Group Theory Methods in Physics
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Lecture – 16
Matrix Representation of Groups – II

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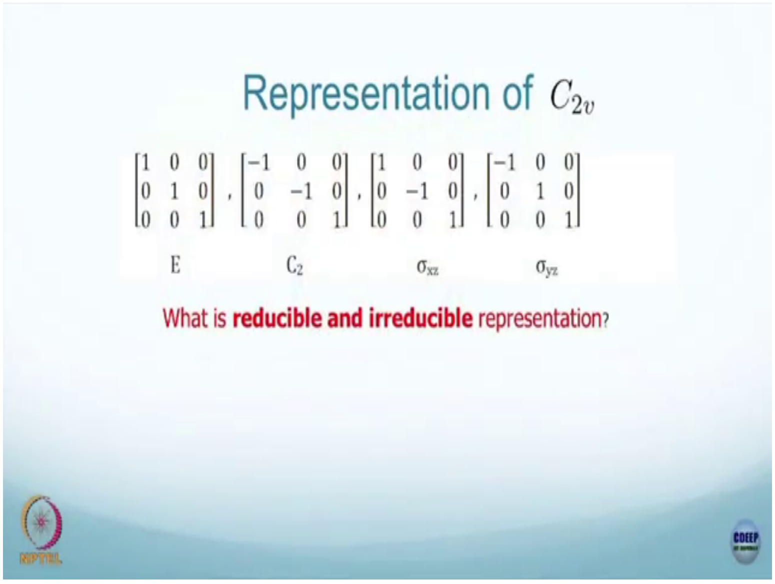

So, what is C 2 v? C 2 v has symmetry which involves C 2, C 2 v has elements identity C 2 then C 2 then sigma v and C 2 sigma right. Essentially it is made of E and C 2 right and you take a product with sigma v with that right, you take it as a product with E and sigma. So, this product also turns out to be a direct product in the context of C 2.

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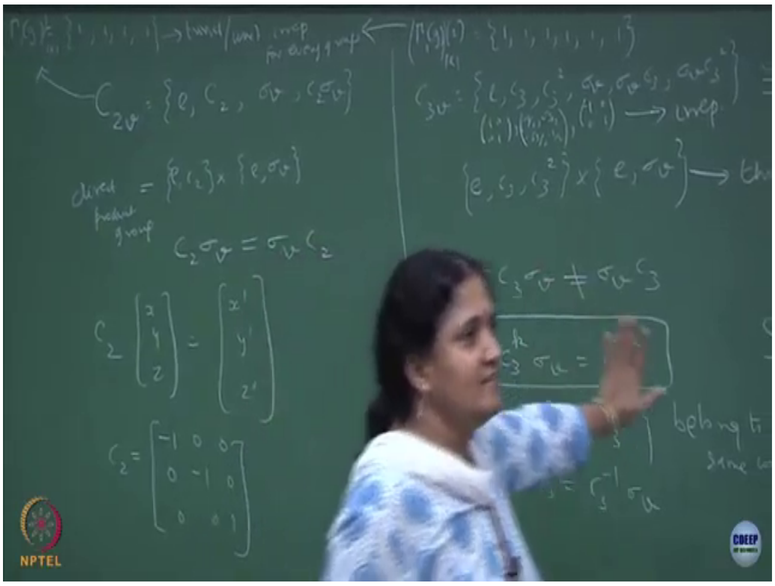

Why? You can show that whether you write C 2 sigma v you can show that it is same as sigma v C 2 clear; v is the vertical mirror sure all of you have synced the notation following the earlier lectures. Absentees or miss some lectures you should please go back look at the previous lectures otherwise the notations will be continuous like whatever I have followed in the previous lecture will continue ok. I am not going to specifically again define the notation.

So, the vertical plane, mirror, the operation is commuting. So, this commuting property will imply that it is a direct product ok. So, this is a direct product group, this will not continue when you do it for C 3 v; this also I spend lot of time in the class still I see lot of errors in the paper.

