Physics of Biological Systems
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Lecture - 19
Life at low reynolds number

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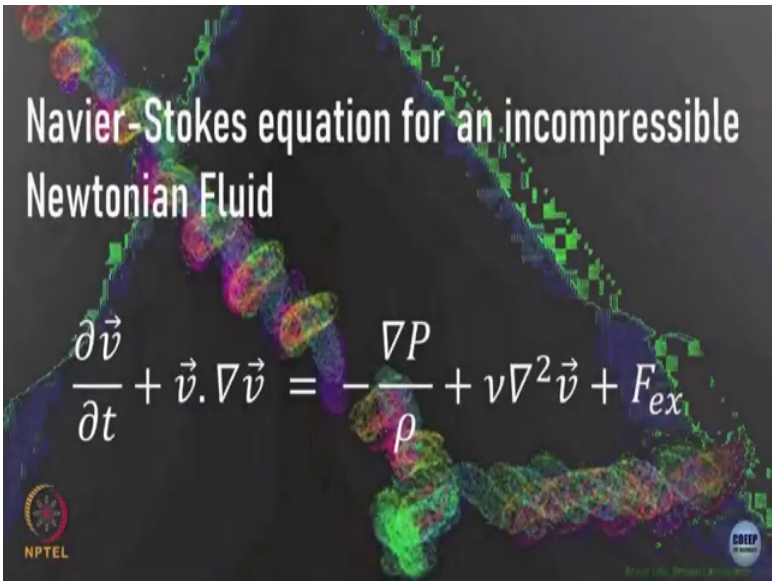

So, what we are doing is this the Navier-Stokes equation, right.

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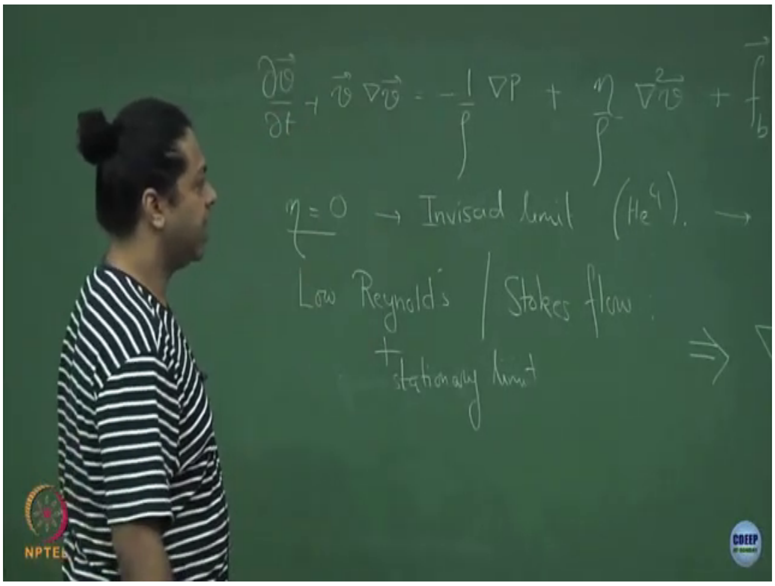

So, we derived the Navier-Stokes equation last class which was del v del t plus v dot del v is equal to minus 1 by rho gradient of the pressure plus eta by rho Laplacian of v plus whatever body force is there on right, like gravity or whatever. So, today what we will look at is we will try to solve this in one or two simple cases, we leave the complicated cases for later. But before I do that I just want to say about a couple of limits of this equation.

