Group Theory Methods in Physics
Prof. P. Ramadevi
Department of Physics
Indian Institute of Technology, Bombay

Lecture – 01
Introduction - I

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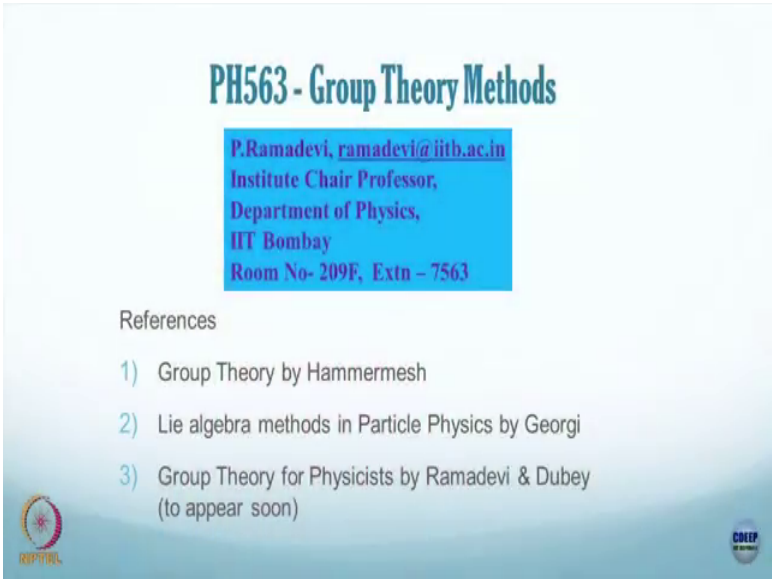

So, what is Group Theory? So, the main thing is Symmetries, ok.

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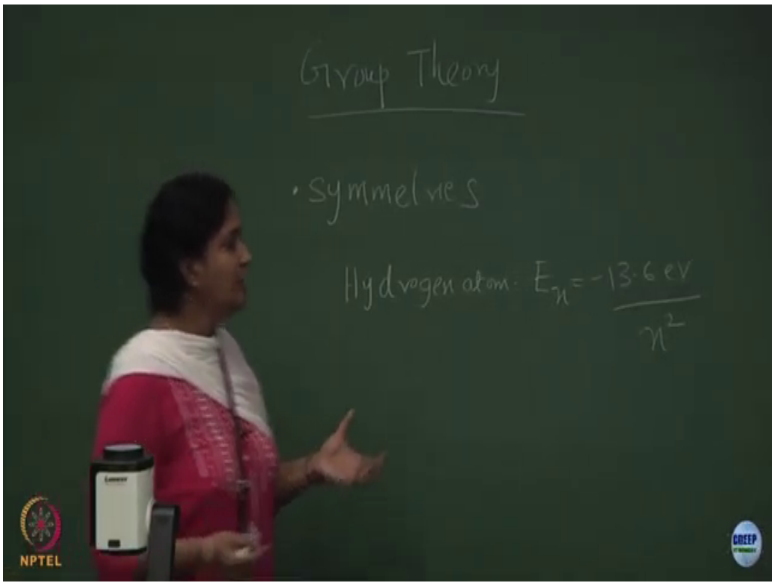

So, symmetries of pattern which you see in nature, that plays a crucial role in trying to get some information about a complex system, ok. So, you can you do not need to try and solve complicated equations to find the solution. Like for example, if you take hydrogen atom, I am sure everybody knows hydrogen atom, right. Hydrogen atom you all know that the energy is 13.6 ev divided by n square, right. So, the energy in the hydrogen atom, so they are quantized and how do you find the solutions for e n? You solve the Schrodinger equation, if you solve the Schrodinger equation, it is a second order differential equations and that gives you minus 13.6 ev divided by n square, right.

So, you have to solve that equation. And what is the symmetry available in that system. It has a spherical symmetry or the potential which you write for the hydrogen atom is going to be invariant under rotations.

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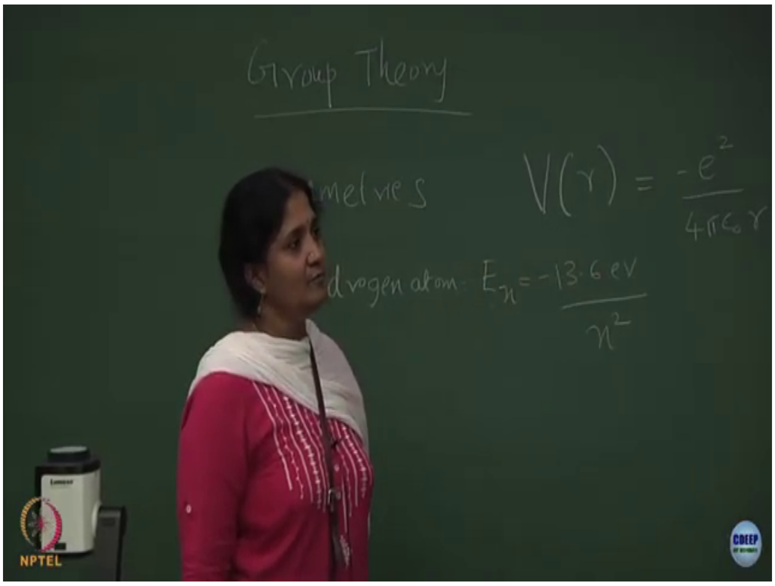

So, the potential energy which we write is only dependent on the radial distance, and the radial distance what is this one? r, r is the radial distance, radial distance between electron and proton looking at hydrogen atom, ok. You can take electron and proton to be point like or you take center of nucleus to the electron.