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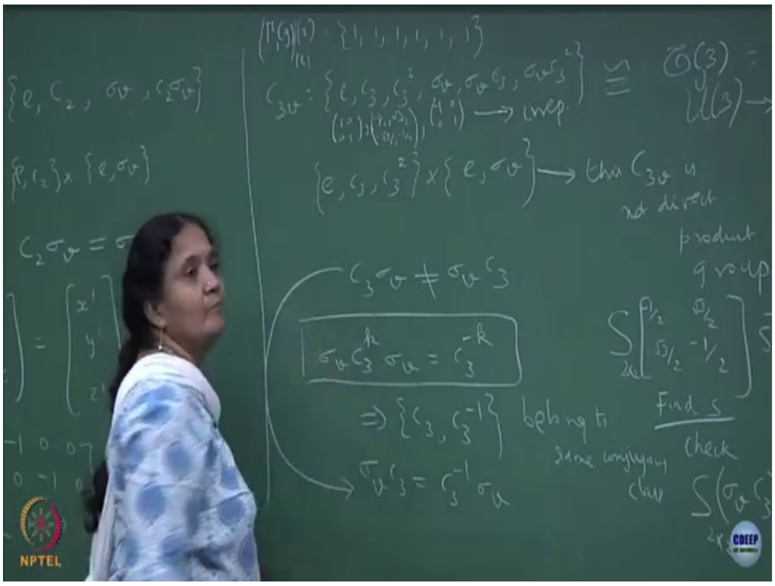

So, C 3 v what are the elements? Identity C 3 C 3 squared sigma v sigma v C 3 sigma v C 3 square ok. You can write the elements as if you are doing e C 3 and C 3 square you can multiply with e and sigma v, but you can show that C 3 sigma v it is not equal to sigma v C 3 ok. So, which means the C 3 v is not a direct product clear. So, if C 3 sigma v is same as sigma v C 3 you would have got C 3 equal to C 3 is like a self conjugate element right you could have.

But, now you also know because this is a bilateral axis C 3 is a bilateral axis whenever you have a mirror containing the axis; you know that C 3 k sigma v sigma v will give you C 3 minus k. So, also implies C 3 and C 3 inverse which is nothing, but C 3 squared belong to the same conjugacy class. So, this is true for a principal axis which has a vertical mirror plane, then it becomes the bilateral and you can use the condition that the conjugation sigma v inverse is same as sigma v.

The conjugation will take the element to the inverse of the element; so, that is why C 3 and C 3 inverse belongs to the same conjugacy class; using this also you can show sigma v C 3 is ok. So, this is another way of validating that it is not equal to this is not equal to this. Once it is not equal order matters then this direct product which I am writing like I can find the elements by using this ok.

That set will be a complete set whichever order you take, but the claim to be a direct product group is that you have to get the same element whether you do C 3 with sigma v or sigma v with C 3 ok; that is not happening, it is not a direct product group. In fact, you can show this to be a semi direct product, this also we discussed. How did we discuss? We said that this is like a symmetric group right of degree 3, it is isomorphic to it.

This group is isomorphic to this and then in that symmetric group you can have alternating group which is a subgroup, this is a subgroup of the alternating. Alternating group is a subgroup of the symmetric group and the elements of this are nothing, but the 3 cycle which has even permutations right, am I right?

