Path Integral and Functional Methods in Quantum Field Theory

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Lecture – 08
Correlation Functions – II

So, now we go on to the more formal aspects and luckily this book of Raymond has a nice bridge to going from single particle Quantum mechanics to the Field Theory. So, towards transition to the quantum theory and let me also make some pre remarks and then repeat them again because these are very important remarks without which we will all get lost in formalism. So, some remarks can be repeated.

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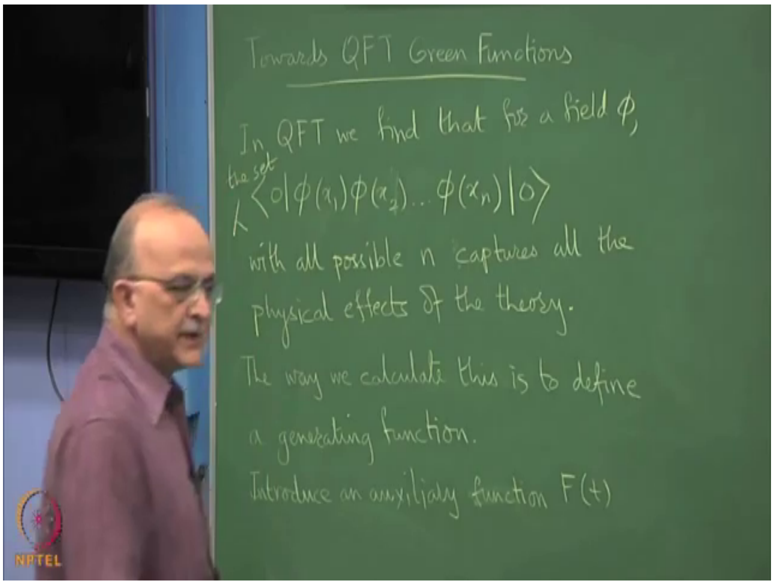

So, we can say towards QFT Greens functions. So, in QFT we find that for a field ϕ_1 the set $<0|\phi(x_1)\phi(x_2)......\phi(x_n)|0>$ with all possible values of n captures all the physical effects of the theory. So, the purpose of quantum field theory or any quantum theory that you have is to be able to calculate endpoint functions. What the mean and how we use them will come out later, but you can just think about the fact that if you had some very general function if you know all its moments in terms of statistics then you can always recover the original function.

So, the contents are all in all the n-point functions and that is what we are going to move towards and towards that let me also give away the other main trick. So, the way we calculate this is to define a generating function. Now generating function will mean that we introduce an auxiliary function some F(t) we have q(t), p(t), but there is some forcing function F(t).

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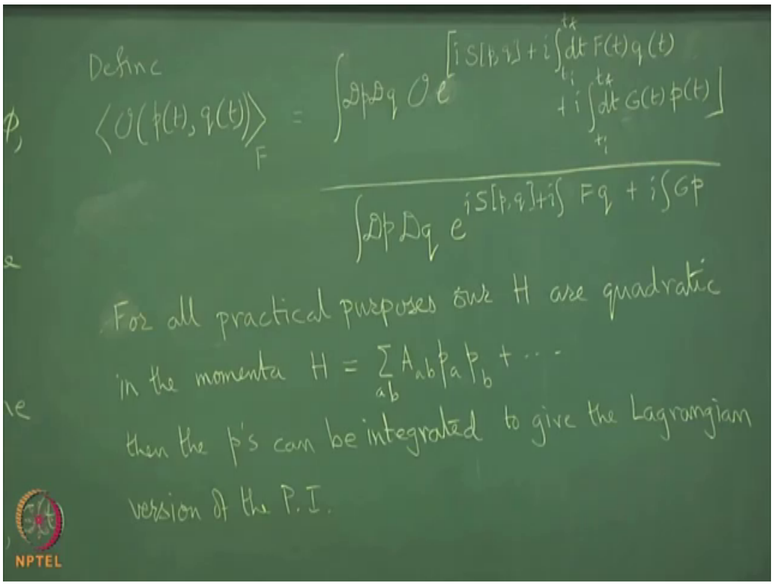

And then we define in the presence of F

$$\langle \mathrm{O}(p(t),q(t))\rangle_F = \frac{\int Dp\, Dq \,\mathrm{O} \exp\left(iS[p,q] + i\int_{t_i}^{t_f} dt\, F(t) q(t) + i\int_{t_i}^{t_f} dt\, G(t) p(t)\right)}{\int Dp\, Dq \exp\left(iS[p,q] + i\int_{t_i}^{t_f} dt\, F(t) q(t) + i\int_{t_i}^{t_f} dt\, G(t) p(t)\right)}$$

So, you have to divide out by the thing without the insertion of O. Now we shall be using the path integral only in the q formalism.

