Path Integral and Functional Methods in Quantum Field Theory
Prof. Urjit A. Yajnik
Department of physics
Indian Institute of Technology, Bombay

Lecture - 09
General Functional, Forced Harmonic Oscillator – I

Now, we are on to a developing a formalism which brings out what field theory is about and how to deduce result from field theory. So, what you are doing now is forced Harmonic Oscillator or rather Greens function.

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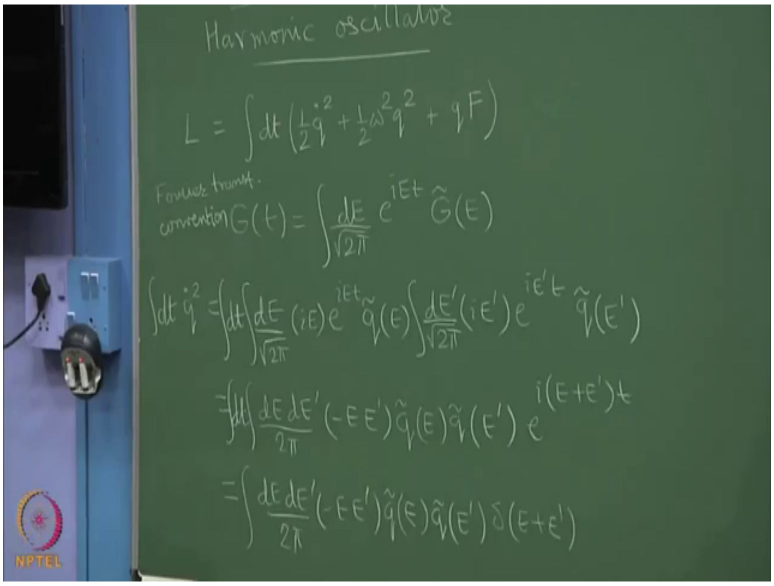

So, we use the Lagrangian formalism and the Lagrangian is of course,

$$L=\int dt\,(\frac{1}{2}\dot{q}^2-\frac{1}{2}\omega^2 q^2+qF)$$

and we had set m = 1. This is what it was and then what we do here is that we propose a Fourier transformed variable for q and similarly for F right. Our definition will be

$$G(t)=\int \frac{dE}{\sqrt{2\pi}}\,e^{iEt}\,\tilde{G}(E)\quad .$$

So, this is our definition of Fourier transform convention.

Now we try to calculate what happens to the $\dot{q}^2$. So, $\dot{q}^2$ will require you to take something like this and differentiate it under the integral sign. So, if you carry out this exercise, all you have to do is take this expression twice. So, we will have

$$\int \frac{dE}{\sqrt{2\pi}} iE\, e^{iEt} \tilde{q}(E) \int \frac{dE'}{\sqrt{2\pi}} iE'\, e^{iE't} \tilde{q}(E')$$

but from this we get,

$$\int \frac{dE dE'}{2\pi} (-EE') e^{i(E+E')t} \tilde{q}(E) \tilde{q}(E')$$

So, if we look at this under the integral dt sign so, then under integral dt,

$$\int dt\, \dot{q}^2 = \int \frac{dE dE'}{2\pi} (-EE') e^{i(E+E')t} \tilde{q}(E) \tilde{q}(E') 2\pi \delta(E+E') = \int dE\, E^2 \tilde{q}(E) \tilde{q}(-E)$$

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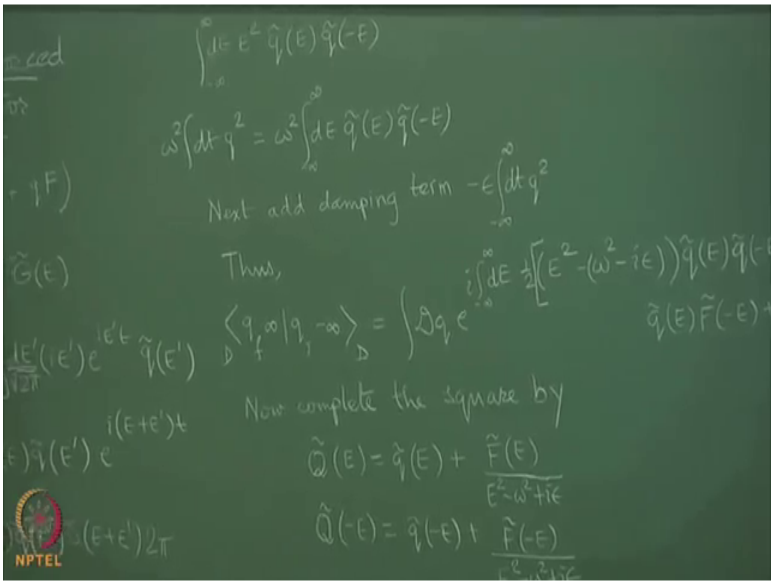

and that is it. Now we have to put it alongside this $\omega^2 \dot{q}^2$. But $\dot{q}^2$ is just going to have without any iE appearing, this thing twice. So, it does not seem to have E and -E; however, you can always reverse the integration direction in one of them because it does not change the meaning of what this expression is and then, you can write

$$\omega^2 \int dt\, q^2 = \omega^2 \int dE\, \tilde{q}(E) \tilde{q}(-E) \quad .$$

