Path Integral and Functional Methods in Quantum Field Theory

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Lecture – 07
Correlation Functions – I

So, we now continue with the path integral formalism and what we are trying to achieve is to create a formalism to calculate Correlation Functions in quantum field theory. So, there are several steps that we go through to get to that point; and I we will explain that in a few minutes. Let us begin with winding up some interesting points about the path integral formula.

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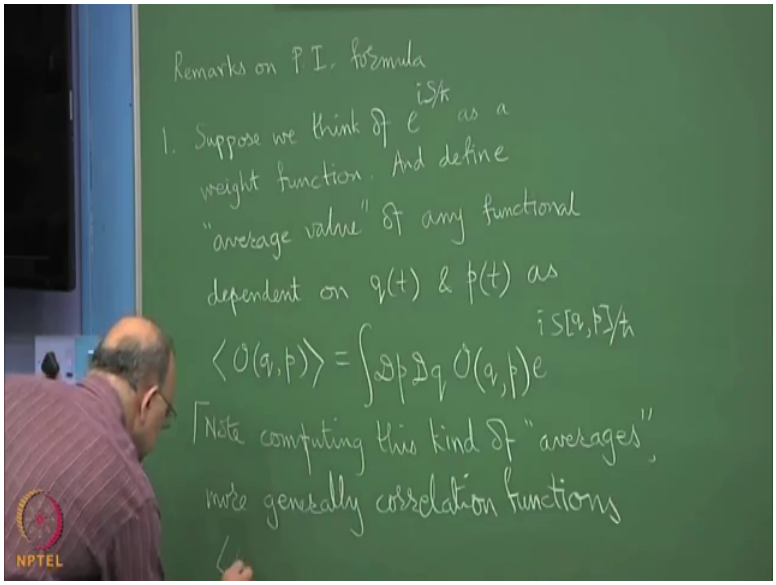

One is that it automatically gives time order products ok. So, the point is suppose we think of $e^{iS/\hbar}$; we might be dropping $\hbar$ very soon so, but let me put it this time. We can think of $e^{iS/\hbar}$ as a weight function. And define some kind of average of any function dependent on q and p

$$\langle O(q,p)\rangle = \int Dp\,Dq\,O(q,p)e^{iS[q,p]/\hbar}.$$

So, it would be actually a functional. We will seen what sense this is an average value and yet not your ordinary average value, but certainly a very important quantity. So in fact, this is how mostly path integral is used ok, for all practical purposes the main

purpose of the path integral is not to calculate a transition amplitude from initial to final state, nobody does that.

What one what does is actually use it to compute certain kinds of a expectation values which we eventually we will call Green's functions.

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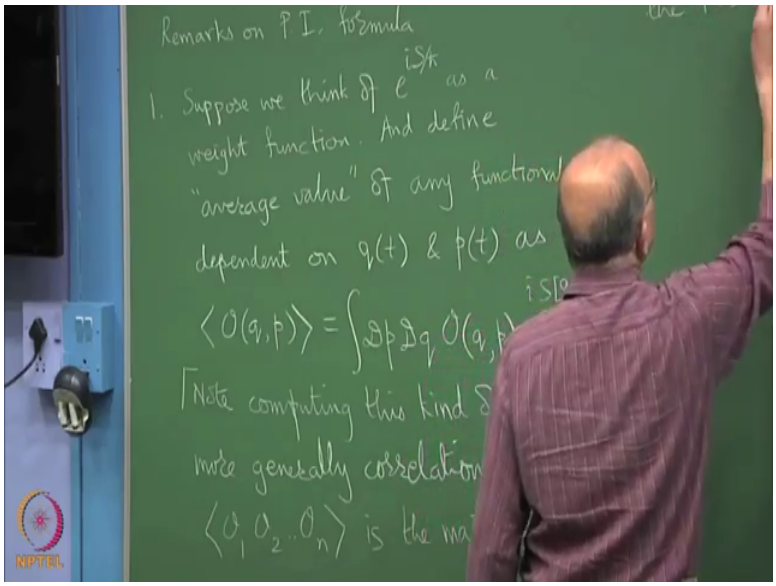