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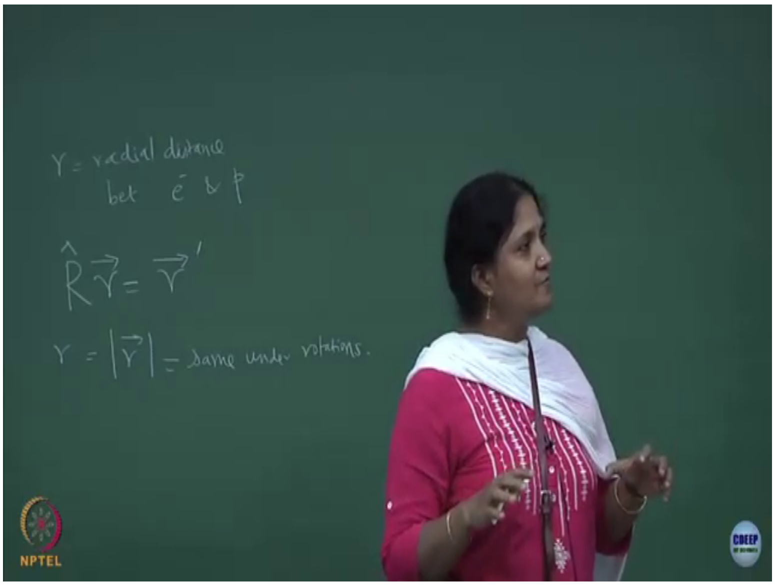

And if you do a rotation operation on an r vector, what happens? r changes to a new r prime vector, but magnitude of r which is r radial distance that is going to be same under rotation, is that right. Rotation does not alter your radial distance. It takes the same radial distance, but theta and phi will change in the spherical polar coordinates and r goes to r prime under rotations. This is the rotation output. I am just giving you a flavor of what group theory can do.

As a conventional quantum mechanics problem you applied in the Schrodinger equation putting this potential energy, try and solve it, right. So, you solve it. You know how difficult or how complex it is, it still mathematically solvable. Sometimes if you have complicated potential energy you may not be even able to solve it, right. Solving means finding the energy of that stationary states of the hydrogen atom, ok. But what we could do is exploit such symmetries you exploit such symmetries and then you get the answer without even solving the Schrodinger equation, ok.

So, this is the sometimes powerful elegant way of doing complex situations by exploiting symmetries of the system, ok; so, this is what is the theme of group theory. It is a tool. You know if you want to break open this table you may need a good tool which is a screw driver then you can easily remove it, part by part. Of course, I can come crudely and start breaking with whatever I have, then that is where you know you rigorously try and use all complicated mathematical equations.

So, these are two different ways of looking at things and that is why group theory has become very important. And this course is just to make you people appreciate such a tool. And there are limitations, you cannot go more and more saying that I will not learn the other methods, I will also tell you till what extent this tool will be useful, ok. So, sometimes suppose, I ask you whether a particle in the ground state, suppose let us take harmonic acid I am sure you all done harmonic oscillator, 1-d harmonic oscillator.

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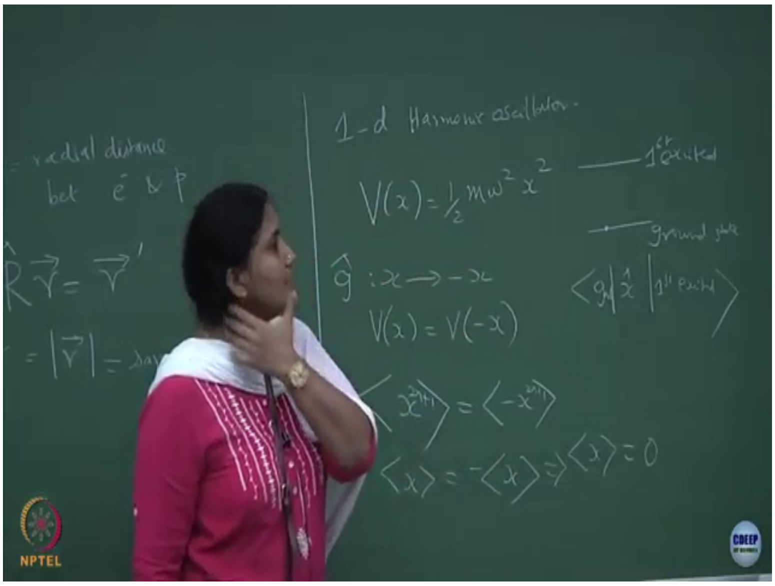

So, 1-d harmonic oscillator. What is the symmetry? Potential is x square. So, this has what symmetry? x goes to minus x, V of x equal to V of minus x, right in this harmonic oscillator, you solve the Schrodinger equation find a wave function, you can try and show that the expectation value of x. Yes.

Student: (Refer Time: 07:09).

How do you show that by using symmetries? Under this operation, so let me call this operation as some g operator. Under this operation my physics which is contained in all the expectation values should not change, ok, but under this operation I know wherever x is I have to replace it by minus x. This expectation value should be same as this expectation value, ok. So, what does it mean? I can pull out this minus x out because if you remember the wave functions when you write, if there is some sign here you can pull out sign out of that expectation value and you can show that expectation value is minus of expectation value, right which means expectation value of x is 0.

Here I did not do any calculation. I just exploited this symmetry, that this is the symmetry of this potential energy for the harmonic oscillator. In fact, even you can say for. What all powers you can say? All odd powers you can say that it will be 0. Even powers, can you say anything? Even powers you can say it is nonzero, but actual nonzero value you cannot predict. If you want to find the nonzero actual value you have to know the explicit wave function otherwise 0. So, it is like a process which will tell you that something is 0, something is nonzero, a process of jumping from the ground state to the first exciting state due to some dipole moment interaction is allowed or forbidden, ok.

So, suppose I have a ground state I have a first excited state, right. Ground state and the particle in the ground state which is undergoing which is subjected to this potential suppose I make it interact due to a dipole moment interaction which is like this x operator, dipole moment is 2 times x such an interaction can you get from a ground state an x operator to the first excited state. Is this going to be 0 or nonzero? Ok. Such questions can be 0 or nonzero can be predicted. You cannot find a nonzero values, ok. This is what is called as the selection rules.