So, you can recast it in that same form and then, we had added damping factor which is $-\varepsilon\int dt\ q^2$, where these integrals are all from $-\infty$ to ∞ because this can be a between arbitrary times, in the Fourier transform, we certainly need E going from $-\infty$ to ∞ because otherwise we do not have Fourier transform. And yes therefore, these integrals are all otherwise we do not get delta function. So, these are all integrals from $-\infty$ to ∞.

So, this is also integral $-\infty$ to ∞. So, the point being that this thing added to the exponent in the path integral. We will add a positive definite quantity with a negative constant in front and it will cause damping for large values of t. If you go to large values of this q^2, then this begin becomes larger and we will damp everything out.

So, we do not have to worry about very large q's and you can focus on a finite region. So thus, the transition amplitude

$$\langle q_f, \infty | q_i, -\infty\rangle_D = \int Dq \exp[i\int_{\infty}^{-\infty} dE \frac{1}{2}[(E^2-(\omega^2-i\epsilon))\tilde{q}(E)\tilde{q}(-E)+\tilde{q}(E)\tilde{F}(-E)+\tilde{q}(-E)\tilde{F}(E)]]$$

Ultimately, it can also be again made a Gaussian. So, if you can just look at the structure of this. Aside from a time derivative there which in the energy domain will just become multiplication the energy. This is quadratic in q or $\tilde{q}$ variable and this is linear in q.

So, all you have to do is a term that is of the form F^2 and you can make it into a complete square. So, that is what we do. By choosing

$$\tilde{Q}(E)=q(E)+\frac{\tilde{F}(E)}{E^2-\omega^2+i\epsilon}$$

$$\tilde{Q}(-E)=q(-E)+\frac{\tilde{F}(-E)}{E^2-\omega^2+i\epsilon}$$

So, we will have to just subtract the square part of this term.

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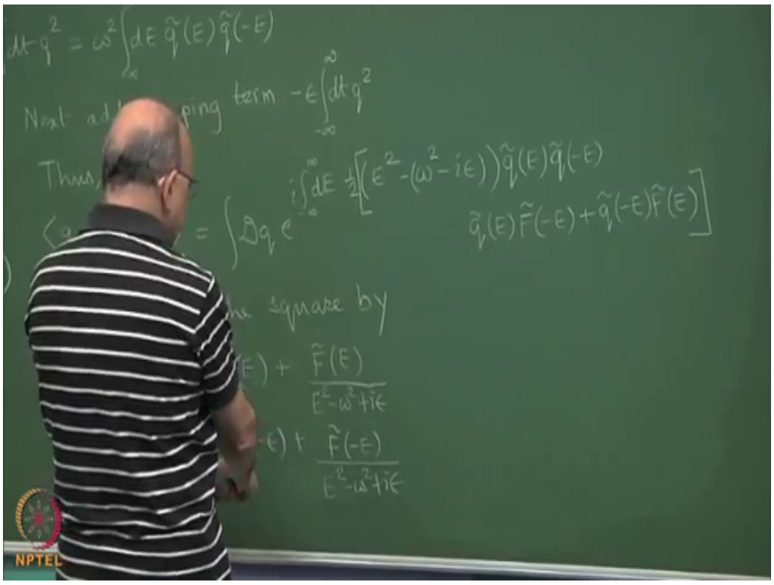

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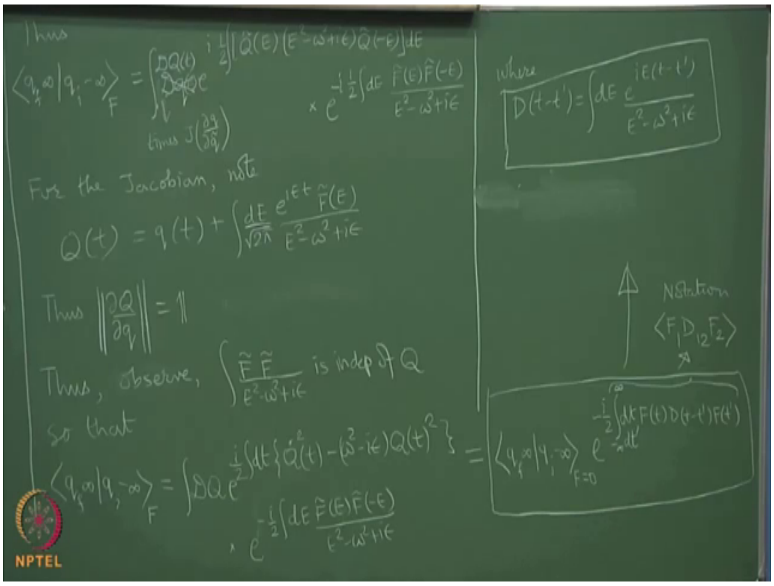