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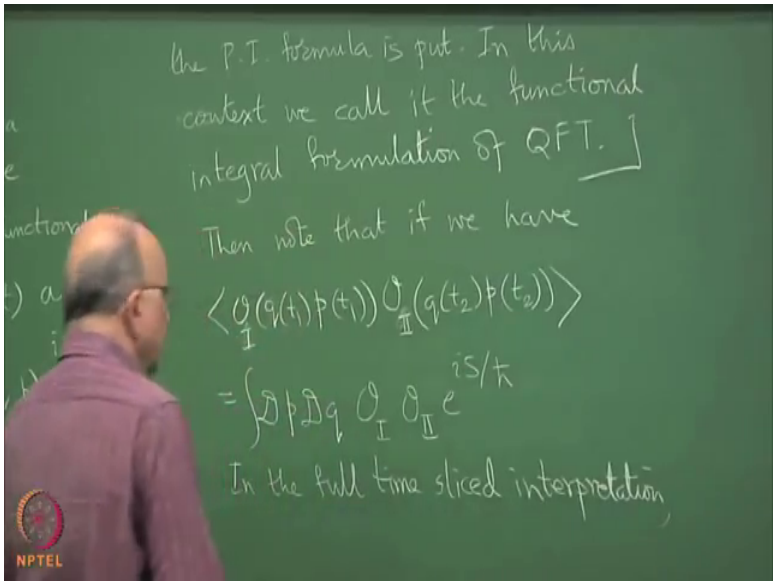

So, that is the end of that side comment. Now, suppose you have two operators at two different times which you are trying to calculate $\langle O_I(q(t_1)p(t_1)) O_{II}(q(t_2)p(t_2)) \rangle$.

But there is a time coordinate we are using for the first operator and another time coordinate you are using for the other operator. Then this will become

$$\int Dp\,Dq\,\mathrm{O}_I\,\mathrm{O}_{II}\,\mathrm{e}^{iS/\hbar}$$

this is what it asks us to do.

This is the definition of that averaged product ok, but now we see what happens if we implement the detail of this, we get a diagram like this.

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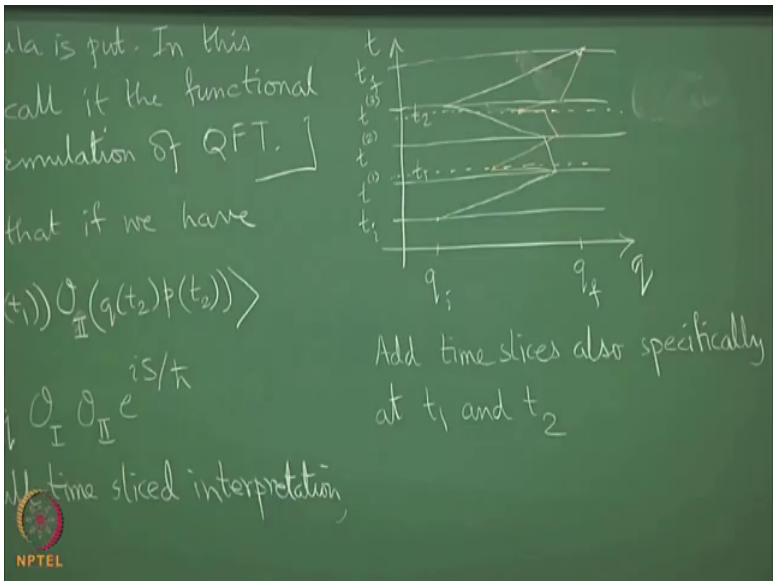

That I have q_t and we are suppressing p because we do not have that many access without getting confused. So, this is t_i and t_f and then let us say I draw t_1, t_2 something like this now I change my notation because I unfortunately used ok.

So, I have time slices, but let us say t_1 is over here and t_2 is over here. So, the instruction says that you will take starting with t_{1i}, q_i to q_f , t_f you; you are suppose to do this right t_2 is here right. So, wherever you like some one path is like this.

