irreps that will freeze my character table. So, number of irreps is equal to number of classes which I am denoting the number of classes as p. So, in C 3v the number of classes p is equal to 3 in C 2v the number of classes is also the order of the group it is 4 ok. Then I am also saying that there is a condition on this dimensions of these matrices. So, this 1 cross 1 matrix, 2 cross 2 and I do not know what this is ok.

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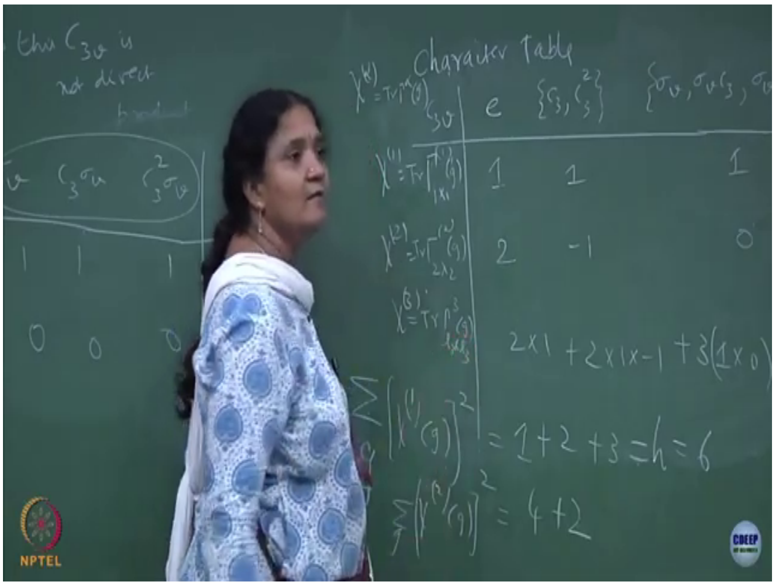

So, let me call it as l3 cross l3. l2 is 2. l1 is 1 cross 1 ok.

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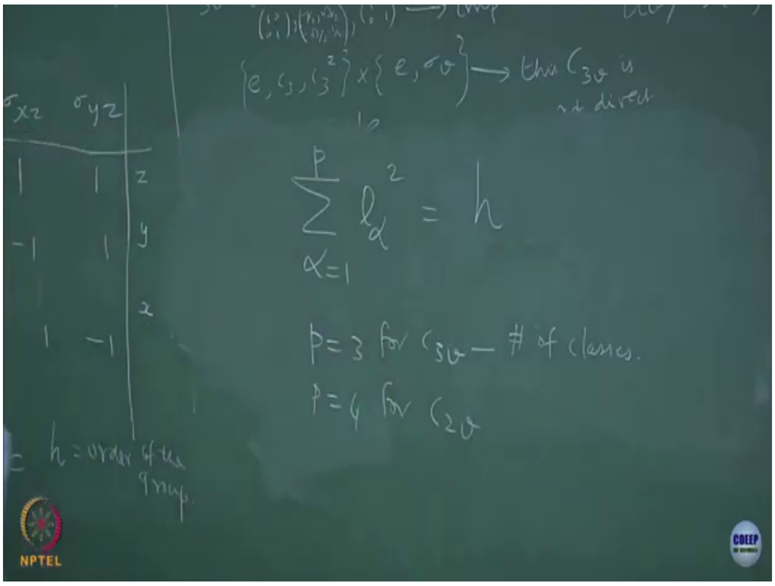

So, I want to put the condition here that is another postulate. May be this I can rub. Summation over alpha l alpha squared is equal to order of the group ok. Remember that l1, l2 and we also said alpha has to be the number of irreps has to be equal to the number of classes p is the number of classes for the group. So, p is 3 for C 3v and p is 4 for C 2v. This is the definition of number of classes. So, in general the number of irreps alpha rep 1, rep 2, rep 3 is denoted by alpha.

The number you can have is number which you have here. It is always a exact square kind of a table. Is that clear? So, here what will it be. There should be.

