Path Integral and Functional Methods in Quantum Field Theory
Prof. Urjit A. Yajnik
Department of Physics
Indian Institute of Technology, Bombay

Lecture - 06
Path Integral Formulation – IV

Suppose we want to start with the simplest thing possible which is try to calculate the kernel for a free particle.

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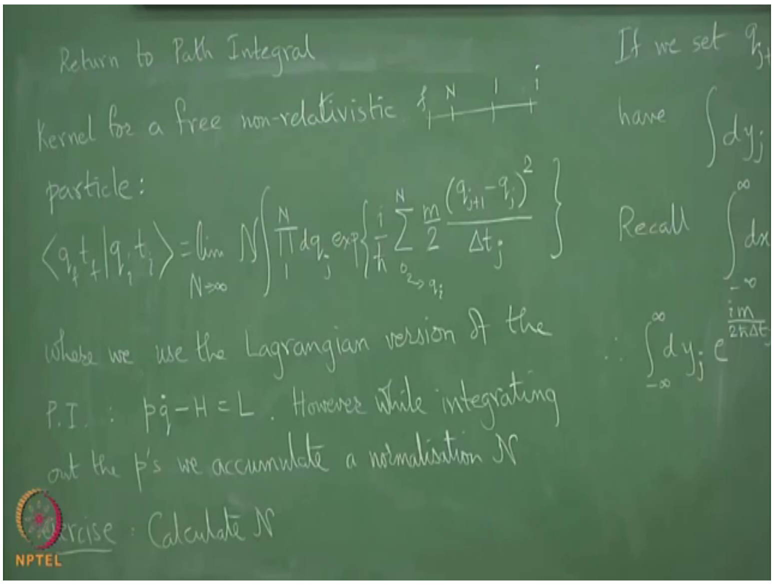

So, we return to the path integral. Sorry this notes are written in terms of classical mechanics terminology q rather than x, so I will not try to change back to x. So, we had this $L = p\dot{q} - H$, but what I am going to do now is to write the Lagrangian version of the thing directly. However, the p integrals are there and that is what this unknown and is for it is actually not an unknown you can calculate it exactly.

So, exercise calculate N. So ignore all these technicality right now, what I want to demonstrate is that this thing can be calculated from an expression like this and the key trick is to use or rather abuse the Gaussian integral many-many times. Note that

$$\langle q_f t_f | q_i t_i \rangle = \lim_{N\to\infty} N \int \prod_1^N dq_j \exp\left[\frac{i}{\hbar}\sum_0^N \frac{m}{2}\frac{(q_{j+1}-q_j)^2}{\Delta t_j}\right] \quad .$$

Now, what we are going to do is, this looks somewhat easy to do because instead of q, I should use differences of q, then it is just of a Gaussian form right, say suppose it is $q_{j+1} - q_j = y_j$ as the integration variable, then it is

$$\int dy_j \exp[\frac{i}{\hbar}\frac{m}{2}\frac{y_j^2}{\Delta t_j}]$$

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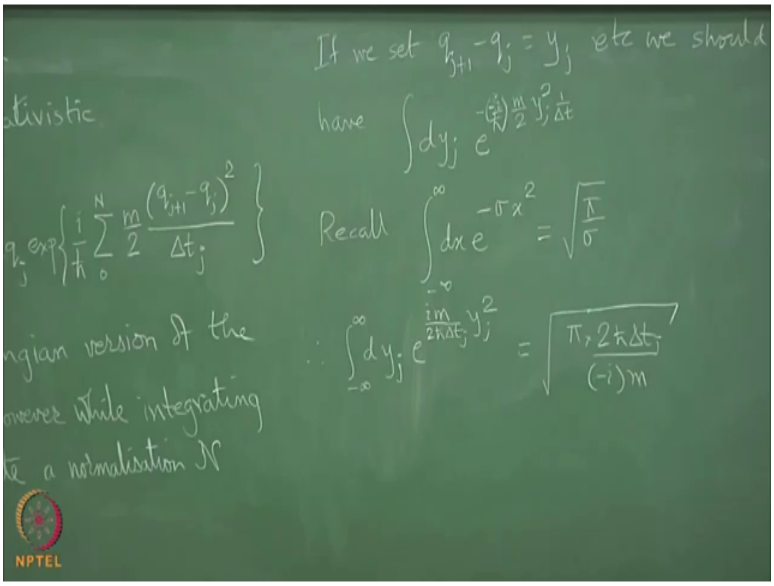

and that is a bit like the Gaussian integral if between friends we imagine that if i is actually a minus sign. So, we say that is minus sign times -i, so I will treat the -i as some constant and just do a Gaussian integral.