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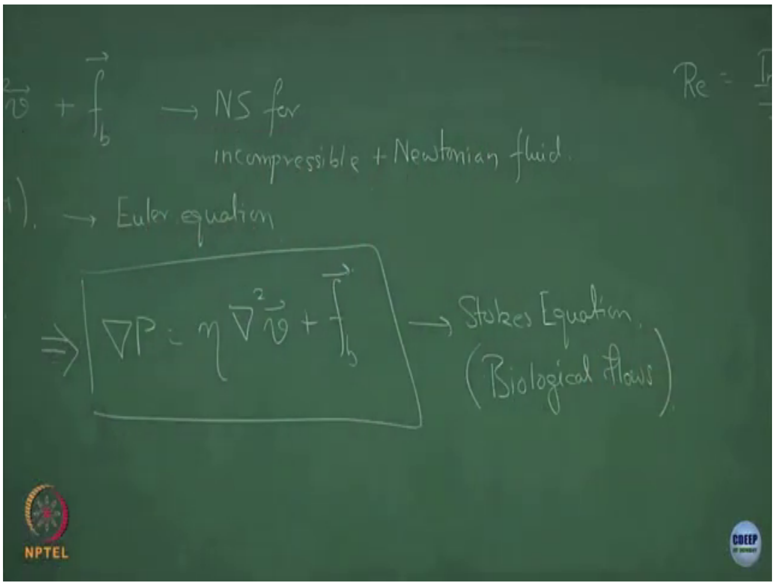

So, this is the full Navier-Stokes. So, this is Navier-Stokes remember for an incompressible fluid, for an incompressible fluid otherwise we will have another second derivative term and Newtonian fluid, right. So, one of the limits that is one of the limits of this equation is when you have a fluid with no viscosity.

So, eta is equal to 0 which is the inviscid limit this is called the inviscid limit. So, this term sort of drops out right and things like superfluid helium and so on come under this unit, ok. The other limit which is what we will be focusing on is this limit of low Reynolds number this. So, once we have taken out the eta from this equation this is called the Euler equation I think, this is called the Euler equation, the other limit is low Reynolds number or Stokes flow, ok.

Where I say that I neglect the inertial terms which means I neglect this v dot del v and I work in the stationary limit which means I neglect this del v del t, right. So, this is low Reynolds number plus quasi stationary, this is stationary, right. And, then the only terms that survive are these terms on the right hand side which means that this then gives me the equation that gradient of P is equal to eta Laplacian of v plus whatever body forces that you have.

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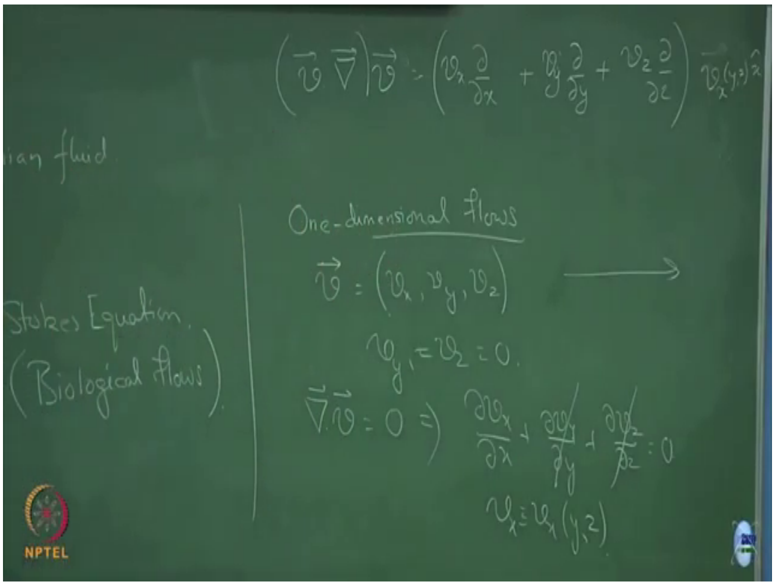

So, if I can neglect the inertial terms in this equation remember my Reynolds number was a measure of how strong my inertial forces were compared to my viscous forces, inertial forces compared to my viscous forces. And, if this Reynolds number is very small which means that the inertial forces are negligible compared to the viscous forces.

I neglect these inertial terms in the equation and then what I get is these terms on the right hand side which means that my Navier-Stokes equation now reduces to this equation which is called the Stokes equation this is called Stokes equation, ok. And the nice thing about this equation is that unlike this full Navier-Stokes this is a linear equation, I have gotten rid of the non-linear term this v dot del v right which was the main problem in solving them which is the main problem in solving the Navier-Stokes.

So, once you have neglected that this is now a linear equation; I mean, you can often write down solutions depending on the geometry or at least (Refer Time: 04:42). So, when we talk of low Reynolds number physics, the hydrodynamics is sort of governed effectively not by the full Navier-Stokes, but by this limiting case of the Stokes equation, ok. So, this is the equation that is more relevant for biological flows mostly, ok.