So, we find

$$\langle q_f,\infty|q_i,-\infty\rangle_D=\int D\tilde{q}(E)\,J\left(\frac{\partial q}{\partial \tilde{q}}\right)\exp\left[\frac{i}{2}\int_{\infty}^{-\infty} dE\,\tilde{Q}(E)\left(E^2-(\omega^2+i\epsilon)\right)\tilde{Q}(-E)\right]$$

$$\exp\left[-\frac{i}{2}\int dE\,\frac{\tilde{F}(E)\tilde{F}(-E)}{E^2-\omega^2+i\epsilon}\right] \quad .$$

Now, I have to argue to you that this Jacobian is 1 that is what we want to check right. So, what we observe is that this shift amounts to I can now Fourier transform it back. So, now, note

$$Q(t)=q(E)+\int \frac{dE}{\sqrt{2\pi}}\frac{e^{iEt}\tilde{F}(E)}{E^2-\omega^2+i\epsilon} \quad .$$

So, this Jacobian is 1 because this was not dependence on q and in which are a dimensional space you want to think of some kind of infinite dimensional space, but it is still identity. All I have to do is make sure that I can still write without any loss of generality

$$D\tilde{q}(E)J(\frac{\partial q}{\partial \tilde{q}})=DQ(E) \quad .$$

We know that this is function of E and this is t, but we know there is a relation.

So, we can always workout. Main thing is that we want to move over to doing an integration over big Q rather than small q. If you try to go from time domain to the energy or frequency domain, then again the Jacobian will be variation of this with respect to variation of G˜ which will leave behind delta function of 0 or something, you know some in ill-defined quantity, but which does not depend on q itself. So, whatever the Jacobian it is a constant ok.

So, this constant overall constant, I am not going to worry about. We will leave it outside. The point is you will leave this measure in the time domain, this you can always convert back and to give you the usual t expression but in q. So, we do this only the big Jacobian, we do only in the known variables in known time domain and now observe that the front first factor

$$\frac{\tilde{F}(E)\tilde{F}(-E)}{E^2-\omega^2+i\epsilon}$$

doesn’t depend on of Q and

$$\langle q_f,\infty|q_i,-\infty\rangle_F=\int DQ\exp[\frac{i}{2}\int_{\infty}^{-\infty}dt[\dot{Q}^2(t)-(\omega^2-i\epsilon)Q^2(t)]]\exp[-\frac{i}{2}\int dE\frac{\tilde{F}(E)\tilde{F}(-E)}{E^2-\omega^2+i\epsilon}]$$

Because here I can certainly convert back to time domain without worrying and it will give me back my word Lagrangian except in the variable big Q instead of small q and times this additional exponent. We will next try to convert that term also to time domain. This front factor is simply seems to be the answer when forcing function is 0. So that becomes

$$\langle q_f, \infty | q_i, -\infty \rangle_{F=0} \exp[-\frac{i}{2} \int dt\, dt' \, F(t) D(t-t') F(t')] \quad ;$$

where, the D is the time domain propagator

$$D(t-t') = \int \frac{dE}{2\pi} \frac{e^{iE(t-t')}}{E^2 - \omega^2 + i\epsilon} \quad .$$

So, sometimes we will write short form for it which is also equal to $<F_1 D_{12} F_2>$. So, this we will use as a notation for this andwe still seems to be stuck with this q_f and q_i which look arbitrary. So, what is actually done is also to extend the space range.