So, if we do this Gaussian integral what answer do we get? So, we agree that is what will come out of this we just have product of lots of these. So, let us do this elementary Gaussian, so recall that

$$\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-\sigma x^2} = \sqrt{\frac{\pi}{\sigma}} \quad .$$

So, in this case we should get let me write this quite clearly

$$\int_{-\infty}^{\infty} dy_j \exp[\frac{i}{\hbar}\frac{m}{2}\frac{y_j^2}{\Delta t_j}] = \sqrt{\frac{2\pi\hbar\Delta t_j}{(-i)m}} \quad .$$

So, that is what we use as the basic tool. So, we go here and do this, but there is one minor technical problem which turns out later to be interesting, it is that the number of differences is one larger than the number of integration variables right because you have from 0 to N. So, if I have three differences, but then there are 4 points.

In other words this integration goes over only one this and this, but then there are three differences. So, if this is i and this is f, then it is 1 to N in between that is right. So, the number of slices and i and f are fixed, so the number of intervals is one more than the number of integration variables. So, 0 becomes q_i and you go up to N, then you get q_{N+1}, so you will get the right.

So, 0 is equal to same as q_i which is at some I should probably be writing because of that yes ok. So, that is actually not an integration variable, but in summation you have to put it because otherwise you will not get the thing. So, what we do is that we introduce one more auxiliary y. So we define like this $y_j = q_{j+1} - q_j$, so there will be a y_0 and we put a delta function to show that extra one is actually not genuine.

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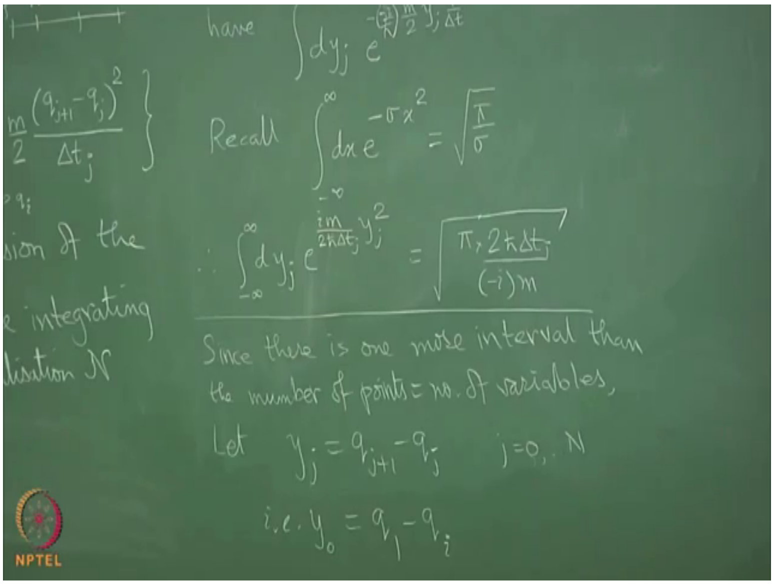

So, $y_0 = q_1 - q_0$ and now we got N+1 y_j is to integrate over. So, j goes from 0 to N.

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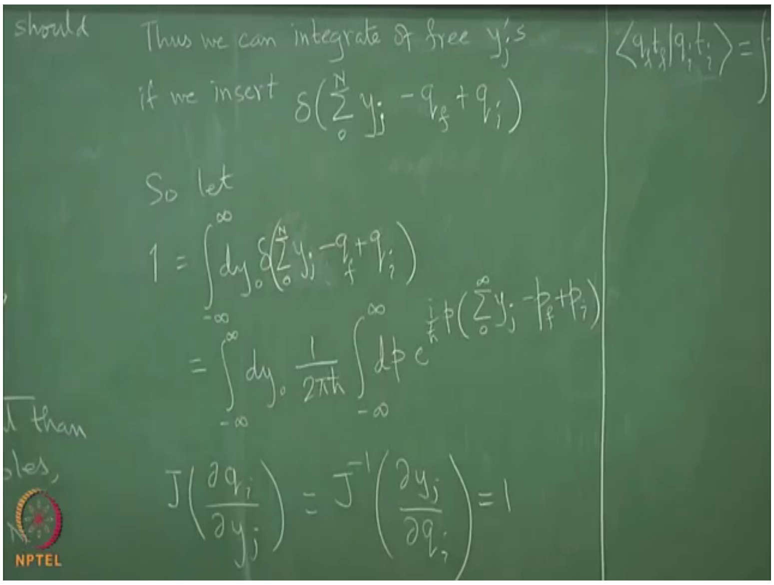

So, we also insert $\delta(\sum_{o}^{N} y_j - q_f + q_i)$. So, we insert an auxiliary p and this we write as equal to, the delta function we write as an exponential representation

$$1 = \int_{-\infty}^{\infty} dy_0\, \delta\left(\sum_{o}^{N} y_j - q_f + q_i\right) = \int_{-\infty}^{\infty} dy_0 \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \exp\left(\frac{i}{\hbar} p\left(\sum_{o}^{N} y_j - q_f + q_i\right)\right) \quad .$$

Yes, so for the delta function we simply wrote the well known representation and we inserted gratuitous $1/\hbar$ and divided by $2\pi\hbar$, so that we match up with all the other $\hbar$ that are occurring here. So, this has to be inserted and once we do this we get.