Now there is a different way you can arrive at this Stokes equation. So, here I arrived by neglecting the neglecting these inertial terms, but I can arrive at it with a different way which is this case of one-dimensional flows which is this case of one-dimensional flows, ok. Which means that let us say I have a velocity vector which has my three components v x, v y, v z for a one-dimensional flow only one of these velocity components will be there, right.

So, if it is along a pipe or something the fluid is just flowing in one-dimension which means that this v y is equal to v z is equal to 0. So, if you have a flow like this only v x is there then we can see what that says for the full Navier-Stokes equation. So, if I still say that my fluid is incompressible which means that divergence of v is equal to 0 right, what that means, is that del v x divergence is del v x del x plus del v y del y plus del v z del z equal to 0, right.

Now, because this is a one-dimensional flow v y and v z are not there, right. So, therefore, the v x is not a function of x; del v x del x is equal to 0 which means that v x is some function of y and z for me, ok. Now, if you work out this term this v dot del v; so, if I now work out v dot del so, this is my inertial term v dot del v, v dot del is v x del del x plus v y del del y plus v z del del z operating on the full vector v which is this v x of y z x hat right, what will this give me 0, ok.

So, for this case for this case if a one-dimensional flow explicitly this inertial term drops out. So, therefore, if you are looking in the stationary limit where this del v del t also you can neglect. This one-dimensional flow reduces back to the Stokes equation where you just have this gradient of P plus eta Laplacian of v plus body force is equal to 0.

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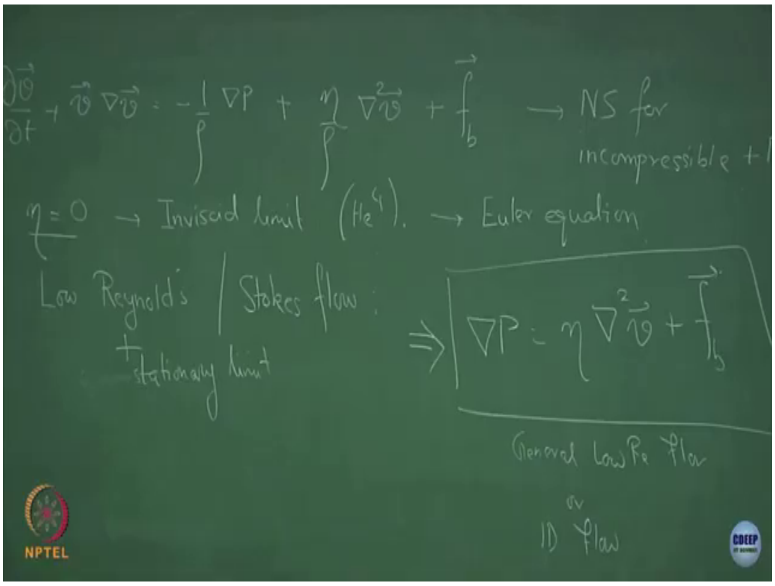

So, this is true therefore, for general flows in the low Reynolds number. So, a general low Reynolds number flow low Re flow or a strictly 1D flow in both cases the Navier-Stokes equation reduces to the Stokes equation where you have gotten rid of the non-linear term. To do this the canonical example is sort of this poiseuille flow. So, we can just quickly work that out.

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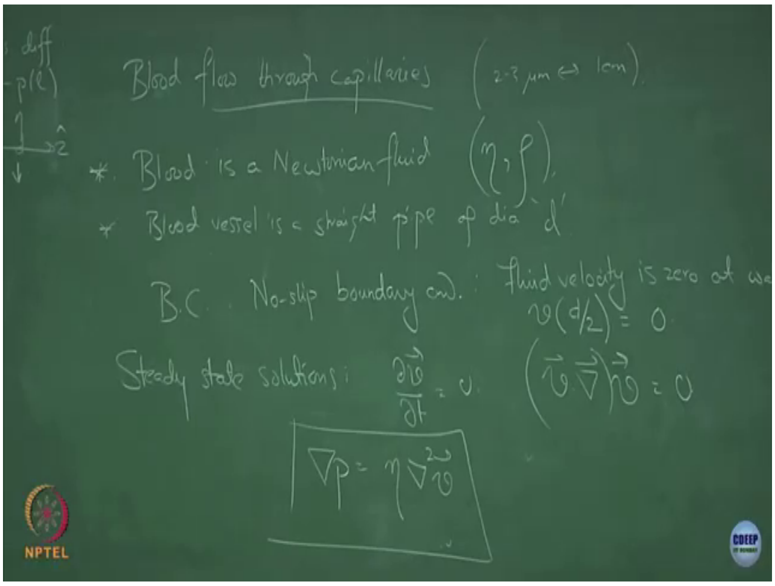