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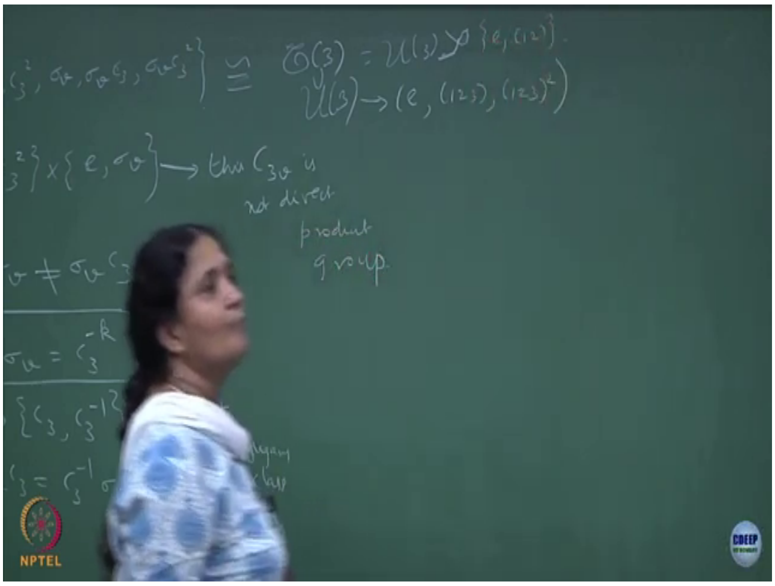

If you remember the elements here will be e 1 2 3 1 2 3 squared and that is isomorphic to e C 3 C 3. So, you can show this to be a semi direct product of, what is the notation? This is the way we write and times the ok. So, the 3 elements will get multiplied with this and that is a symmetry, this also we discussed ok. So, the next thing which we were trying to write is a matrix representation.

I said let us do a matrix representation where we take x y z and operate C 2 on this; you will get x prime y prime z prime and you can write the 3 dimensional representation; so, that is being displayed on the screen. So, C 2 will be; C 2 will be what? It will be taking x 2 minus x y 2 minus y because it is 180 degree rotation z to z. So, you will have minus 1 0 0 that will be your C 2.

So, as shown in the screen I have put it in identity element is identity looking at the how it operates on your x y z components of your position vector C 2 will be the diagonal elements x component will be minus x, y component will become minus y and z will not change. If you do the mirror just the mirror, if you take the mirror to be in the x z plane which components will not change sign?.

A mirror in an exit plane will not change x component or the z component, but the y component will changes on clear. And, similarly C 2 sigma v can be written as if it is a mirror in the y z plane both are equivalent. So, the mirror in the y z plane will change the x component alone, but not the y component and z component, is this clear? How I have written a matrix representation for the C t C to v group I have tried to find the same.

These matrices when you multiply it will satisfy the same multiplication table ok, matrix multiplication if you do the multiplication table of C 2 we will be satisfied by these 3 3 cross 3 matrices clear ok. So, now what you see here is that these are x you know already diagonally ok; this is what you see here, but if you had done a 120 degree rotation, that also we did in class right.

So, see if you want to write C 3 element it will be cos 120. What is cos 120? So, this will be the matrix which you will have to write right cos 120 is sin 30 right sin 30 is half and then.

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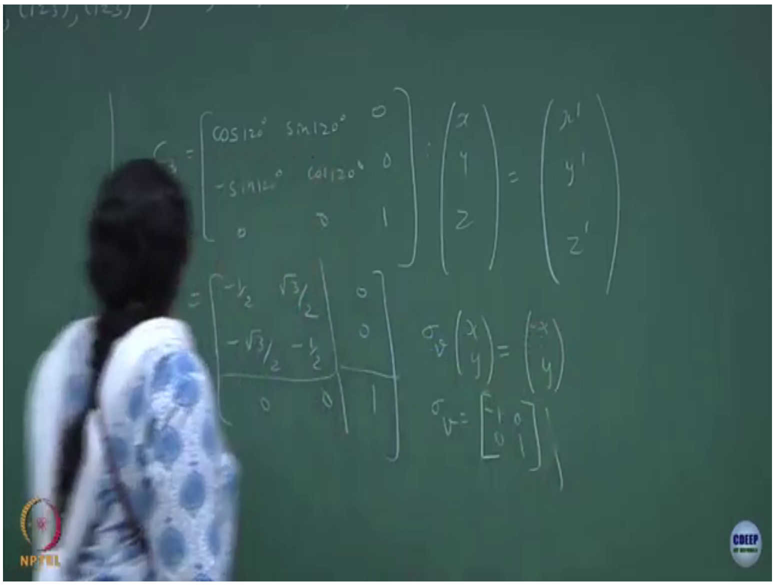

Student: Minus half.