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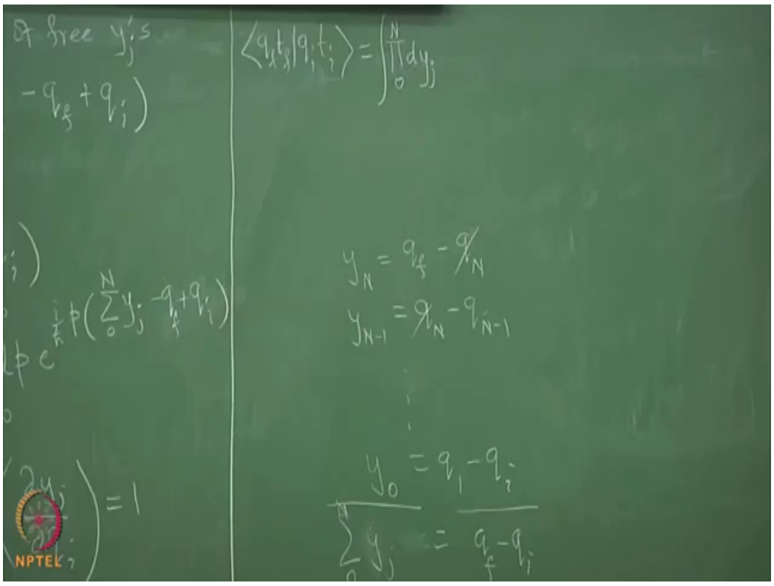

So, ignore the limit N going to infinity part, but we write integral product over instead of i to N, I now write 0 to N and write $\prod_{0}^{N} dy_j$ and I need to put the Jacobean of this transformation. So, that we can calculate here Jacobian of transformation

$$J\left(\frac{\partial q_i}{\partial y_j}\right)=J^{-1}\left(\frac{\partial y_j}{\partial q_i}\right)=1$$

this is what I am putting because I am going to insert it here.

So, that Jacobian is 1 because $\frac{dy_j}{dq_j}=-1$, but $\frac{dy_j}{dq_{j+1}}=+1$ or any other j is equal to 0. So, you just get a product of minus 1,s actually which does not really matter because that is in over all because Jacobian is after all modulus of this, so it is equal to 1. So, the Jacobian of transformation is 1.

So, all you have to do is start writing $y_N = q_f - q_N$, $y_{N-1} = q_N - q_{N-1}$, and going up to $y_0 = q_1 - q_i$,, but if you add then this thing will keep cancelling and so you accumulate that $\sum y_j = q_f - q_i$,.

So, that is what that is and here again you can see directly how the Jacobian will work out because the derivatives are all minus signs, but you do not have to worry about accumulating minus sign because Jacobian is always modulus of that thing, Jacobian as a interpretation of a volume, so it is no sign.

So, we have this integral and then we can put a 1/ $2\pi\hbar$ and then we can put

$$\int_{-\infty}^{\infty} dp \exp\left[\frac{i}{\hbar}\sum_{o}^{N}\left(\frac{m}{2}\frac{y_j^2}{\Delta t_j}+py_j\right)+\frac{i}{\hbar}p(q_i-q_f)\right] \quad .$$

We will have now,

$$\langle q_f t_f | q_i t_i\rangle=\frac{N}{2\pi\hbar}\int_{-\infty}^{\infty} dp \exp\left[\frac{i}{\hbar}p(q_i-q_f)\right]\prod_{0}^{N} I_j \quad \text{with}$$

$$I_j=\int_{-\infty}^{\infty} dy_j \exp\left[\frac{i}{\hbar}\sum_{o}^{N}\left(\frac{m}{2}\frac{y_j^2}{\Delta t_j}+py_j\right)\right] \quad .$$