So, up to an overall normalization N we write

$$\langle \mathrm{O}(q(t))\rangle_F = \frac{N\int Dq\, \mathrm{O}[q(t)] \exp\left(iS[q] + i\int_{t_i}^{t_f} dt\, F(t) q(t)\right)}{N\int Dq \exp\left(iS[q] + i\int_{t_i}^{t_f} dt\, F(t) q(t)\right)}$$

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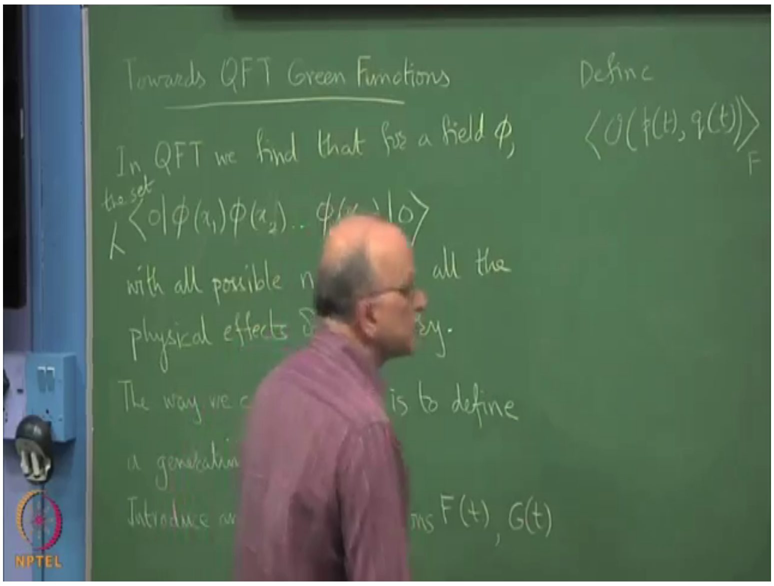

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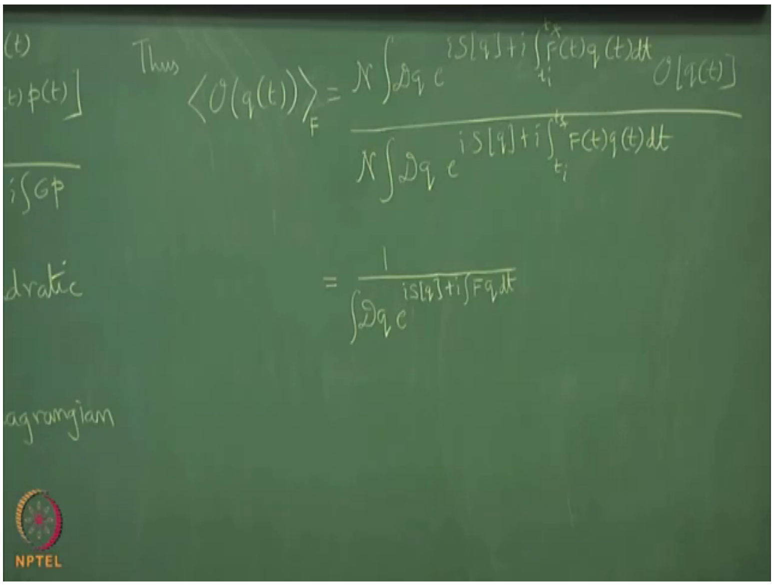