What is a selection rules? You will say that to undergo this transition delta m should be plus or minus 1, some selection rules you would have seen. Those selection rules are consequences that is 0 or nonzero allowed or forbidden process a process of transition to go from here to here triggered by an external interaction is that allowed or not allowed, so, that much you can say. It is allowed, not allowed means value will be 0, but allowed what value we will not be able to predict, but you can say it is not allowed, clear.

So, these are the things where the power of group theory is really very powerful and I want you to appreciate this for that you may have to learn a little bit of those tools. I will take you through the path of what all is happening, ok. So, this is the theme of this course.

So, let me just go on to tell you what is my plan for this course. Group theory by Hammermesh is one book which talk extensively about discrete groups. Then the other group which I have been very impressed is one by George. Even though it is written for particle physics people I will try to give you a, you know the set of silent features how continuous groups have to be viewed. So, this will be one kind of you know text book which you put keep it at the background while you go through those lectures.

So, in the process of teaching this course for the last you know I have thought this 3 times in the institute with one of my former student Varun Dubey, I have been trying to get this book to come in print form. You can also you know take a look at this and I am trying to follow those writings which I have put it in this book in this lectures, ok.

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So, the syllabus and the plan discrete groups in the first half of the semester. So, essentially I will try to focus on cyclic group, permutation groups, point groups, then with matrices which you all play around how to focus on irreducible representations, great orthogonality theorem, character tables and some applications in solid state physics. So, this will be the theme for the first half of the semester.

And there will not be rigorous proves like a mathematician we will only try to give a statement and supplemented with examples and you know exercises. So, great orthogonality theorem will be only said as a statement, we will not prove it, ok. So, proves are not part of this course. The part of this course is to show how the tools can be applied to solving problems in physics, ok the group theory tools; is that fine.

The second half will be continuous groups. So, the simple example is like space translation, time translation, rotational symmetries which I have already indicated to you for hydrogen atom problem will lead us to slowly get the formal aspects of what the continuous groups are sometimes called as lie groups and the corresponding algebras which are lie algebras. So, I will do focus on SU 2, SU 2 is not very different from the angular momentum algebra which you have studied and rotation groups which you have studied and that sets the paths to go SU 3, ok. Once we understand SU 2 clearly you can follow how to go to SU 3 and higher groups, but we will focus on SU 3.

And in special theory of relativity you all done Lorentz transformations. So, as more applications we will also look at the group which is called as a Lorentz group and its applications, ok. In one of these applications I will try and explain how to get this e n as 1 over n square without solving Schrodinger equation. So, these are the motivations and aim which I have.

Of course, it is every lecture is connected to the previous lecture and please do the assignments, sometimes many of the problems will be for your own hands on experience and that will help you to grasp things in a neat fashion otherwise you know you may not be you may be completely not in sync with the lectures, ok.

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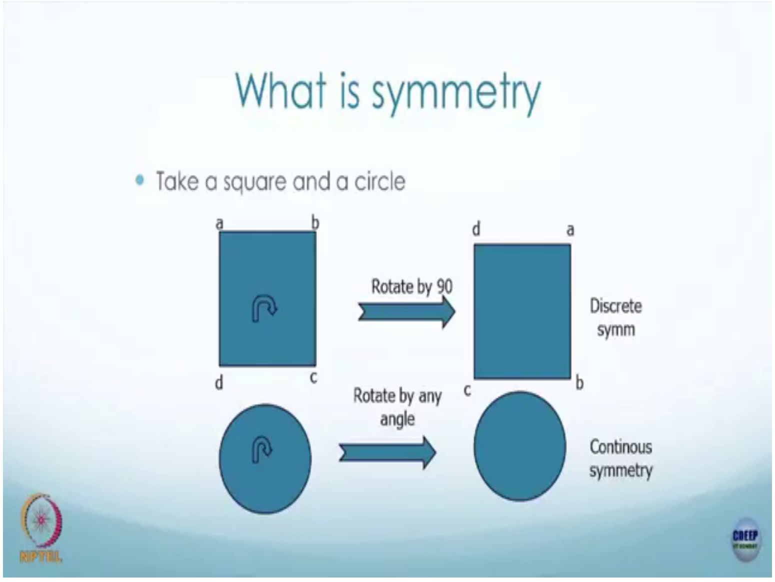

I just said discrete symmetry and continuous symmetry as a simple example I am showing that there is a square which has, so you have a square, so the square you all know, all of you know. You cannot do 35 degree rotation and make it look like as it was like the original configuration, ok.

The original configuration has 4 vertices and when you do 35 degree rotation the vertices points were all will all be distorted. So, the vertices will go into each other only if you do a 90 degree rotation, right. So, it is a discrete rotation. So, the symmetry of a square is 90 degrees. Suppose the square, I have put in some disturbance and it becomes a rectangle then what will happen? Will the symmetry be still 90 degrees?

Student: 180 degree.

180 degree; so, these are things which you can start thinking, you know any complicated polygon is given to you, what is the kind of symmetries you have, ok. So, basic rotation fundamental rotation has to be 90 degrees, of course you can do 180 degrees, you can do 270 degrees and finally, if you do 360 degrees it is like an identity rotation, ok.