Now, because to take account of this t_1 and t_2; what we will do is we can introduce additional slice exactly at that point. So of course, then you can get something like go up to here, then here and then you go there and then there and then there then at t_2; you know of course, you have to reach the same final point.

So, so you could add additional slices and then it is just the matter of the detail of setting up the time slicing. Ultimately, you are suppose to take the limit of the all the slices coming very close to each other; infinite number of time slices. So, automatically the t_1 and t_2 are going to get covered in that, but we are just illustrating that also strategically; in this sliced version before having got to the limit, make sure that t_1 and t_2 are included in the list of slices, but once we do this we see that if t_1 is.

So, the point is the slices are ordered; so if t_1 is less than t_2, then t_1 slice will appear before t_2. In above example t_1 is less than t_2; therefore, in the averaging the operator O_I will automatically appear first when you begin to average it will get averaged first.

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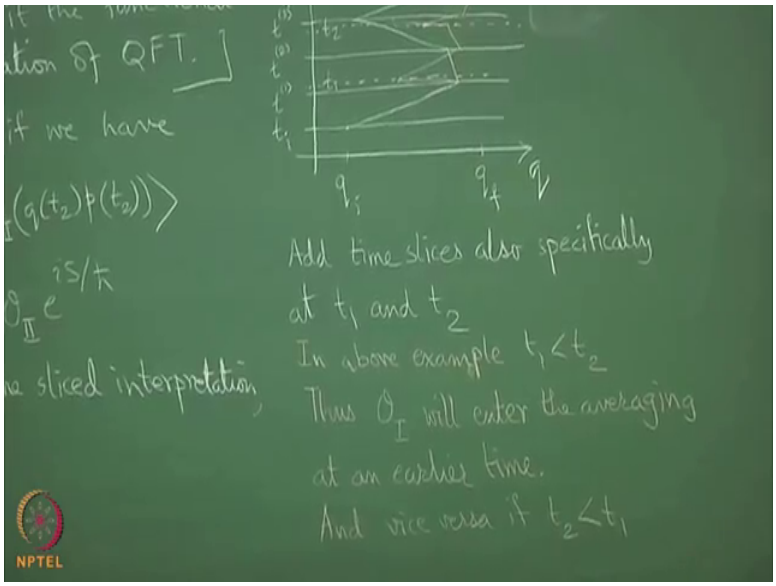

And similarly visa versa; so in quantum mechanics we are concerned about ordering of operators; what we will find is that automatically if t_1 is less than t_2 then t_1 will appear actually to this side because you are slicing like this. So, it will go like this and then if it is t_2 that then II will appear later.

So, in the time ordering prescription what we say. So, what we do is that we write a compact formula for this ok.

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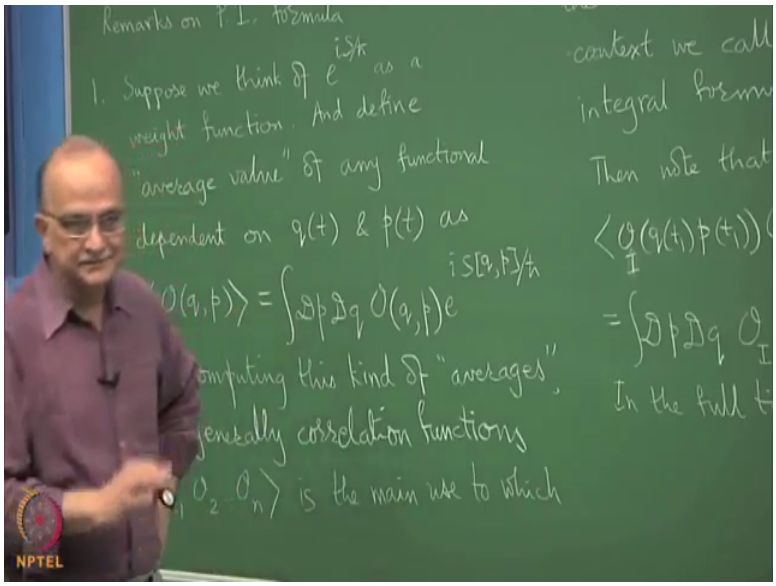

One more important comment to make; note that the path integral is a very interesting formula because it actually involves no operators; the path integral formulation of quantum mechanics does not talk about any operators. It does talk about initial state and final state between which the transition amplitude is calculated.