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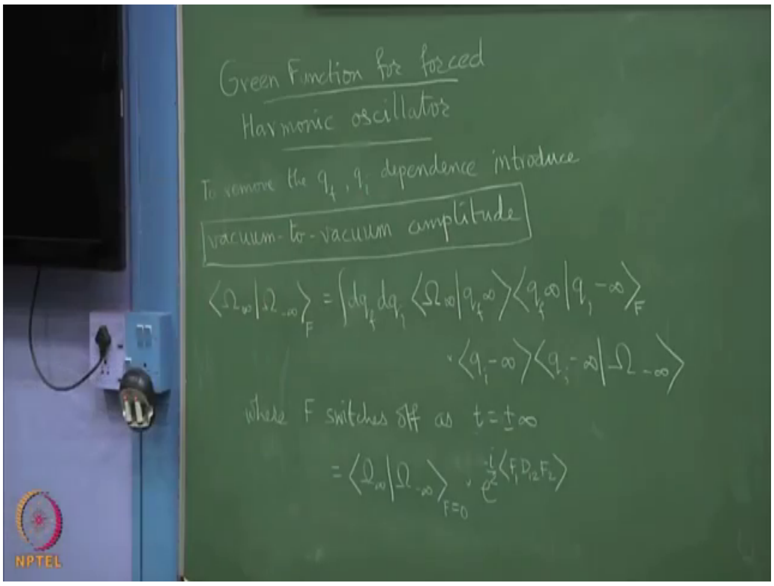

So, what we do is introduce vacuum to vacuum amplitude. Just for the possibility that after you have done some drama that two vacua may differ by some overall phase. So, they may not be exactly identical, but they will be same, they will both will be normalized to 1. But this then is

$$\langle\Omega_\infty|\Omega_{-\infty}\rangle = \int dq_f\, dq_i \langle\Omega_\infty|q_f \infty\rangle\langle q_f \infty|q_f - \infty\rangle\langle q_f - \infty|\Omega_{-\infty}\rangle \quad .$$

This vacuum to vacuum amplitude is a very important concept. So, we can show that this also becomes equal to similar kind of thing and overall thing without any j. So, suppose we do this with the presence of F; then, we would have the presence of F over here, but F is supposed to switch off as +∞ and -∞. In fact, F is in time domain.

And therefore, it doesn't appear here does not affect the overlap over there. So, then that factor that we have there the FDF just comes out of the whole thing and we get that this equals

$$\langle\Omega_\infty|\Omega_{-\infty}\rangle_{F=0} \exp[-\frac{i}{2}\langle F_1 D_{12} F_2\rangle] \quad ,$$

but this is what we wanted because we had made that proposal that any average is can be calculated by varying this now with respect to this external current. So, F is sometimes well, right now it is not a current it is just forcing function in field theory it becomes a current. Many of these ideas are actually developed by swinger and they have eventually come back although in early developments for some time it was ignored.

So, this idea of constructing vacuum to vacuum amplitude writing a generating functional and by introducing external force in function with respect to D, F will bring down while a q. So, we can calculate a qq correlator by varying which respect to F twice. So, this can be used.

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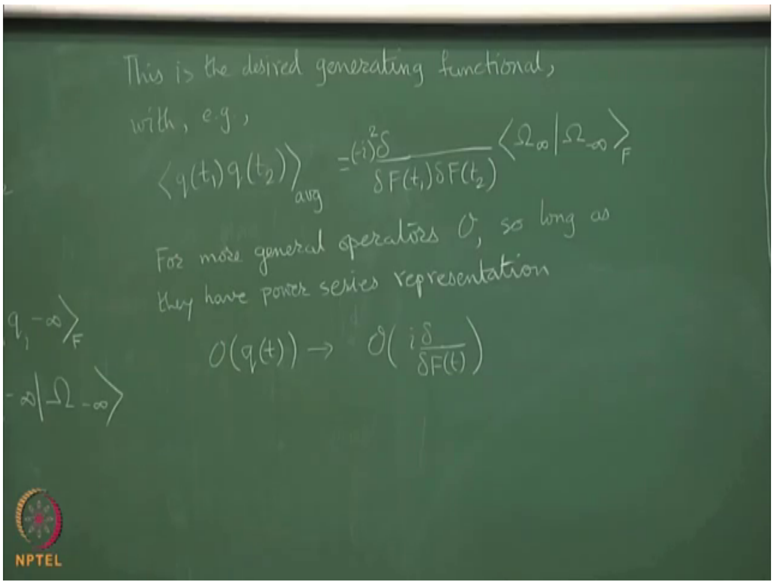

So, this is the desired generating functional, with example

$$\langle q(t_1)\, q(t_2)\rangle_{ava} = (-i)^2 \frac{\delta}{\delta F(t_1)\,\delta F(t_2)} \langle \Omega_{\infty} | \Omega_{-\infty} \rangle_{F=0} \quad ,$$

this is now the average; this is not quantum mechanics. If you have any operators all you have to do is write

$$O(q(t)) \rightarrow O\left(i \frac{\delta}{\delta F(t)}\right)$$

whatever power series you have in q, you just construct that same power series in variations with respect to absolute. You get higher powers of variation and now that we have committed so many since we even if you have a transcendental function all you do is replace wherever you see a q by id/dF is the same answer ok. So, you can calculate the average value of any operator or products of such operators by doing this.