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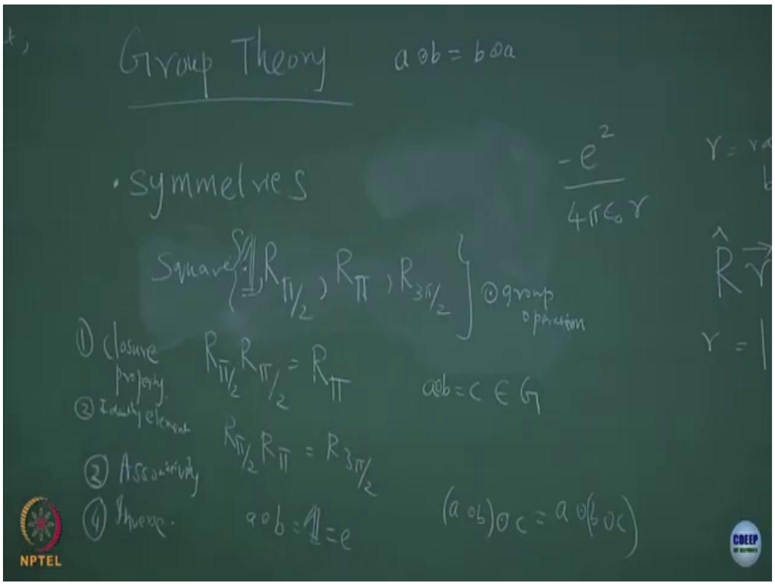

So, for a square you have rotation by 90 degrees, rotation by 180 degrees, rotation by 272 and you will also have an identity operation which you can call it as rotation by 360 degrees, ok. So, these are the 4 distinct operations. As operations identity is do nothing then rotation by 90 will change the vertices the way it is shown here for clockwise, and you can do 180 then the vertices will further shift as shown here, right. You see that d has gone from here to here, a has gone here, b has gone here, c has gone here, ok. And if you do one more get a distinct state, but it look like an original state that is why it is called as a symmetry, ok.

So, these 4 elements together this forms a set, ok. So, you have a set, set of 4 elements and you can introduce an operation which is the rotation operation as an operation. So, R pi by 2 if you do it twice two rotation by same will give you rotation by 180; is that right. Similarly, you can do R pi by 2 with R pi you can get R 3 pi by 2. So, what have I shown? A physical rotation by 90 degree is a group operation and if you do it amongst the set, ok, you can do it amongst the set. These rotations are going to give me an element which belongs to the same set, clear.

So, you have a set of elements and then you give you a group operation and then operation you do it twice know you can take one element from here, another element from here, you just multiply those two rotations ordinary multiplication. You get back an element which is still belonging to the set, ok. So, that is one. This is what we call it as a closure, closure property, ok.

So, group operation is just I would say that I put a group operation which is just ordinary rotations which I am doing here, ok. It could be sometimes if I write matrices, suppose I write matrices that also I am sure you all know. How do you write the matrices?

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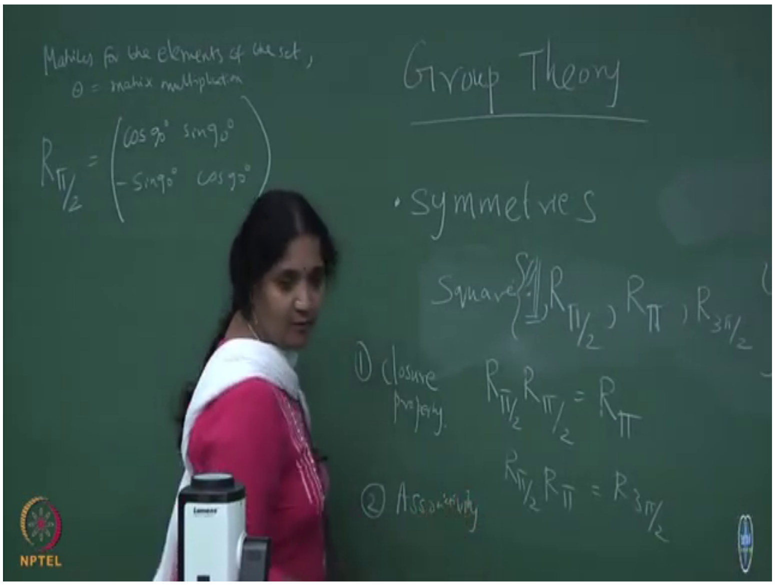

If I want to find out rotation by 90 degrees, am I right, you can use matrices then the group operation will be matrix multiplication, right, it will be matrix multiplication. So, if I use matrix, matrices for the elements of the set group operation will be matrix multiplication, ok. So, the group operation depends on what exactly you are going to do. In this case, I am doing physically rotations. So, you just do rotation then after that again another rotation that is the group operation or the product rule here is like this. But you can also do it like writing the explicit matrices and do matrix multiplication, ok. So, this is another way of doing it, but both are going to be representing the symmetry of the square, ok.

So, this set satisfies the closure property that I have already showed you. That if you multiply any two elements you will get an element which will be in the same set, if it exceeds 360 you will bring it back it to the first domain between 0 to 360, correct. So, closure property is there. What about associativity property? Ok.

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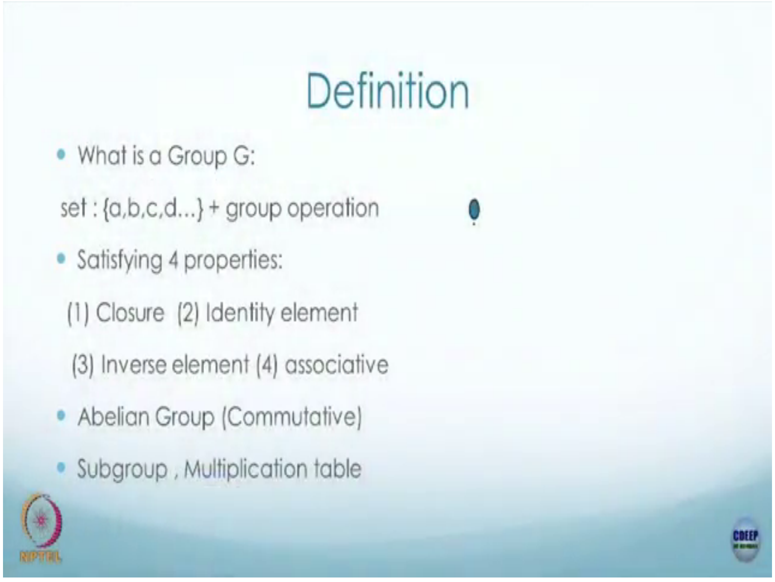

So, there is also associativity. So, let me write I think, I have this rule here. So, what is a group? So, you start with a set of elements, ok. In this particular case of the square there were only 4 elements and you have to also define a group operation and that group operation is dependent on the situation. If you give the a matrix kind of a representation, by representation I mean it is like you know dress which you associate. R pi by 2 you can treat it like a rotation by 90 degrees on a physical square.