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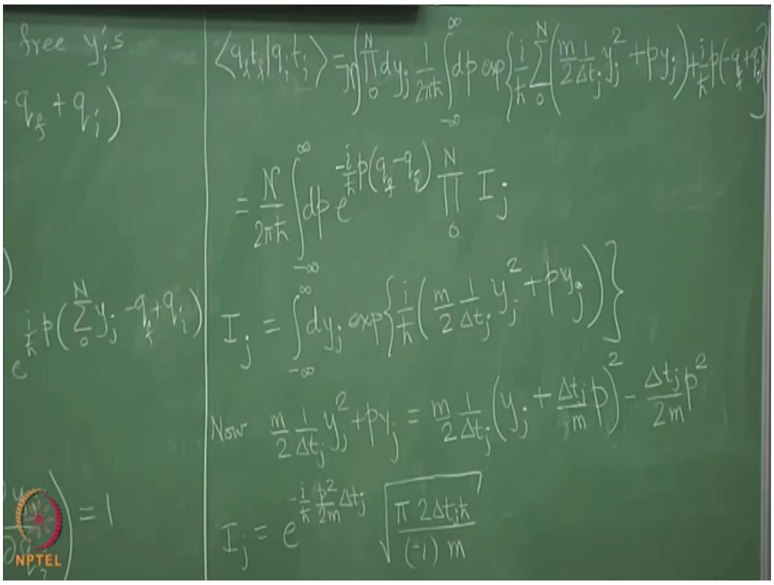

Now, this we do by completing the squares and converting it into a Gaussian right there is a $y_j^{\,2}$ and there is a y_j.

So, yes now to do this we need to add and subtract,

$$\frac{m}{2}\frac{y_j^2}{\Delta t_j}+py_j=\frac{m}{2}\frac{1}{\Delta t_j}\left(y_j+\frac{\Delta t_j}{m}p\right)^2-\frac{\Delta t_j}{2m}p^2 \quad .$$

So now, what we have to do is to shift in this in this integral we have to shift the y_j by this amount which is some constant, but the integration range is minus infinity to infinity, so it does not matter. So,

$$I_j=\mathrm{e}^{-\frac{i}{\hbar}\frac{\Delta t_j}{2m}p^2}\sqrt{\frac{2\pi\hbar\,\Delta t_j}{(-i)m}} \quad .$$

That is what I_j. No we do not have to be frightened by all these because all this masala will cancel that N in front ok.

So, if you do the p integral in the exercise correctly then you are exactly cancel this strange accumulating factors you remember that this is going to get a product and infinite product of Δt_j's in the limit that N is going to infinity, but it is all because it is just an overall factor and actually you will find that if you have done the exercises correctly that the first p integration generates exactly this thing, then they just all cancel.

But we are left now with a nice piece $e^{-\frac{i}{\hbar}\frac{\Delta t_j}{2m}p^2}$ and then there is a product of these. So, there is a summation in them, so it will just become t_f - t_i and so the answer is,

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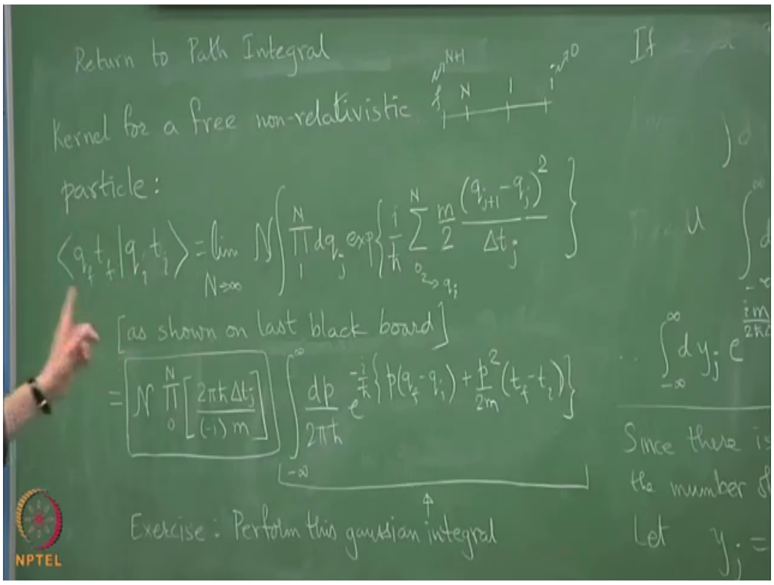

$$\langle q_f t_f | q_i t_i \rangle = N \prod_0^N \sqrt{\frac{2\pi\hbar\Delta t_j}{(-i)m}} \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{-\left(\frac{i}{\hbar}\right)\left[p(q_f - q_i) + \frac{p^2}{2m}(t_f - t_i)\right]}$$

Because now you are become experts at doing this do this one last integral, it is just one more Gaussian there is dp there is p^2 and there is a linear term. So, complete the square and then you can beat it down to a final answer.

So, this part as I am telling you will become 1, if you do exercise number 1. So, if you do this exercise number 2 you will find and answer which is given in Schiff's book as an exercise ok, but not through path integral. So, Schiff is a very very clever book, it has so lot of hidden gems in it. Schiff's and Merzbacher these two books really very nice books for theorist to read.