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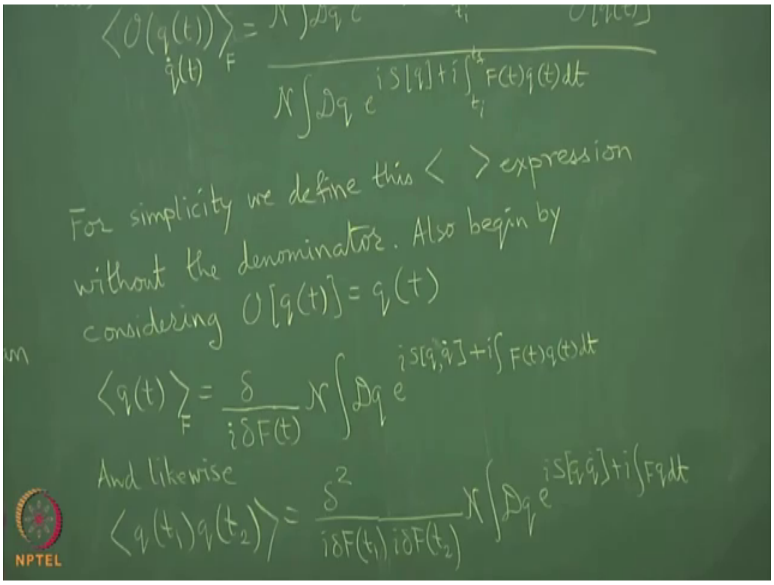

So, this is technically how it is defined I mean in real life to calculate averages, but for the time being we define this expression without the denominator, also begin by considering just q. So, the main trick time about to talk about is that the expectation value of q of t then is simply equal to variation with respect to F(t) of this object. So,

$$\langle q(t)\rangle_F = \frac{\delta}{i\,\delta F(t)} N\int Dq \exp\left(iS[q,\dot{q}] + i\int_{t_i}^{t_f} dt\, F(t) q(t)\right) \quad .$$

So, this is clear right because if you vary with respect to F then a q will come down and it will automatically reproduce this where O is q and then the exponential will remain as it is. So, in this is standard state matrix right. So, that is what we are using. So, and likewise let us say

$$\langle q(t_1) q(t_2)\rangle_F = \frac{\delta^2}{i\,\delta F(t_1)\, i\,\delta F(t_2)} N\int Dq \exp\left(iS[q,\dot{q}] + i\int_{t_i}^{t_f} dt\, F(t) q(t)\right) \quad .$$

So, now what we will do is to introduce the harmonic oscillator in this language as a warm up to the real thing. We will find that there is a nice analogy of non relativistic harmonic oscillator to the relativistic propagator because the relativistic propagator is of the form $(p_0^2 - (\vec{p})^2)$ that acts like $(\omega^2 - p^2)$ the harmonic oscillator; Hamiltonian is quadratic in q and p so, it matches exactly.

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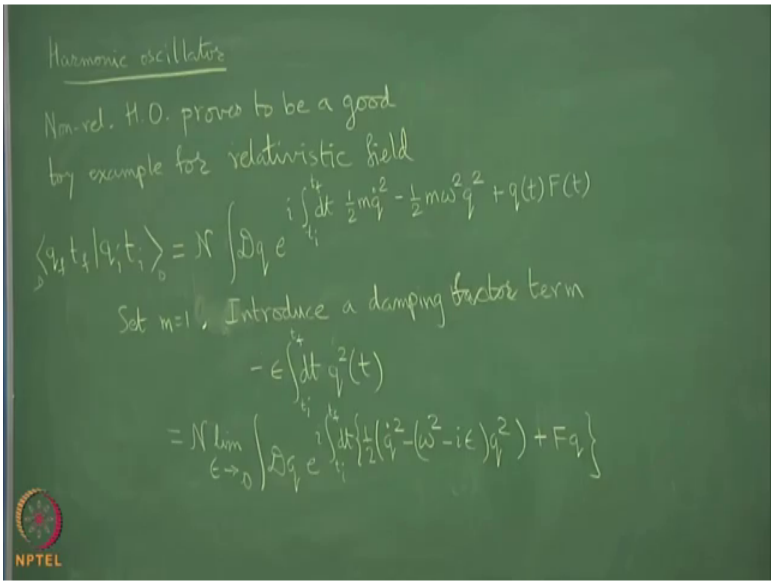

So, now consider taking the harmonic oscillator. So, this transition amplitude in this case is aside from the N which we accumulate due to the momentum space integration

$$\langle q_f t_f | q_i t_i \rangle = N \int Dq \exp\left(i \int_{t_i}^{t_f} dt \left(\frac{1}{2} m (\dot{q})^2 - \frac{1}{2} m^2 \omega^2 q^2 + qF\right)\right) \quad .$$

So, for convenience we set m=1 and we also introduced a damping factor $\exp(-\epsilon \int dt\, q^2(t))$. So, then this is equal to

$$N \int Dq \exp\left(i \int_{t_i}^{t_f} dt \left(\frac{1}{2} (\dot{q})^2 - \frac{1}{2} (\omega^2 - i\epsilon) q^2 + qF\right)\right) \quad .$$

So, note that this ϵ is such that as if you extend the t_i to t_F to very large values and it picks up a large contribution to this then it will make the path integral go to 0 it will make it converge in the limit of t_i and t_F going to infinity.