You can also say that I want to do the rotation by giving a presentation which is a matrix presentation for the rotation by 90 degrees. These are the two ways of looking at. There can be many other ways of looking at. Depending on that your group operation here, the group operation is just doing physically 90 degree followed by 90 here it will be matrix multiplication. So, the group operation depends on the situation, ok.

So, you have to have a set and you have to also give a group operation and then it has to satisfy 4 properties, you should have a closure property which I have already explained. Take any two elements according to that, take any two elements with the group operation, ok. You should get an element c which should belong to that set, the set which I calling it as G, it should belong to that set that is what happening here, right. If you take any two elements the combination will give you another element which also belongs to that set is that clear.

And then, identity element is what? I have already said do not do anything, do not do any rotation, rotation by 0 degrees is an identity element. Any of these elements if you multiply with an identity element what happens? It is the same. So, there should be an existence of identity. Identity element has to be there, in that set, if you do not have an identity element then that set will not be satisfying the properties to qualify for a group.

So, you should have an identity element then you should also satisfy associativity property and you should also have an inverse, ok. So, associativity is what? If you take a combine it with b and then combine it with c, it should be same as someone it combine b and c and then an a, this property should also be satisfied. So, you can check it out that it satisfying that.

Inverse is every element if you combine with another element let us say b and if you gives you identity element, so most of the group theory books identity written by the letter e, ok. So, I am going to follow the letter e for identity element. You have to find at least for every element some element in the set a and b should be in the set such that this combination is an identity. So, in this particular case for R pi by 2 what is b, someone, which will give me identity R 3 pi by 2. So, for every element you will find an inverse, this element it is its own inverse, it is also in the set, ok. It can also be self inverse.

So, inverse is the, this is associativity and you have to find inverse, ok. So, inverse for every element should be in that set. If all the 4 properties are satisfied then you call that set to be a group, otherwise it is not called as a group, ok. So, I will give you few examples. Yeah I agree, but it has to be also complete you know, for every element there should be an inverse in the set. Closure property will be trivially satisfied once you find that, but even if every element does not have any inverse the product should give you a another element, right that is always possible. You have to get another one which is saying that a b should equal to e, right.

Student: (Refer Time: 28:31).

There is no necessity that a b has to be e condition is to be required for closure property. Converse is true. If a b is equal to e inside the closure property, but just a closure property does not necessarily mean that every element should have an inverse. What is minus R pi by 2 mean?

Student: We should have (Refer Time: 28:53) anticlockwise.

Yeah. Anticlockwise will be R 3 pi by 2 is R of minus pi by 2 if you want to call it, ok. So, it is not going to give you any new information. You have to find a distinct element which gives a new information, is that ok.

So, I put it a everything as clockwise going till 360, but you could also done let me go 180 degrees clockwise, the other 180 in anticlockwise then you would have got R 3 pi by 2 as R minus pi by 2; any other question? Ok. So, over on top of it if you have this requirement, if you have a b equal to b a, ok, the order if you do first the operation a and then b or you do first b and then an a if these two are equal then that set with this 4 property satisfied which we call it as a group will be called as an abelian group, ok.

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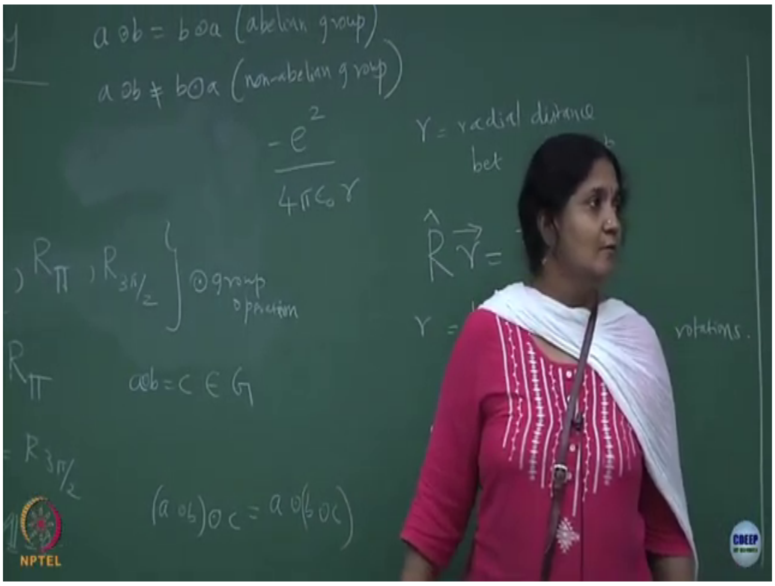

Abelian group satisfies this group or all elements of the set, not just from specific element, it should happen for any element of the group then you call it as an abelian group. If a dot b even atleast one element you find where this is not equal to this even if you find one element then you will say it is non-abelian, ok, is that clear?

So, I started with a simple example of a square which has a discrete rotation which is 90 degrees and I found that set has 4 distinct element, 4 distinct configurations and I try to give you a formal way of looking at these expressions of closure associativity and so on, and every element should have an inverse there should be an identity element which is like no rotation operation and, we also have this property satisfied by the square. You all agree. If I do rotation by a 90 and then a 180 it is same as doing 180 and 90. So, it is going to be an abelian group. There could be some cases where you will find it to be not an abelian group, we will see all these things, ok; so, just to give you some more examples.