So, the idea is this that we look at fluid flows inside a tube, with some velocity and one of the cases where you might imagine this is relevant in biology it flows like this through tubes is for example, blood flow through capillaries, ok. So, blood generally flows through all these arteries veins capillaries and so on, if you look at all these sort of vessels. The diameters of these would be in the ranges of a few microns 2-3 microns ranging up to for the major blood vessels up to few centimeters maybe, ok.

So, these of course, these are branched networks and so on, but if you look at a small segment of this network that is like blood which that is blood which is flowing through this parallel tube of some diameter microns or whatever, ok. So, that is the sort of biological picture that we have in mind. So, this we can solve using the Stokes equation and we can see at least what; so, this has many sort of approximations, but even within those approximations we can try and see what it tells us a little bit, all right. So, here is the idea.

So, this is I will assume that so, blood is a Newtonian fluid I will assume that blood is a Newtonian fluid with viscosity eta and density rho. This is not true, blood is not Newtonian one can do stress strain measurements and show that blood is not a Newtonian fluid. But, let us make this assumption since that is what we can do a little bit analytically. It is not too bad at least it will give us a sense of how bad this approximation is.

So, this is my assumption one that I will assume that blood is a Newtonian fluid and secondly, I will assume that this blood vessel is a pipe, blood vessel is a straight pipe of diameter d, ok. So, in principle of course, capillaries or whatever will be flexible. So, they can be in some shape, I will just assume for the time being that these are straight pipes.

So, what I want to solve is this Navier-Stokes equation and I need a boundary condition. So, I will use the no slip boundary condition, no slip boundary condition which means that the velocity at the walls of the pipe, the velocity of the fluid or blood at the these walls the walls of the pipe is going to be 0, ok. So, this is fluid velocity is 0 at walls, ok. So, that is that and so, let me draw this a little better.

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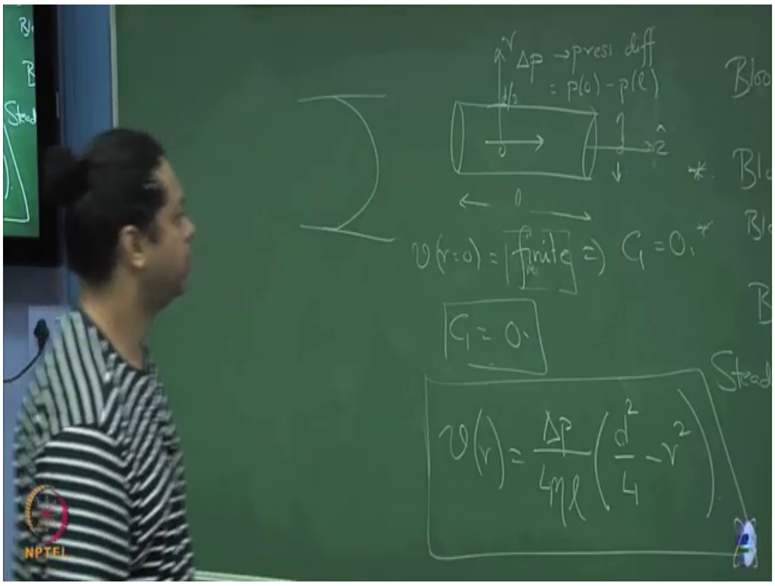

So, here is my blood vessel, it is a cylinder like this of some length let us say l, and some diameter d. Let us say there exist some pressure difference of delta P between the two ends of the pipe. So, this is the pressure difference between the two ends of the pipe, this pressure difference competing with viscosity is what causes the flow.