Minus half this one will be root 3 by 2 ok. So, it is already in a completely diagonal fashion there. But here you see a something where it is a block diagonal fashion ok. So, that is the first observation, C to v all the matrix for all the elements there are 4 elements in C to v. All the four elements are diagonals and C 2 v is also an abelian group or a non-abelian group? The direct product groups each one is abelian, the product is also abelian.

So, it is an abelian group, C 3 v is isomorphic to the symmetry group of degree 3 which is a semi direct product and we have seen many times that it is not an abelian group ok. So, I am just taking these two simple examples whatever I am doing for the simple examples you can extrapolate for other groups. So, it is completely diagonal. And, we saw this definition of what is reducible and irreducible representation is what is my aim today ok.

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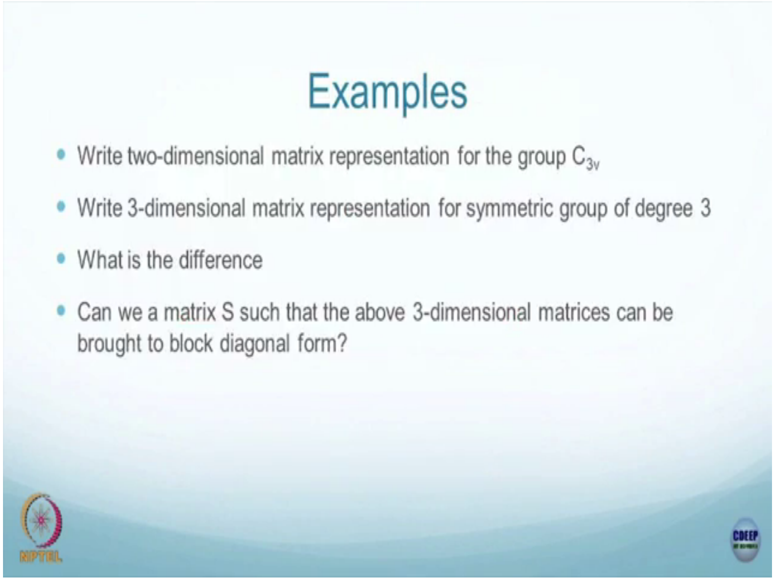

So, instead of working with x y z basis; so, this one is acting on x y z giving you x prime y prime z prime and you see that the z component is never changing you could also work with these matrices acting on the x y plane ok. So, you can do things like what is the C 3 maxes which takes x y to x prime y prime that is nothing, but this 2 by 2 block. If you just take the 2 by 2 block you can say they are 2 dimensional matrix representation for the same group, which group?.

Symmetric group of degree 3 or C 3 v, this is 1 element I have written; you can write it for sigma v also. You can write it with C 3 times sigma v, you can start writing those matrices right. How will you write? C 3 sigma v is what? So, let us take the sigma v to be the y z plane for example, if suppose I take the y z plane then this is supposed to give me and y right and then write the matrix form for this.

So, the matrix form for this will be x goes to minus x y y remains same and this is a 2 by 2 matrix sorry this is a 2 by 2 matrix which you have to write. And, this is a 2 dimensional representation for the sigma v and if you want to find C 3 times sigma v you just multiply.

You can do the multiplication and write what happens, similarly you can write C 3 square times sigma v and you can write all the matrix elements; you can write the 2 cross 2 matrix elements. So, this will be 1 0 0 1 this will be minus 1 0 0 1, but then there are elements which are minus half root 3 by 2 minus root 3 by 2 minus half for this. And, that will be the inverse element which is like rotating by minus 120 degrees right.