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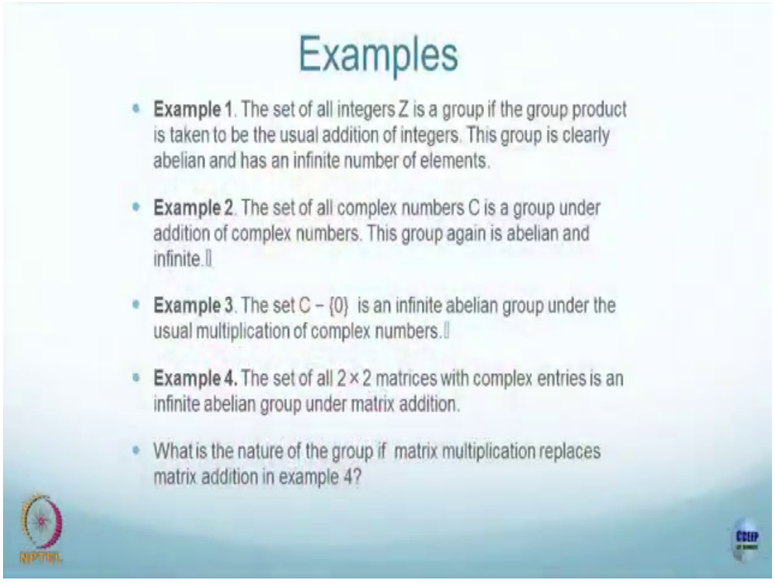

Take the simplest example. Set of all integers, ok; so, let us take the set of, so you are going to take a set which is all integers, and have plus or minus, right. So, this is the set. So, what is the identity element? Identity element is 0. So, the set has an identity element. Closure property any integers if you had, it will give you a p which is also an element of the set, right.

Then the third property is if you add m plus n, m 1 plus m 2 plus m 3. In an any way group operation is what? I should have written the group operation, group operation is addition set plus group operation, ok. This is same as.

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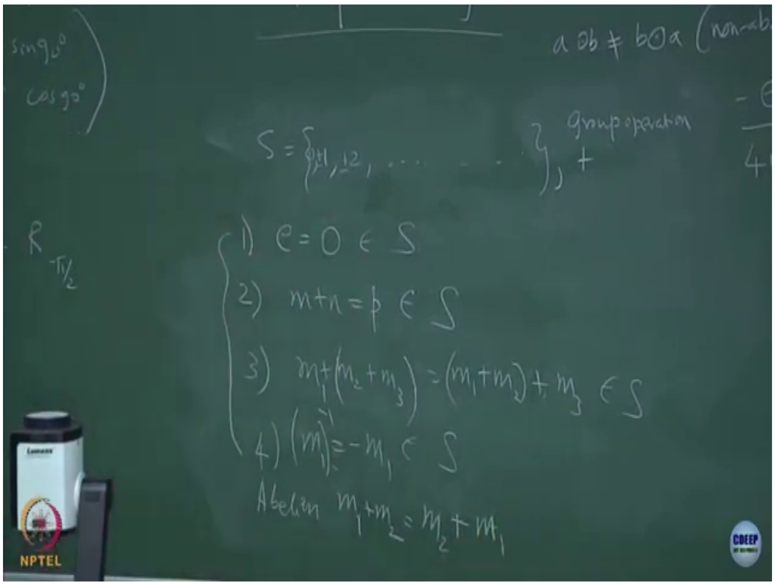

What is the inverse of every element? Right, m 1 inverse is minus m 1 which is also in the set. And what more, is it an abelian or not abelian group? It is abelian, right. So, this is qualifies for it to call set as a set of integers under group addition is a group is that, right, under addition, group operation which is addition is a group. If I change this to multiplication what will happen?

Student: (Refer Time: 34: 21).

So, you can see. So, you can have a set with closure properties, but you cannot have a under multiplication product of two integers will give you an integer, but you will have problems with that identity. Identity is also there, but inverse will not be there, ok.

So, this is one example which forms a group and further you can say that it is abelian because m 1 plus m 2 is same as m 2 plus m 1, ok. To multiplication it is not a group because inverse is not there is that clear, ok. So, it is abelian but that set has infinite element. So, it is an infinite number of elements.

Other one is also set of complex numbers is a group under addition of complex numbers; again you can show it to be abelian and infinite. This notation C minus 0 basically means it is a set of complex numbers excluding 0, ok. That is the meaning of writing it as C minus 0.

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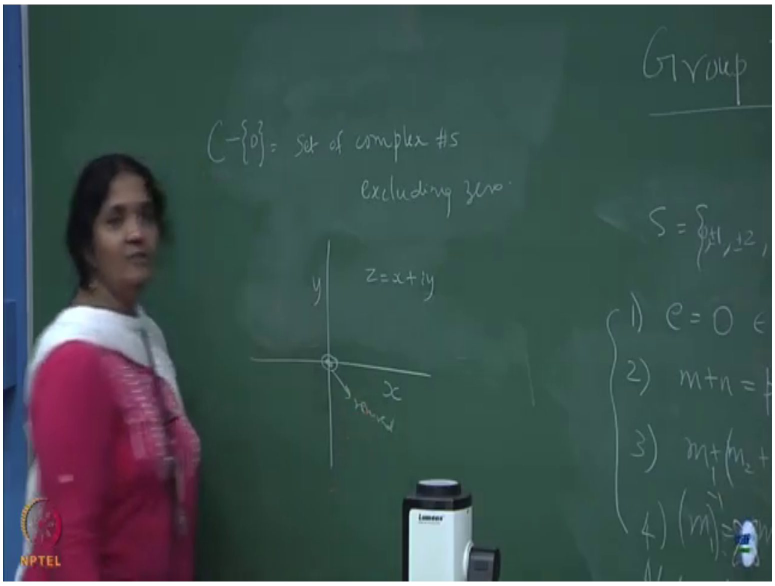

C is the set of complex numbers, complex numbers can also include 0, you remove the 0, ok. So, the complex numbers can be plotted on a two-dimensional plane is y-axis and x-axis, all points on this two dimensional plane where every complex number z can be written as x plus i y you all know this, right. I am going to remove this point out of this plane, ok.