So, this is basically P 0 minus P l, this is 0 that it is a and my blood is flowing let us say in this direction and let us say this is my z axis. So, my velocity, the fluid velocity vector that has only a component along the z axis, and if you think about the symmetry of the problem the cylindrical symmetry. The magnitude of the velocity can only depend on the distance from these walls. So, it can only depend on this radial distance from the walls.

So, this is r equal to 0, this is r equal to d by 2. So, this is the form of my velocity vector v of r in this traveling in the z cap, and let us say that I only I want to look again at steady state solutions. So, I want to look at steady state solutions. So, del v del t in the Navier-Stokes is 0, I know that again because this is a one-dimensional flow my inertial term will also drop out. So, v dot del v is equal to 0. So, what I will saw there are no other external forces. So, the equation that I need to solve is gradient of p is equal to eta Laplacian of v, ok.

So, this is the setup of the problem, it is a standard problem in fluid mechanics pipe flow we will just adapt it or not even adapt it. We will do it with the background of this sort of blood flow through capillaries at the back of our mind. So, when we put in some numbers we will put in numbers that are for this viscosity diameter whatever, we will put in numbers that are that correspond to this sort of capillary blood viscosity and so, on and then see what that tells us.

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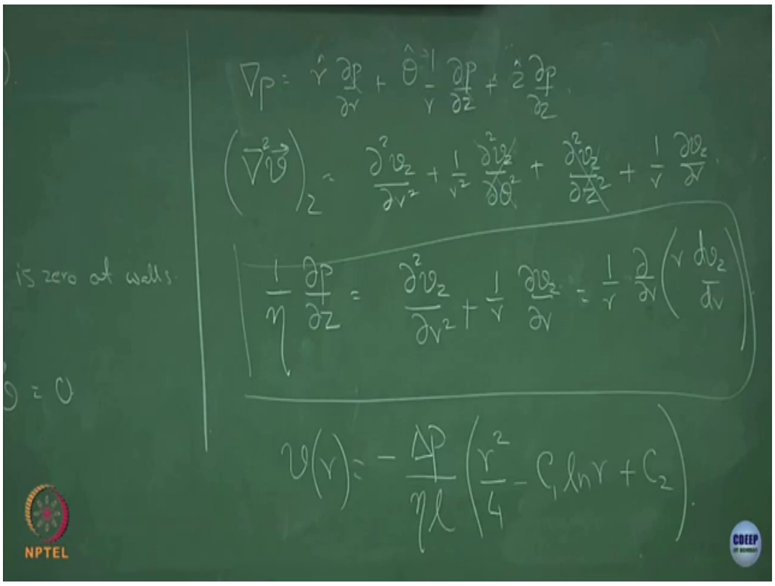

So, the obvious choice is to work in cylindrical coordinates, right because the problem is cylindrical symmetry. So, the gradient of p in cylindrical coordinates is r hat del p del r plus theta hat 1 by r del p del theta plus z hat del p del z. And, the Laplacian the z component of the Laplacian is del 2 v z del r 2 plus 1 by r square del 2 v z del theta 2 plus del 2 v z del z 2 plus 1 by r del v z del r.

So, this if you go back your curvilinear coordinates the cylindrical coordinates and check this should be the z component of the Laplacian, ok. This term does not is not there because v is a function of r only, this term is also not there, these terms are not there which means that if I substitute this gradient of p and eta Laplacian of v. The equation sort of reduces to del p del z is equal to eta into; so, let me bring the eta this side del 2 v z del r 2 plus 1 by r del v z del r, ok.

So, this is the equation that I need to solve this two terms I can just combine and write as 1 by r del del r r del v z del v, ok. You can integrate so, this is the equation that I need, this is what the Navier-Strokes equation reduces to for this pipe flow within this assumptions. You can solve this differential equation; so, you can integrate directly once with respect to z and twice with respect to r right there are two derivatives of r.

So, if you for example, integrate with respect to z over the length of the pipe, this will just give you delta p, right; the difference in pressure between the two ends over here and on the right hand side you will get a l obesity which is the length of the pipe and then you can integrate twice with respect to r. So, just do this I will write down the solution, ok. So, you can solve for this v z of r or v of r.