So, you can write all the matrices which acts on the x y coordinates alone and it gives you another representation which is a 2 dimensional representation for the C 3 v group correct. Why it is a representation? The product of this with this will give me identity, I can see that the multiplication table is satisfied by this matrices process ok.

So, in that sense it is a 2D representation. Can I say whether this 2D representation it is reducible or irreducible that is the next question ok. What is the definition of a given matrix if I give you a matrix you have to see whether it is reducible or irreducible. First of all you have to find; so, can we find a matrix S such that you can further break it in two. So, if it is 2 dimension I should see whether there is a matrixes S.

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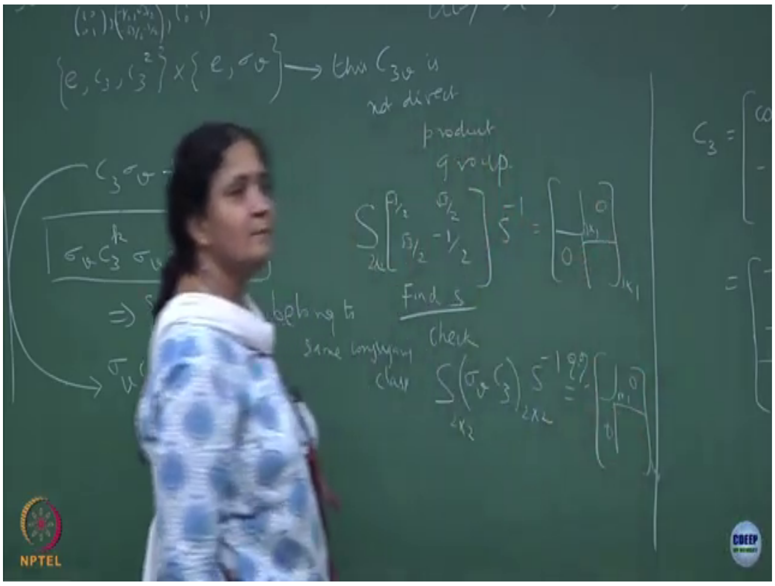

Let us take half I think this plus right S inverse it is 2 dimension. Can you break it up into a 1 cross 1 under 1 cross?. Suppose you find the matrix S which breaks it into 1 cross 1 and 1 cross 1 what will happen to the identity element? Identity element will not be touched right, identity element is already in a complete diagonal form. So, this S which if suppose it diagonalizes it, it will automatically diagonalize this as well as probably this right, it is not really going to matter.

What can matter is the other elements, can all the elements be diagonalized by the same S matrix, the same S elements?. If suppose you find S; so, find S then check S sigma v C 3 I am not writing the matrix, you write the 2 cross 2 matrix here. Whatever 2 cross 2 matrix you find here you put the same matrix S inverse question mark, can you make it into a diagonal form? By this I mean the off diagonals are all 0s ok, that is the meaning of writing it in this shorthand notation.

Student: (Refer Time: 18:32).

In this case of 2 cross 2 there is nothing called block diagonal, but in general if you had a 3 cross 3 or a 4 cross 4 you could break it up into sub pieces ok. So, in this particular example of a 2 cross 2 matrices can you find a S matrix with simultaneously the similarity transmission with this S matrix on all the elements should give me diagonal matrix, can you achieve it is the question. I am telling you the answer, the answer is no ok.

One is you can do it by Brute force and check it out, I am trying to tell you that simultaneously making all the elements to be diagonal starting with this 2 dimensional representation it is not possible ok. So, one is to do it by Brute force, another one is to try to understand from a set of Lemmas theorems and postulates. But, as a you know to get a feel for it please try it out, I would ask you to I would strongly urge you to try out doing this; find an S first; this diagonalization is trivial for you, you can do it.

After you do that and find the S, I am sure all of you have done diagonalization of matrices. After you find that S try to see whether that S when you employ on the sigma 3 v or sigma v C 3 or sigma v C 3 inverse check whether what happens; are the 2 elements are trivial identity and the other one.