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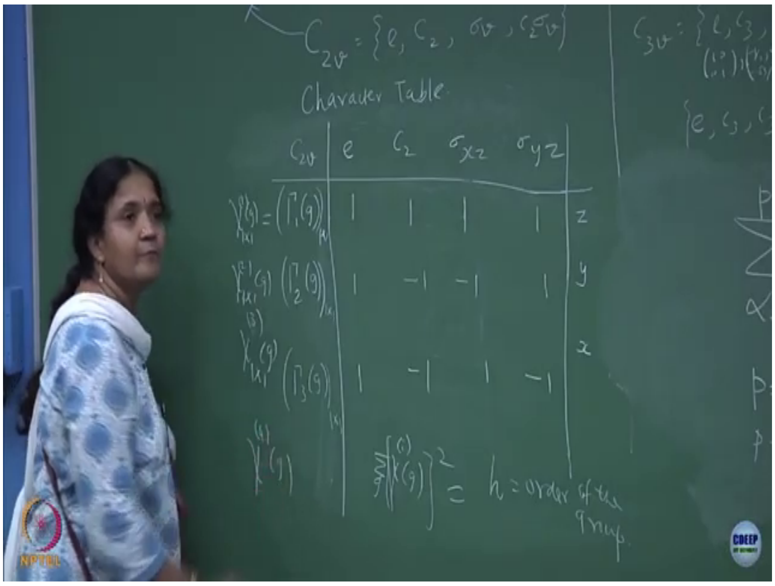

One more you agree. 1, 2, 3, 4 classes are there. So, I should find. I do not know whether it is 1 cross 1. We will come to it, but there is a fourth one because of this argument that alpha has to run from 1, 2. And now I am putting another condition that l alpha squared that is the dimensions of the matrices of every irrep the square of the dimension should add up to give me the order of the group. So, tell me what are the possibilities for C 3v.

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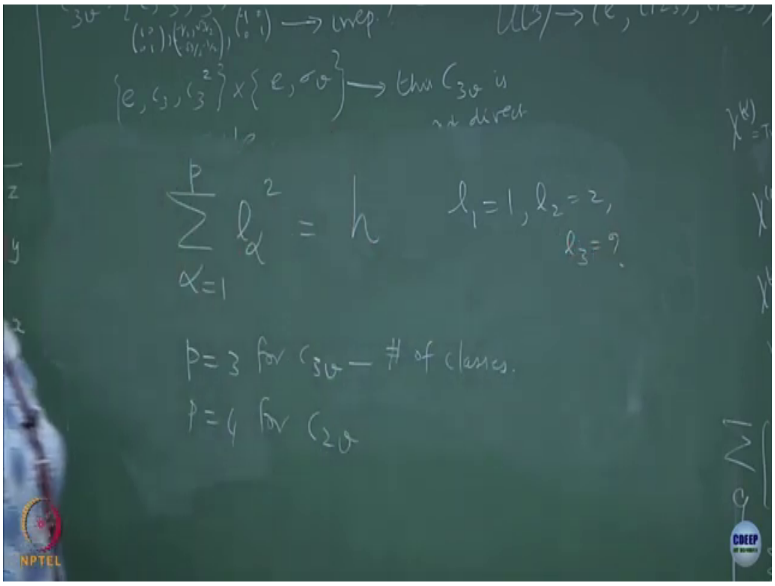

L1 is 1 we have found 1, l2 is 2. What can l3 be which satisfies this condition. All of them are integers is there in more than one solution.

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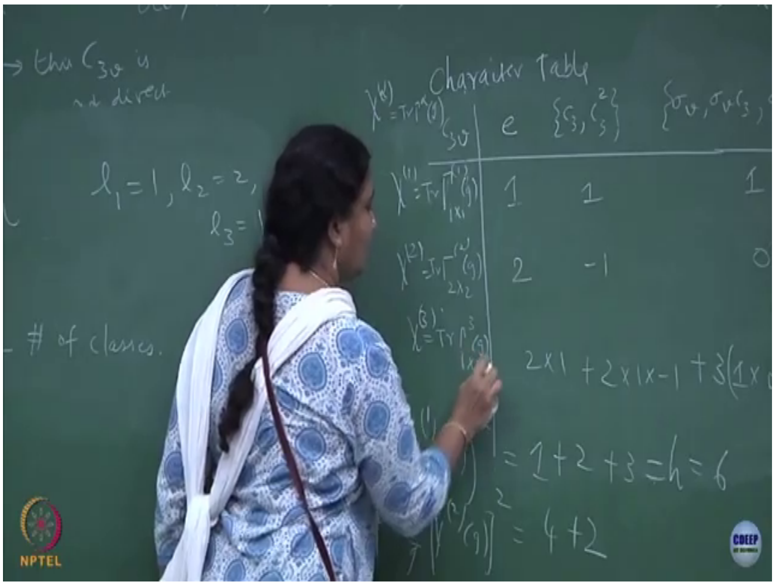