So, this is removed, this I am removing it. What is the advantage of removing it? Why am I removing it? 0, 1 over 0, two problems will not be there. And then for this set without the 0 I can still show it to be abelian group and a multiplications, ok. The earlier one was the addition, you want to do multiplication you can use the space without having the origin showing pictorially the space and equality, ok. Any other question on this?

This is another example with complex entries. So, you can have infinite elements, infinite number of elements and you can show that all the properties are satisfied. Is it satisfied? What is the group operation am going to use? I am going to use matrix addition, ok. The group operation is matrix addition.

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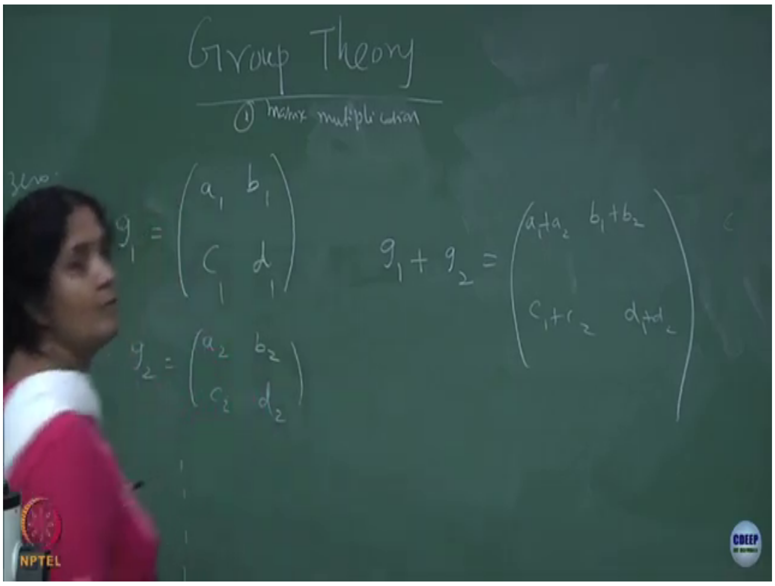

So, g 1 plus g 2 will give me, ok. This also belongs to that set off because a 1, b 1, c 1, d 1 or all complex entries and this one also belong to that set. What is the identity matrix here, identity element? Identity element is identity matrix; inverse will be negative entries, right. So, it forms the group, does it form a abelian or non-abelian?

Student: Abelian.

Addition louder.

Student: Abelian.

Abelian, ok. So, it is a abelian. Let us replace this group operation by matrix multiplication. What happens? All the properties are satisfied. What are the properties, where you have problems?

Student: (Refer Time: 39:48).

Inverse; if it is the singular matrix then you are stuck, you have to make sure that they are non-singular matrices whose determinant is nonzero then only you can find inverses, ok. So, let us take it to be set of non-singular matrices or determinant not 0, then it forms a group. Is it abelian or non-abelian?

Student: Abelian.

So, matrix multiplication all of you know need not be committed order can matter. Some case cases a b can be b a, it not all elements in that set. So, it is the first example, with set of matrices whose determinant is not 0, so that is invertible you can find inverses, right with matrix multiplication as a group operation it forms a non-abelian group, ok.

So, I hope by now you are now familiar what I mean by a group it is not just said you will have to give an operation and then under that operation all the 4 properties have to be satisfied and then if on top of it if you have commutative property it is an abelian group. 4 properties are satisfied it is a group, on top of it if it is abelian. Many times it will not satisfy commutative the commutative should be satisfied all the elements not just some elements, if all the elements are satisfying it is an abelian group.

So, we have seen the first 3 examples are commutative, abelian group and the forth one is also abelian group and under addition, and then I ask what is the nature of the group if matrix multiplication replaces the matrix addition, and we have already seen it will be a non-abelian group provided they are non-singular matrices or equivalently their determinant has to be nonzero, ok.

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See just confining our self to the square incidentally I should also just take you back on this, before this definition I had said discrete group is rotation of a square, but continuous group is where you can take a circle all the points on the circle under any rotation, any small angle, finite angle it will look exactly the same, ok. So, this is where you will see that it will have many elements and this rotation angle is not a finite angle, it is not discrete, it can be continuous. You can do R of theta, you can do R of theta, you can do R of theta plus delta theta, ok.

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So, the theta which is the rotation angle can be varying continuously also and this will again give you an R of some new angle which will involve rotation by 2 theta plus delta theta and so on, ok. So, in that sense I would say that the theta is can have any value between 0 and 2 pi, unlike your square where theta has to be only n times pi by 2, its discrete whereas this theta is continuous, can be anywhere. In that sense the corresponding symmetric you call it to be a continuous element. Is this set which involves rotation by; this set is going to be infinite elements, is it abelian or not abelian?

Student: (Refer Time: 44:19).

Abelian, right. Because I am moving the points on the circle there are infinite elements unlike the square which had only 4 elements, but it still an abelian because I am going to do a rotation about the center of the circle and it is not going to give me anything new, ok, fine, ok.

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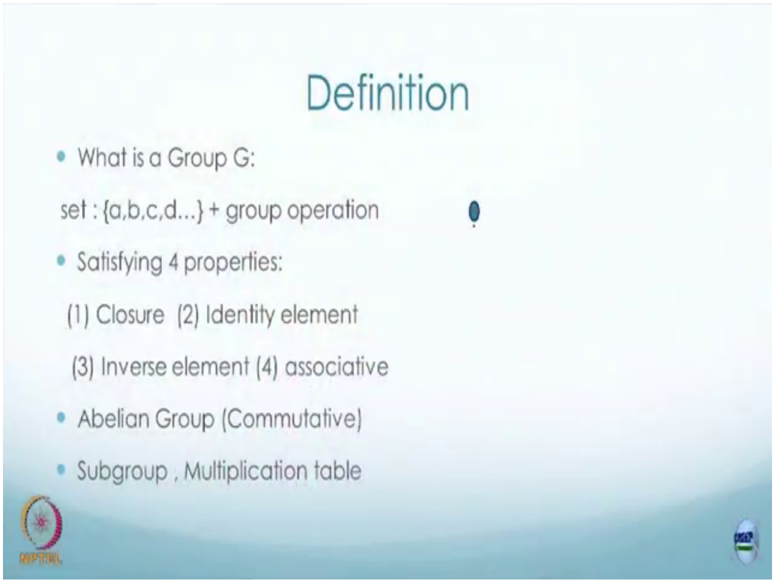

So, there is couple of things which I have still not defined for you what is the subgroup, what is the multiplication table. So, let me spell some prime on that.