But we put it also not because we are worried about our mathematician fringe and convergence for anything it is actually a very clever device which comes useful. So, to tell you because you already know from quantum three in the Feynman propagator you have to have the iϵ prescription (p^2 - m^2 + iϵ) that iϵ comes out exactly correct with the correct sign if you pretend to tell people look I am regulating this by putting a damping factor. So, that is what is nice about the path integral. So, now next thing we need to do is rewrite this in terms of energy integrals. So, since we are run coming closer to end of the class time what I will do is, I will write the what we are going towards and we will prove the formula next time. So, what we can show using this is that

$$\langle q_f \infty | q_i - \infty \rangle_F = \langle q_f \infty | q_i - \infty \rangle_{F=0} \exp\left(\frac{-i}{2} \int dE \frac{\tilde{F}(E)\tilde{F}(-E)}{E^2 - \omega^2 + i\epsilon}\right)$$

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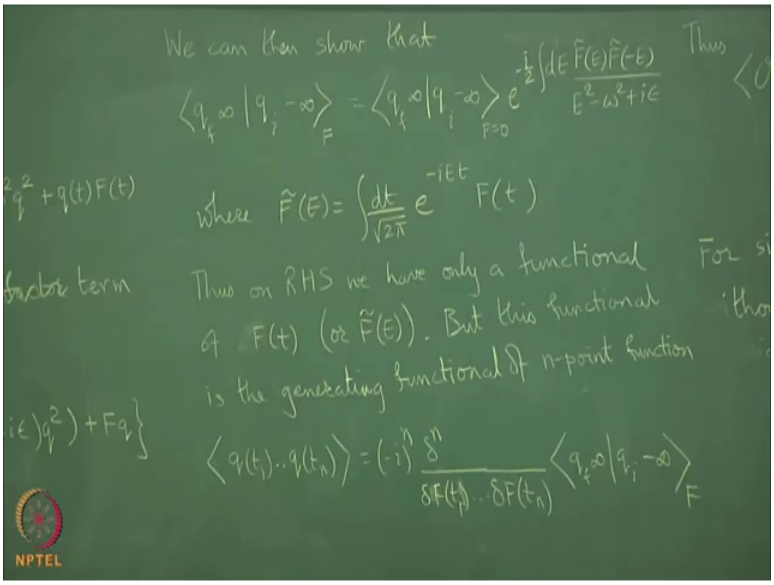

where the F tilde are just Fourier transform.

So, the point is that this object that was to be computed if you just did the without any F then the answer for O cannot depend on any q(t) right because q(t) got integrated out there is a big integral over d[q(t)]. So, the meaning of what we are calculating becomes clear if you insert this F as a temporary device and then let then drop the F in the end after your calculated. So, here I should have said one more step was here these are still in the presence of F, but to get it in the in the end without any F what you have to do is this and then evaluated at F equal to 0 that is the prescription.

So, now we can see that this expression in the presence of F has no q(t) in it and it is a function only of F there is no sign of any q in it, but we can now compute the expectation value of $\langle q(t_1)q(t_2)\rangle$ etcetera by just varying with respect to $F(t_1)F(t_2)$ of course, their Fourier transforms of this variable, but you can extract the 2 point functions out of this.

So, this is the meaning of this is what we call generating functional or generating function from which by varying the auxiliary variable we can extract the n point averages. So, it contains all the information that you need to calculate all the physical quantities of interest ok. So, from here on I am actually just using Raymond's book for next couple of turns. So, you can read it and the comments that I have been making about momentum space the canonical path integral the phase space path integral as being the correct thing is also commented by him.

He derives it using more like the more intuitive Feynman approach and assuming that it is given by Lagrangian, but then before switching on to these topics there is a short discussion of why actually you have to use the p. If you do not then you get some spurious factors which usually do not bother you ever, but for keeping things straight and then you do not have to put any N at all you get the exact answer.