So, the states are there, but we do not have introduce any operators because all the operators are just classical; it just their going to be integrated functionally by this complicated time sliced measure.

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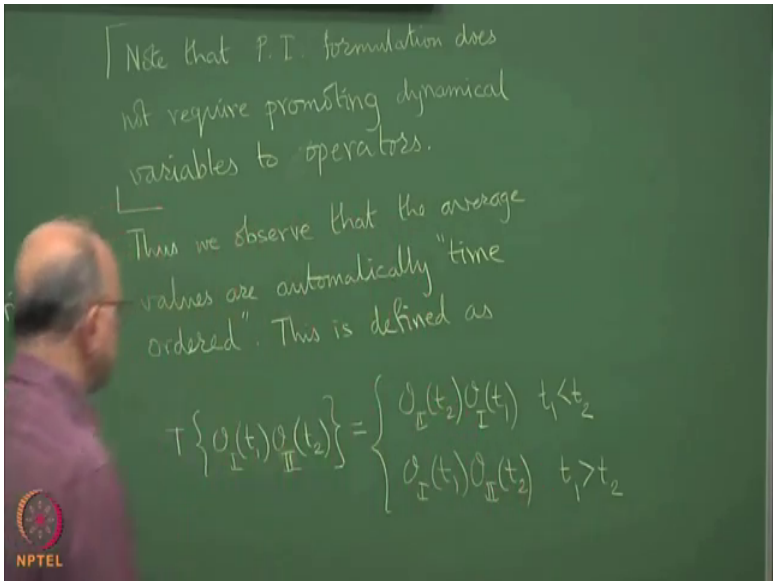

So, one more comment, formulation does not require promoting dynamical variables to operators. We automatically get average values or expectation values by doing the time slice functional integral.

So, by time ordered we mean if t_1 is less than t_2 then it has the meaning of being $O_{II}(t_2)$ $O_I(t_1)$ and if t_1 is greater than t_2 then t_1 will be on this side.

Thus we find that this two point function as you might call it average value will be automatically ordered in the path integral. So, this is still continuing with comment number 1.

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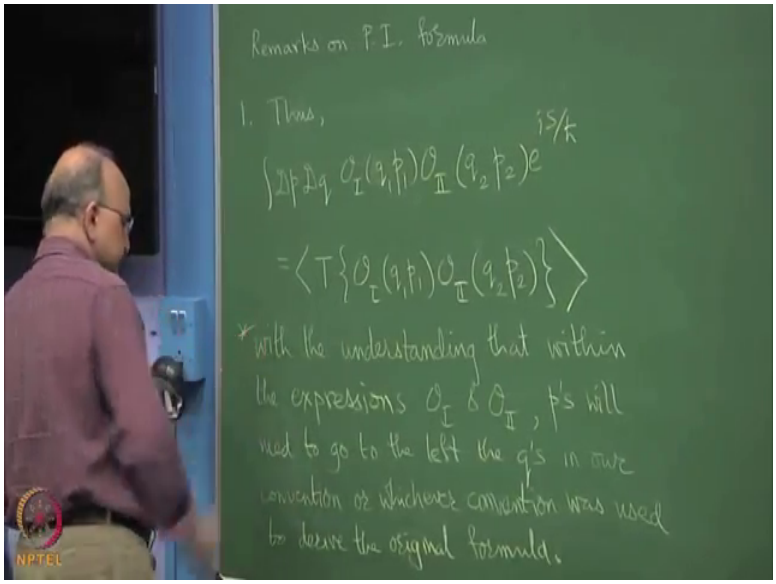

Thus $\int Dp\,Dq\,\mathrm{O}_I(q_1 p_1)\mathrm{O}_{II}(q_2 p_2)\mathrm{e}^{iS/\hbar} = \langle T\{\mathrm{O}_I(q_1 p_1)\mathrm{O}_{II}(q_2 p_2)\}\rangle$.