So, you can check so, this is minus delta P the difference in pressure divided by the viscosity into the length of the pipe r square by 4 minus C 1 log of r plus C 2. So, you can check this and substitute back and check that this is the correct solution. So, now I need to find out this constants which means I need to apply the boundary conditions. So, I have one boundary condition here, what other thing can I use?

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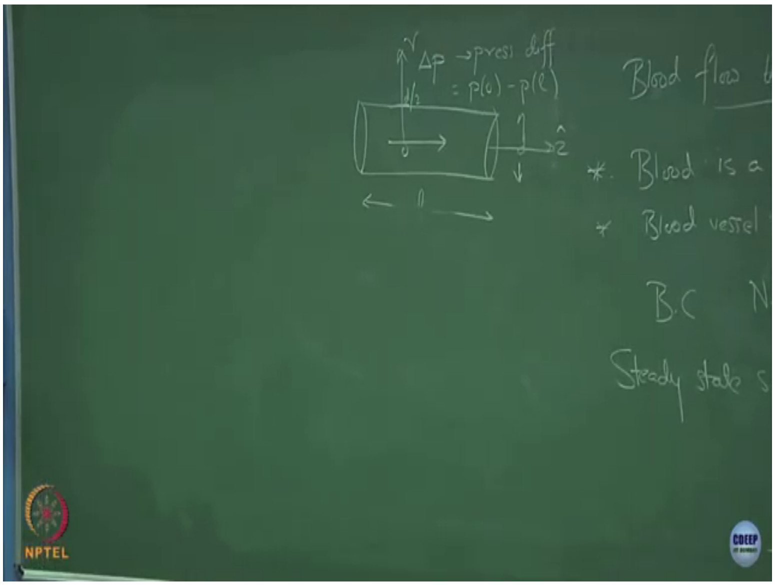

Right, remember this is my radial axis this is 0, this is d by 2 at the surface of the pipe. So, what can I say about this constant C 1 and C 2?

Student: (Refer Time: 20:16).

So, that is one because v at d by 2 is equal to 0 that is one, but I have two constants. So, I need to use something else what other thing can I use?

Student: Maximum velocity.

Maximum velocity.

Student: (Refer Time: 20:36).

Maximum velocity at the center, ok. So, what would I write for the mathematically v at r equal to 0 is v max something like that, but then this you do not know right. So, you are just introducing another constant into the system as such, it instead of C 1, C 2; one you can get rid of by using this, the other instead of writing C you would write in terms of v max it is true, but you have to evaluate that. What else can you use?

Student: (Refer Time: 21:18).

Huh?

Student: (Refer Time: 21:19).

C 1 is equal to 0. Why C 1 equal to 0?

Student: (Refer Time: 21:25).

Right. So, if you put r equal to 0 which is the center of the pipe this term blows up right which is not physically possible you need to have some velocity. So, the fact that this thing is some finite number implies that C 1 is equal to 0, right and now you can put this other condition v is equal to d by 2. So, let me get rid of that term; so, this term is not there anymore. So, now, we can put r equal to d by 2 and at that the velocity goes to 0 which gives you C 2.

So, what is C 2? Yes, right; this comes out to be delta P by 4, it is a 4 eta l d square by 4 minus r square. So, C 2 comes out to be d square by 16. So, if you put that in this is the velocity profile which as you say does indeed have a maximum when you put r equal to 0. The magnitude of that maximum velocity is delta P by 4 pi eta l sorry, 4 eta l d square by 4 and of course, by construction when you put r equal to d by 2 this velocity goes to 0.

So, the profile looks like that of this pipe flow at the walls it is lowest. So, there is no flow at the walls it is maximum at along the axis of the cylinder, this is called specific heat flow, ok. So, this is my velocity, this is my velocity profile it is a profile like this along this is, all right.