But, non-trivial elements which involves off diagonal elements is this same S which you are going to find going to diagonalize it. If you can find that out then you can say that the 2 dimensional representations, suppose you find an S so, let me write it.

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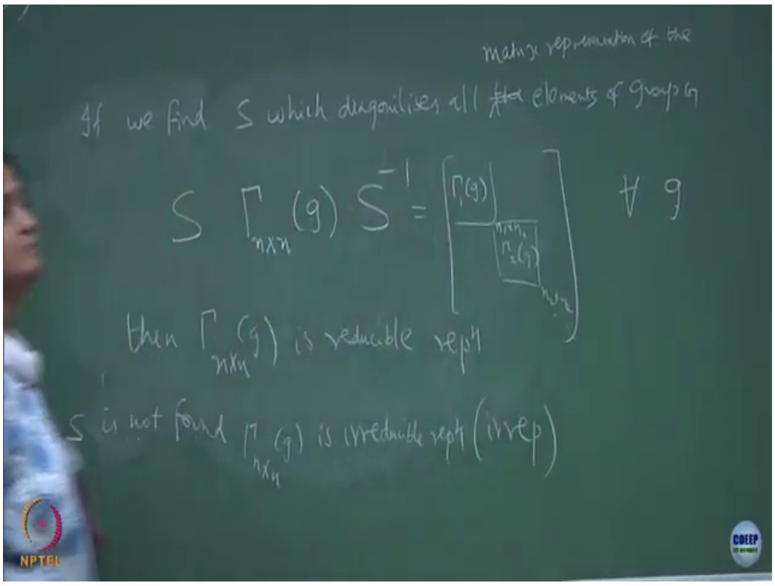

If we find S which diagnosis all the elements of group g ok, all the matrix level all the representing. So, let me write it this way representation of the elements of group g, suppose you find an S and if you take that matrix representation. So, let me denote that matrix representation are some n cross n matrix for a group element g ok. Suppose you find S; S inverse, if you can break this into a piece which is n 1 cross n 1 n 2 plus n 2 and so on such that the n 1 plus n 2 should add up to n ok, n cross n matrix for every group element ok.

So, let me call this as so, gamma 1 for the same g gamma 2 for the same g and so on. So, these are lower dimensional matrices. So, this you should do it for all g, this is not dependent on g should be the same as on all g. If we find an S which satisfies this then gamma n cross gamma n of g is reducible.

Student: (Refer Time: 23:20) that represent can reduce (Refer Time: 23:21) representation (Refer Time: 23:24).

No given a representation I am not saying that this gum.

Student: (Refer Time: 23:30) for other representations you can find that (Refer Time: 23:32).

So, I am just saying in an abstract fashion take an n cross n matrix representation, take that representation matrix entries also and then work it out. If you take another n cross n with some other entries that is a different story ok. So, I am not saying that.

Student: But, that is representing independent if you can find the S (Refer Time: 23:58) for one representation then you can find for n representation (Refer Time: 24:04).

It does not depend on the elements of the group; it does not depend on this, but it depends on what is the entries of this matrix ok. So, in some simple cases there is only one unique matrix which you can write, but there are some complicated situation where there will be two different 3 cross 3 matrices which you can write. I will come to it, some examples I will tell you, but right now you take that just like this example where I have taken a 2 cross 2 matrix or a 3 cross 3 matrix you can try to see whether you can find an S which satisfies this.

Student: (Refer Time: 24:50) if it is (Refer Time: 24:53).

That is another way of arguing it for simple cases.

Student: Higher representation if possible.

Higher representations also it is possible to argue this way good. So, as he is already saying, but that is not really a postulate because I want to do it in general for a n cross n matrix; what are these blocks in which it can break that you cannot generally answer from your just an abelian or non-abelian ok. So, the question that it is abelian or non-abelian will not really help me, but what I want to drive upon you is that there is going to be some kind of an orthogonality theorem which is called great orthogonality theorem which plays a crucial role in saying, that we can find an S not that I am going to find the S.