As you would have done I mean in quantum mechanics the ordered becomes important; so this will automatically give this whether you like it or not. But amazingly you know that is exactly what you like, as you know from your quantum III course right. Now the second comment is about the classical limit.

I write these things out in long details; I am not always very accurate, but it is a marker for you to remember what was said because if you write out formula without the remarks, then you probably not understand what was the motivation for what. So, with the understanding that within the expressions O_I and O_{II}; p's will need to go to left of the q's; you could have of course, defined your whole path integral with some other ordering

prescription; you could have said q's to the right, p's to the left; its matter of slicing, remember how we started inserting the complete set of states? You could have started it from this and instead of that end.

So, whatever prescription you followed their will automatically apply here; p's will go need to the left of the q's in our notation in our convention or whichever convention was used to derive the basic formula. So, so you have to take care of that part and which is where effectively you will be constraint by what quantum interpretation you are giving to you or what is your proposed quantum operator is implicitly there in the formula.

But I mean in terms of formalism when you use it, you do not have to promote, but I guess I agree with you, you will have to remember to order your operator that way. And if your operator is different, then you should have the additional pieces that comes from that normal ordering which should be retained with the operator.

So, this is corrected by this, but I can tell you that; just the where it is written an expression like this which although this is highly nontrivial in terms of real analysis or multivariate analysis and because of the oscillating nothing, Mathematicians will faint several times over looking at this expression because it just makes no sense in anyway at all. But anyway if you look at this then nowhere do you have to put any quantum variable.

And you get a answer for a transition amplitude between one quantum state and the other. And so John Wheeler who had the; who always used to create jokes about this something without something; so he used to say path integral is quantum mechanics without quantum mechanics; Feynman was his student who wrote the thesis with him.

So, path integral is quantum mechanics without quantum mechanics ok; not really as we just learnt. But it looks; it is good to tell people like that when they do not fully know the integrities alright.

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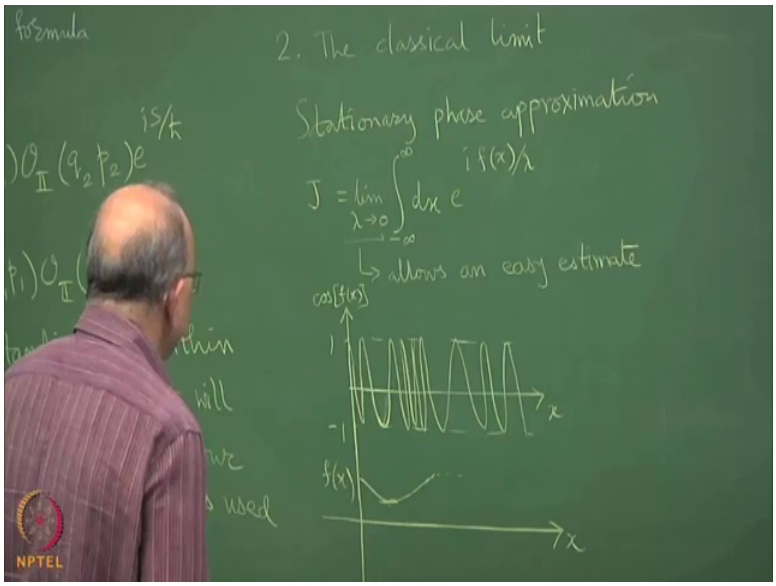

So, the other comment was that the classical limit. To understand this we need to understand what is called stationary phase approximation in integration. So, suppose we had an integral $J = \lim_{\lambda \to 0} \int_{-\infty}^{\infty} dx\, e^{if(x)/\lambda}$.

Suppose, I am integrating something that has e raised to i times sum function of x; which is what the form of our path integral is. To keep track of this $\hbar$ or putting some scale in the problem, we insert a book keeping constant λ. But then consider this in the limit of λ going to 0 or λ small ok; you do not have to take the technical limit, but what we mean is as λ gets smaller, it is possible to or as at an estimate of this integral.