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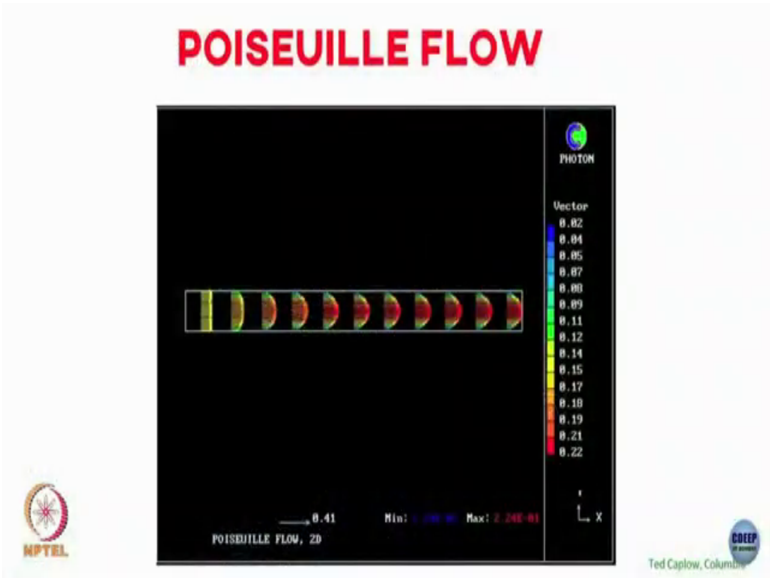

So, now if I say that let us calculate a few; now that I have the velocity you can try to characterize it different ways. So, for example, I could ask that what is the average fluid velocity across the cross section without.

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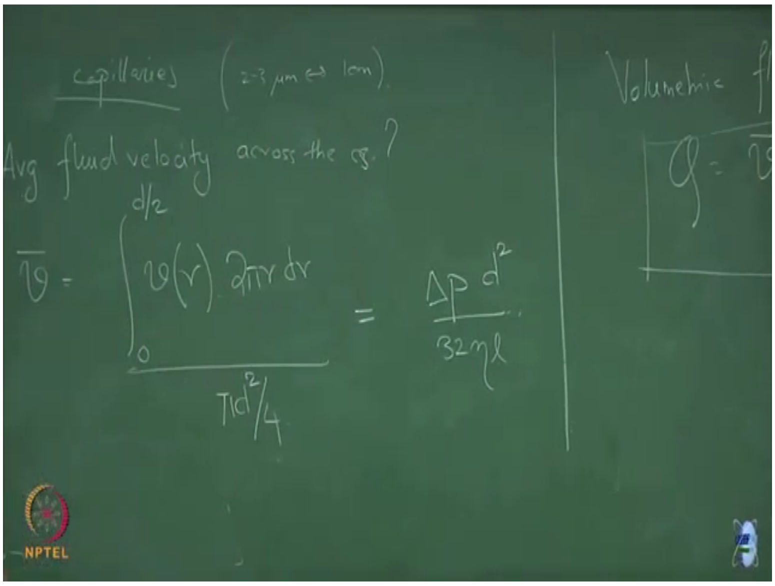

So, what is the average fluid velocity across the cross section cross section? Which means that I need to take the average. So, v average across this cross section is this v of r into 2 pi rdr right r growing from 0 to d by 2 divided by pi r square. So, pi d square by 4, ok. So, this is right this is the average fluid velocity over the cross section v r 2 pi rdr divided by the cross sectional area.

So, if you put it in this v of r and you do this integration, this is an easy enough integration you put in the limits. Again, I will just write down the answer this comes out to be the pressure difference times v square by 32 eta l, right. this is the average fluid velocity.

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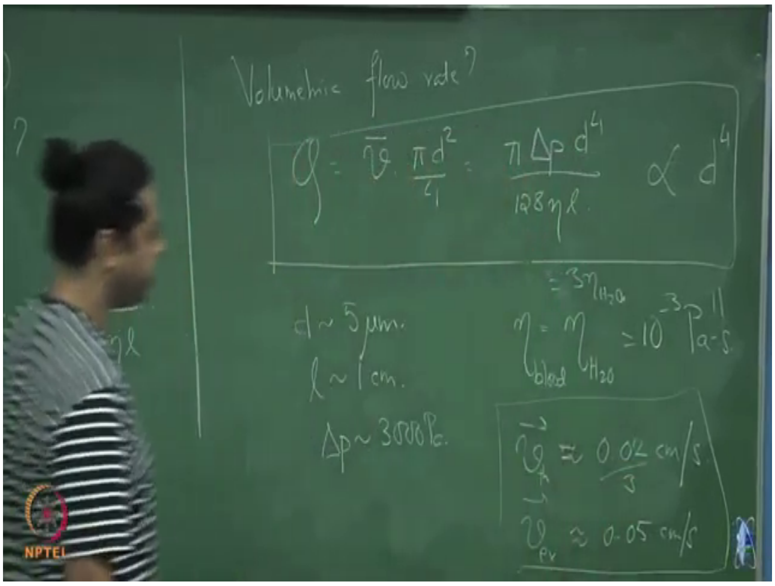