But, that theorem tells you that yes I can find an S which does this job in which case I can say that this one is a reducible representation. If I cannot find an S, if I do not find an S, if S is not found then this element itself what does it mean? If you do not find an S I cannot break it up like this which means this one will itself be a irreducible representation ok. I am using the shorthand notation refn, but the representation we will start using further I will shorten this and call it as irrep ok.

So, I am only claiming to you that this one is an irrep, this 2 dimensional matrix representation is an irrep or in other words I cannot find an S which will block diagonal has no meaning in a 2 dimension; it should become completely diagonal and one of the ways to see it is that it is non-abelian. You can show that you cannot simultaneously diagonalize all the elements of a non-abelian group that is another way of arguing so hm.

Student: Maam abelian we can (Refer Time: 27:23).

Good, but that is what is happening in your if you see that earlier representation that is a good point. So, you do see that abelian is trivially a block diagonal form and you can pick up the irreducible blocks out of it ok.

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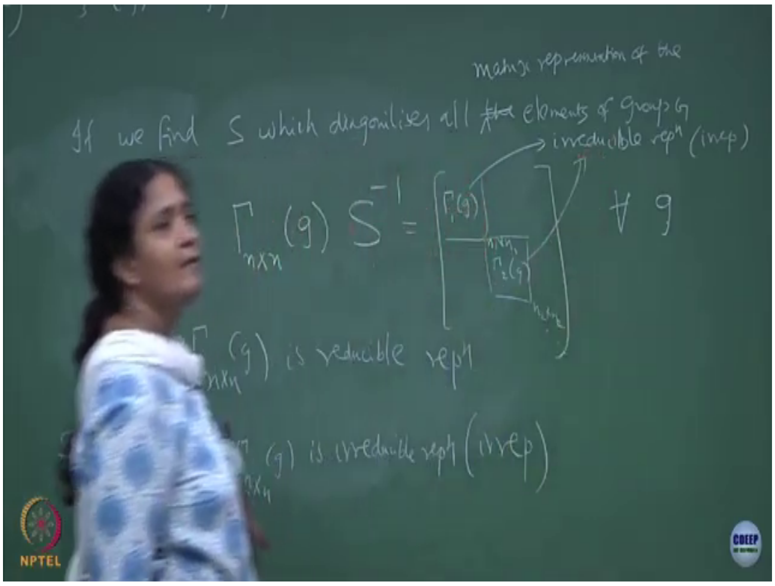

So, here let me also say that if this these are what we call it as a irreducible blocks representation or irreps. So, gamma 1 gamma 2 are all different each of these blocks block diagonal are called the irreducible representation for the same group. What I mean by that is that you can play around with gamma 1 of g alone and write the group that is what happens in C 2 v, in C 2 v I can say that 1 minus 1 1 and minus 1 ok.

So, let me write that here it is a 1 cross 1 matrix ok. So, this is a 1 cross 1 matrix with the elements which is 1 1 1 and 1, it will trivially satisfy any group actually ok. In fact, here for the C 3 v you can still take it on the z axis e C 3 C 3 squared sigma v sigma v C 3 sigma v 3 square you can apply it on the z axis. If you do it on the z axis what happens?

Identity nothing, C 3 also is the z axis itself. So, here also you can have a representation which is acting on 1 dimensional z coordinate correct. And, what will that element be? 1 1 1 1 1, it is a 1 cross 1 matrix right because it is acting only on the z axis.

Student: (Refer Time: 29:57).

For C 2 v there are only 4 elements right, this is for C 2 v, this is a matrix 1 cross 1 matrix 1 cross 1 matrix is just the number 1, it is like a trivial representation which is going to be there for every group ok. So, this is a trivial sometimes called unit representation irrep for every group ok. So, the number of elements is the order, in this case it is 4, this case it is it cases is 6 that representation is there.