So, the point is that now we look at this; it is the integral dx. Suppose it was not any complicated f, but just e^{ix}; then you know that it is (cosx + isinx). If you integrate over the whole range, you are going to get nothing; it is just going to keep cancelling, you will get some oscillatory answer depending on where you stop. But overall it is just it just gives you nothing definite; as you go through several cycles, they keep cancelling each other. So, you do not get any very definite answer, but certainly not any divergent answer either.

So, if you then try to think of what happens to this $e^{if(x)}$, you can try to plot let us say the real part of it. So, suppose we plot real part of or that is to say cos[f(x)]. Now whatever f is; cos can only osculate between +1 and -1. So, without worrying too much where the 0

of f is and so on, I just draw some random curve. It just keeps oscillating between -1 and +1, but now let us specialize on some points.

So, such a thing might arise by let us say if f was monotonically progressing from here to here; so over this it is just as if it is cosx right; it is happening rather regularly. So, you might say it is just going something straight like this and then it may turn over ok. So, suppose f is drawn here; what you will find is that when f is doing something nontrivial, this will keep oscillating, but precisely where f actually goes through a minimum, it is stationary; if f is not advancing then this cos function will stay put there.

So, what you say is that; I will look at the points where f has extrema is where the progress of the argument of cosine is going to slow down and for some time it will stain here a constant value.

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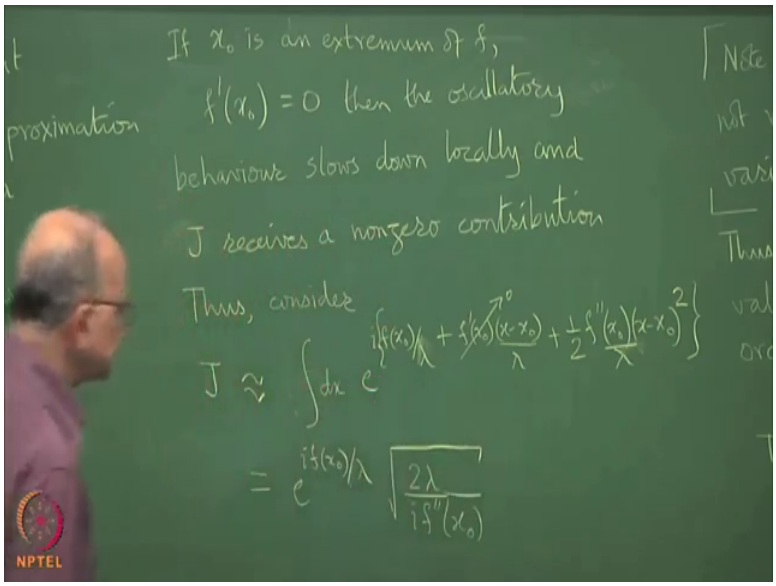

So, we; so if x_0 is an extremum and J receives a non-zero contribution. And the point is as you make λ smaller; f will become much the exponent will become much larger. So, small changes in f will make f/λ run over a long range. So, it will just make the oscillations much faster; as λ goes to 0 except at the point where it has actually hit a minimum. So, we get

$$J \approx \int dx \exp\left(\frac{if(x_0)}{\lambda} + \frac{1}{2}\frac{if''(x_0)}{\lambda}(x-x_0)^2\right) \quad .$$

So, it is approximate; I did forget the higher order terms and we proudly retain this term because we know how to do fake Gaussian integrals.

So, all we have to do is that therefore, this becomes equal to $e^{if(x_0)/\hbar}\sqrt{\frac{2\lambda}{if''(x_0)}}$, but now the consequence of this in our case is that the path integral receives one. And, so we can generalize to several instead of one extremum, if at several all it becomes is a sum over each of them with contribution like this from each of them. Aside from the stationary phase there the Gaussian fluctuations you get there.