You can also ask what is called as the flow rate or the volumetric flow rate, what is the volumetric flow rate which is the amount of fluid flowing through a cross sectional area per unit time volumetric flow rate, right. And, that is just that is called Q and that is just this let us say this average velocity across the cross section into the cross sectional area. So, pi d square by 4 which therefore, comes out to be pi delta p d to the power of 4 by 32 into 4; so, 128.

This is the amount of fluid that flows past a cross sectional area per unit time. The thing to sort of note is that this is proportional to the fourth power of the diameter or the radius of this pipe that you are flowing through, ok. So, this is poiseuilles result actually it is a very famous result in flows, but if you think about in the context of blood flows for example. What it says is that very small changes in your diameter or the radius of the capillary that you are flowing blood through can impact this flow rate by a large amount, right; this goes proportional to the 4th power of the d.

So, if you have a blood capillary and where your blood is flowing and you have some plaque deposition or something which reduces the effective radius of these tubes that effect even if it is a very small change in the radius. Say, if you have a capillary and you have some deposition in this, even if this amount of change is not a lot; that means, sorry this reduction in the radius is not a lot that affects the flow rate sort of disproportionately ok, because of this decreases this flow rate immense significantly, ok.

So, what I wanted to do was to take this result which is the standard result and just put in some numbers and see what sort of velocities I get in the context of blood flow, ok. So, let me say I take a typical capillary diameter of some 5 microns, it is roughly 2 to 10 microns the capillary; so, I take 5 microns.

Let us say I take a capillary length of around a centimeter and a pressure difference which is close to the experimental value 3000 Pascal's, and I need so, I have a delta p, I have a d so, I need a viscosity. So, let me just assume that I take the viscosity of water, ok, it is slightly different, it is somewhat different from water, but what let me take the viscosity of water which if you remember it is 10 to the power of minus 3 Pascal seconds.

If you put in all of these numbers into this into let us say the velocity expression, the average velocity expression. You can get what is the average velocity of this blood flow in a capillary of in a pipe like capillary of 5 micron diameter. So, if we put in these numbers, what we will get is roughly a wrong so, you can check. What we will get is roughly around 0.02 centimeters per second and measurements of blood flow in actual capillaries gives you a number.

So, let us say so, this is my theoretical estimate. The experimental range is something like 0.05 centimeters per second, ok. So, which is actually pretty good, we have done our extremely sort of simple naive modeling. We have thrown away any complication we have taken a pipe straight pipe and so on and so forth, but still we get an estimate which is pretty close at least order of magnitude, it is correct to whatever is the actual flow, ok

So, even though we have made these simplifications the analysis is not too bad. If I wanted to go from this sort of a value to this, if I wanted to increase the velocity and I say that this is so, I have taken that eta blood is equal to eta water, let us say I say that this is not correct, ok. I need to take the correct viscosity of blood should I take a larger viscosity or a smaller viscosity to get the correct average velocity? Smaller viscosity, right.

So, if I wanted to get this number closer to the experimental estimates, it would say that I need to take a smaller viscosity. Do you know what is the viscosity of blood, anyone any idea, is it more viscous than water, is it less viscous than water?

Student: More viscous?

More viscous than water. So, actually blood is around if I it is around 3 times the viscosity of water ok, which means that this estimate that I did by doing this approximation is actually is actually slightly misleading. The actual estimate is worse than this because I have taken the viscosity of blood to be the same as water and you cannot really do a correct estimate unless you. So, now, if you want to get the numbers, right; so, this is actually the theoretical estimate should be something like 0.002 by 3, if I take the correct viscosity of blood.