It has to be only one possibility which is 1. So, this has to be a again a 1 cross 1 matrix. I have not written the character for it, but we will come back to the writing the character. Now you can do something more if it is a 1 cross 1 matrix.

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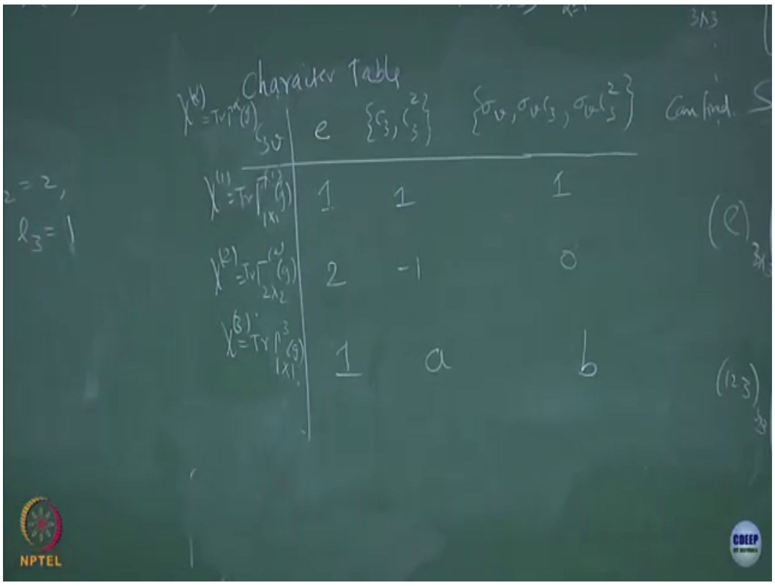

Identity element is 1 always. 2 cross 2 matrix there are traces 2. So, identity element character you can always write. What about these 2 elements let me call it as a and b ok. How do you fix that a and b? It is the next question. That is where you get into that great orthogonality theorem.

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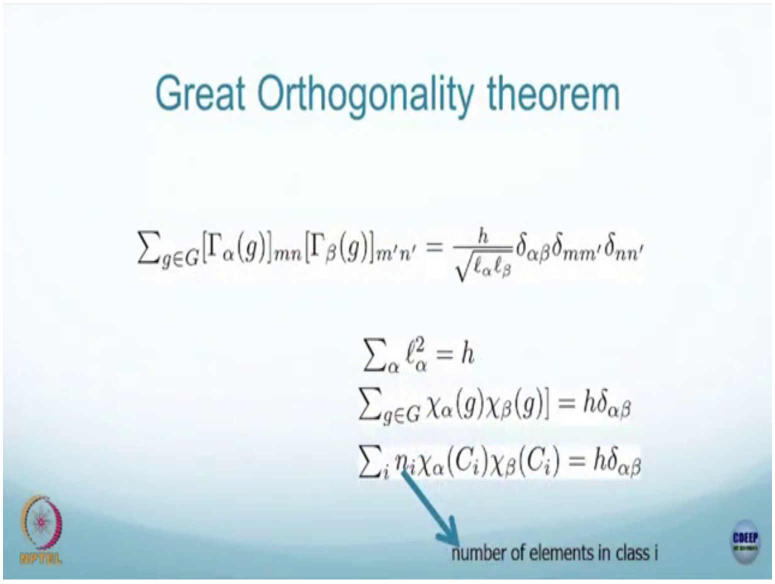

So, the great orthogonality theorem involves either the matrices with matrix elements which you can multiply and write it in terms of that it should be alpha and beta have to be same that is what I said between 2 different rows alpha is different it has to be orthogonal right. The delta alpha beta takes care of orthogonality between row 1 and row 2, row 1 and row 3, row 2 and row 3 and so on and l alpha is the dimension of an irrep alpha, l beta is the dimension of the irrep beta and this condition I have already said in the previous slide about the postulate.