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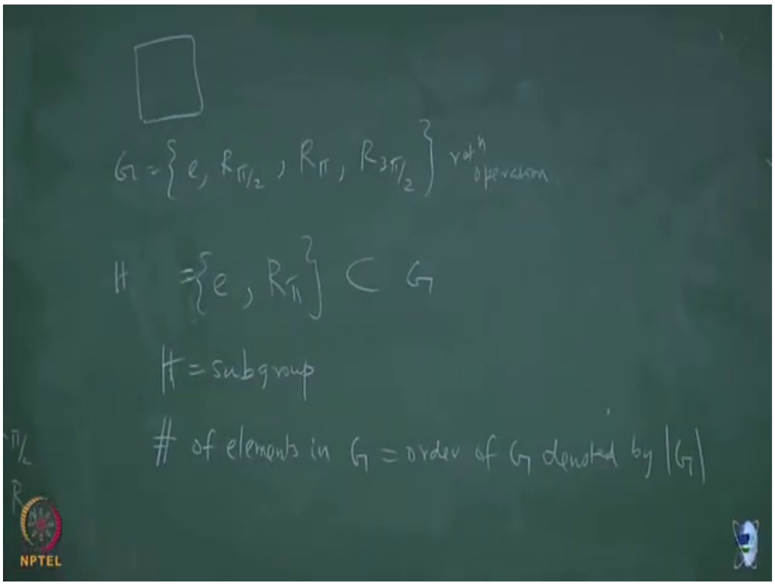

So, we saw the rotation on a square, ok; so, this one the symmetric group the group involves identity R pi by 2, R pi R 3 pi by 2, throughout that mean let us some group operation which is just multiplying the rotation operation, ok. What is a subgroup? You have to find a subset; you have to find a subset with again all the 4 properties. Can I find a subset here? So, this by this I mean subset of G, ok. Which one will it be? R pi, right. So, this one is what I call, this is what I call it as H, let me write the H is the subset of G satisfying all the 4 properties. What is the inverse? Self-inverse, identity is there, associativity and closure property has trivial. So, it is still a satisfying the group properties. So, we call this H as a subgroup, fine.

So, you can start doing this for other cases, other examples, you can see when there are subgroups in it, ok. I am not really, I gave you few examples for a group, it start seeing whether there are subgroups and then find out the subgroups which are sitting inside the bigger, exactly like the subsets inside sets, ok.

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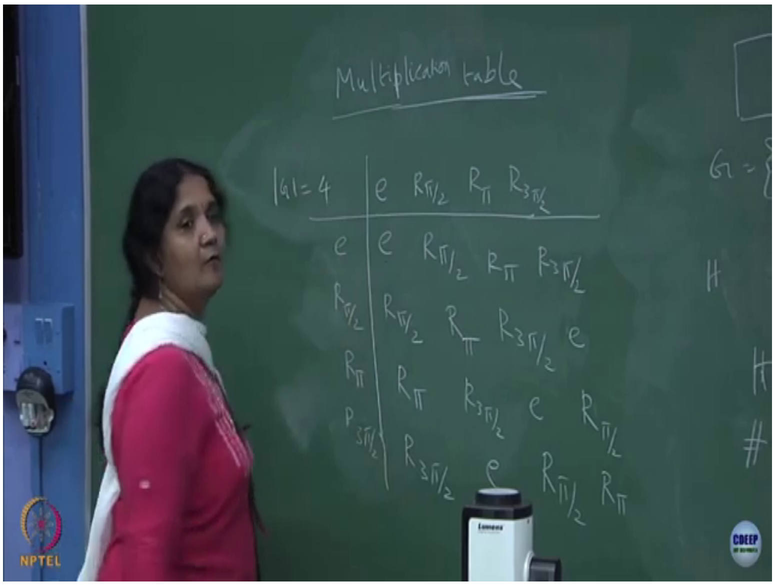

And then we also can write the multiplication table, ok. So, the multiplication table goes this way, this is one thing which will show you all the properties. You write all the elements, both along the row and along the column, ok. It is a kind of a grid and then do the group operation these elements. So, identity with identity is e, identity with R pi by 2 is this, R pi, R 3 pi by 2. If you do R pi by 2 with identity it is again R pi by 2. R pi by 2 with R pi by 2 will be R pi and then this one will give you R 3 pi by 2 and R pi by 2 with R 3 pi by 2 is identity. What do you see in the second line is as compared to the first line.

What is the second line showing? The same 4 elements with some permutation has happening, no reputation of any element, each element in that set appears only once along a row or along a column of this multiplication table, but there are some permutation. So, that tells you that when you do this. What is this? Am I right? This is again a permutation, the last one; you all with me. So, this is what is called as the multiplication table.

Typically, it is useful if you have finite number of elements in a group, ok. So, the number of elements in a group in G, in any group G we call this as order of G and we denote it by, ok. So, in this particular case the G has order; order is the number of elements in that group. So, for that particular multiplication table order of the group is 4, so it be a 4 by 4 grid, and each row and each column will have all the 4 elements in suitable permutations, ok. So, that is what is a multiplication table, ok.

So, I have gone through these examples, I have also said what is the order of a group by giving you the specific example where the number of elements is finite. If it is infinite the order will be very it will be infinite, ok. We are going to confine ourselves to finite groups. Subgroups I have already said subset satisfying all the 4 axioms of the group.