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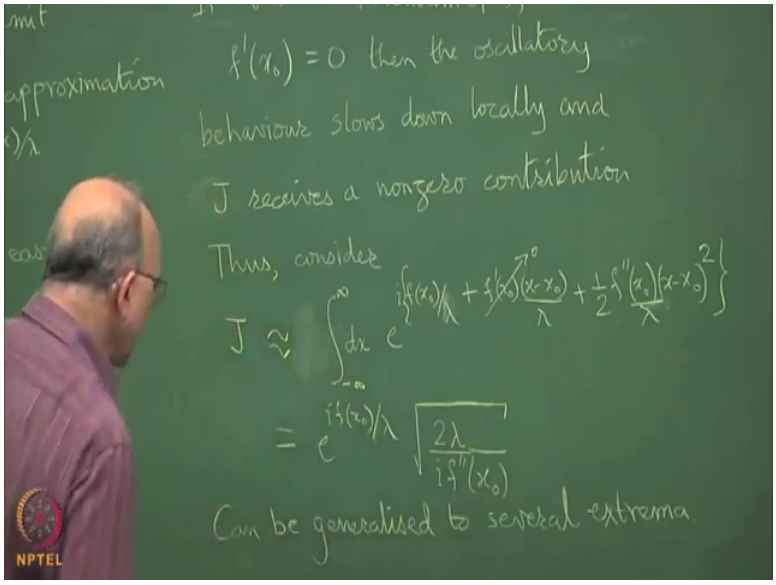

So, can be generalize to several extrema; note that at any one point you did this; now you go to another point you say, but when they when I am integrating there this enters; no, you do not have to worry. Do you put this expression for one and because all the nearby points is just going to give you in 0 because of oscillatory behaviour. If you once estimated like this, it will actually become just sum over all the extrema happily with integral minus infinity to infinity because everything else at each of them is just giving 0's. So, you can just generalize it by putting sum over n; so it can be generalized to several extrema.

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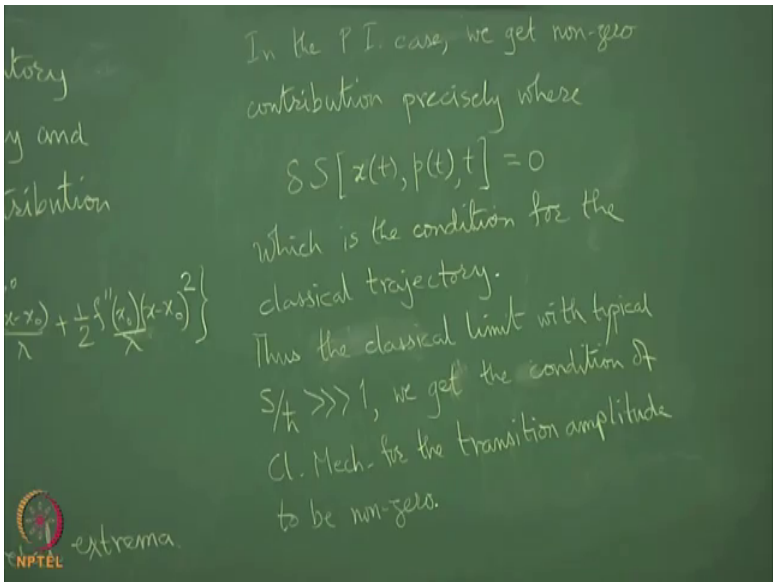

So, in the path integral case, we get maximum contribution or we get non-zero contribution precisely where $\delta S[x(t), p(t), t] = 0$. So, which not put any below; you can put with respect to x or with respect to p both, but whenever the variation of S is 0 and now S is a functional of the trajectory x(t), p(t). So, this is the classical condition; so, on the classical trajectory the path integral will give this answer some overall constant which is not of importance. But the point is that is what will dominate; the classical trajectories will dominate.

And, remember that S has to be measured in the units of $\hbar$ let us has this f was measured in the units of λ. So, if you are living in a world where S is much much bigger than $\hbar$, then any tiniest variation of the path from classical will make this f run so fast that it will wiggle so fast that it will contribute nothing to the integration.

So, that is the answer ok; so we can leave it at that. Quantum mechanics is this strange theory where you can superpose various states. So, if you have a cricket ball then its location here it momentum this can be superposed which its location there with momentum that and so on. But the fielder out there is managing to catch the ball where is supposed to because all the deviations that happened on the way are all canceled out ok; gets it exactly with correct q and p.