And now you have a condition in terms of characters which is what you verified. You multiplied these 2 and it was orthogonal. You can square this and sum it up and that adds up to be the last equation ok. So, what are the conditions I have given you now? This is 1.

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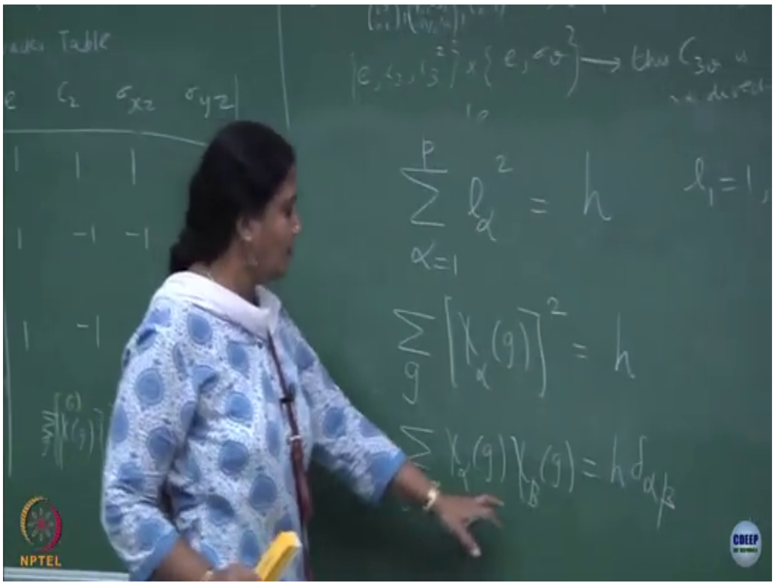

Summation over g. For a particular row alpha denotes the particular row that would be equal to. So, this can be fancily written as summation over g, chi alpha of g, chi beta of g. This takes care of orthogonality between rows.

And if you put the same row the square should be equal to the order of the element which is what you checked right. We check for this row and this row that they added up to give you the answer which was. Is that clear? Equivalently instead of writing it as summation over all the elements you could write it in terms of summation over classes.

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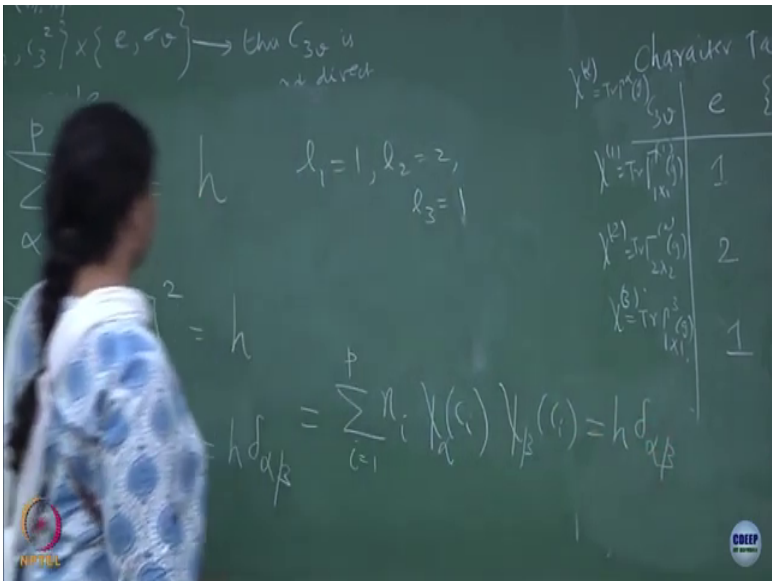

Classes I will denote it by i, number of elements in the classes ok. The classes is running from 1 to p; p in the C 3v is number of elements in the class and then you can write it as a character for the class alone right. You can write it for the character in the class Ci per candidate in the class Ci for an irrep alpha that will also be ok. Both are equivalent. Here I sum up over all the elements I just put because the characters are same for every within a class I can sum up over the class and put the number of elements ok.

So, let me stop here.