Path Integral and Functional Methods in Quantum Field Theory
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Lecture - 04
Path Integral Formulation – II

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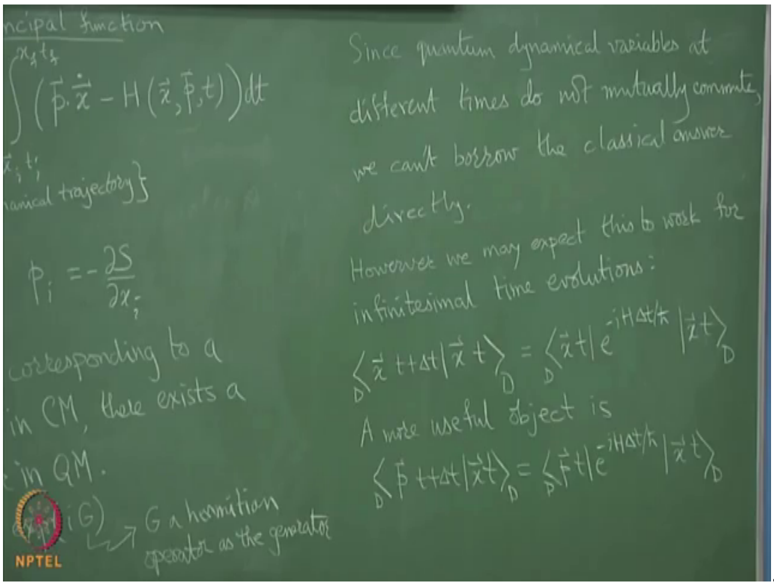

Variables at different times however; we can do this as an infinitesimal transformation. So, we can attempt to calculate Dirac picture state

$$\langle \vec{x}\, t+\Delta t|\vec{x}\, t\rangle_D = \langle \vec{x}\, t|e^{-iHt/\hbar}|\vec{x}\, t\rangle_D .$$

So, this however, is also not very useful is momentum in this because ultimately x will not the time to by this up using only x will not work. So, what we do is, we compute

$$\langle \vec{p}\, t+\Delta t|\vec{x}\, t\rangle_D = \langle \vec{p}\, t|e^{-iHt/\hbar}|\vec{x}\, t\rangle_D .$$

So, what we are doing is assembling the pieces of the whole kernel and they will put together in the end.

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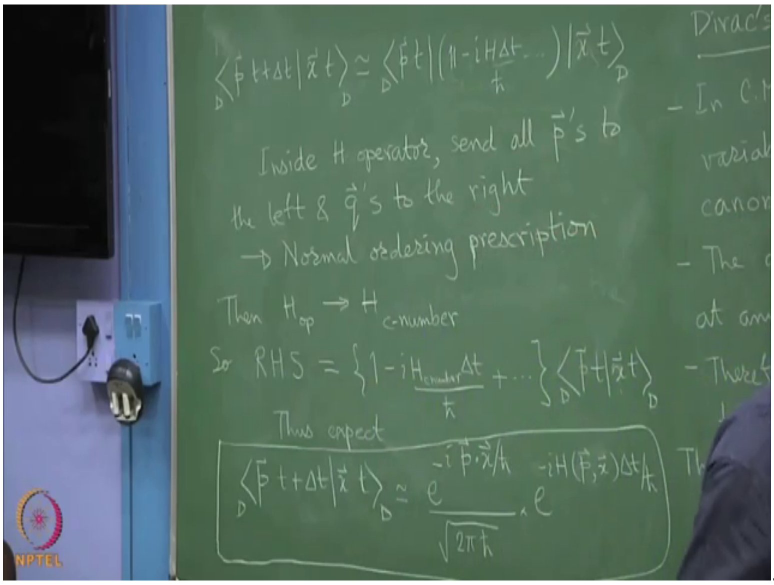

So, this is also the time-dependent momentum basis done in the same way, then

$$\langle \vec{p}\, t+\Delta t | \vec{x}\, t \rangle_D = \langle \vec{p}\, t | 1 - \frac{iHt}{\hbar} + \ldots | \vec{x}\, t \rangle_D$$

Now we come to the point of why we put p and x, the point is this is an operator H, but now I can replace it with it is eigenvalue between p and x and to do it inside the H, I take all the p to the left, all the x to the right, then the p's will act to the left, x will act to the right, they will become numbers.

So, and this is the logic behind doing p x, this is the normal ordering prescription, remember postulate number 5 quantum kinematics. Quantum kinematics said that we have a great convenience of expressing any operator in terms of the basic canonical set q p and except for the problem for the little irritation that when we try to do it for operators that are higher powers of x and p, there is problem of ordering the x' s and p's.

So, that same problem occurs here as well, you will find in literature some people are saying because the path integral is just some over some paths and it is all classical expression there is no operator ordering problem. That's not true. If you just follow everything properly, this is the precise point at which operator ordering also enters in path integral. This makes the path integral formula exactly equivalent to the Heisenberg formulation; no more and no less ok.

So, we have to do some operator ordering here. So, after we do this normal ordering we get it equal to it becomes c - number and we can assemble this to be equal to

$$\text{RHS} = \left\{ 1-\frac{iH_{c-number}\,\Delta t}{\hbar}+\ldots \right\} \langle \vec{p}\, t|\vec{x}\, t\rangle_D \,.$$

So, expect that

$$\langle \vec{p}\, t+\Delta t|\vec{x}\, t\rangle_D \approx \frac{e^{-i\vec{p}.\vec{x}/\hbar}}{\sqrt{2\pi\hbar}}\, e^{-iH(\vec{p},\vec{x})\Delta t/\hbar}$$

So, this is going to be one key component of the final part of the derivation, which now becomes very simple except for the complication of writing it out. So, then I will just tell you the answer I mean what is done next.

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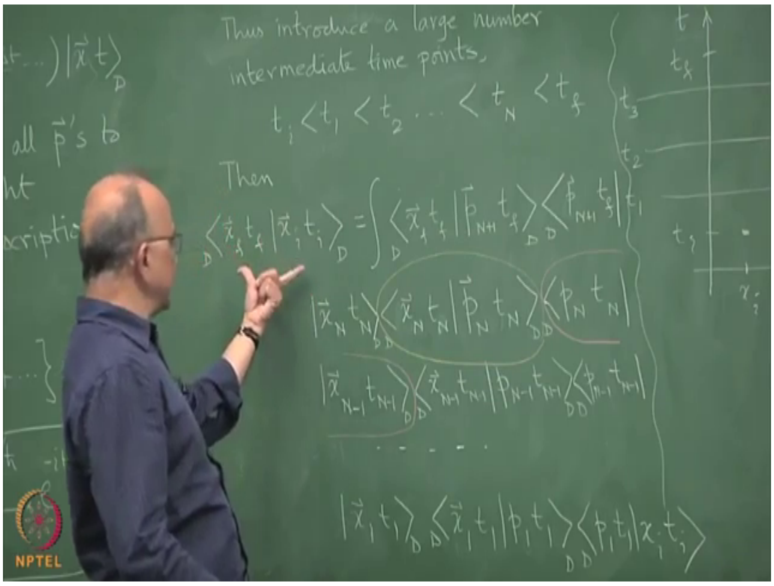

So, what we so, instead of just saying finally, consider let me see, let me say thus introduce a large number of intermediate time points with values so, t_i so, we will introduce N of them less than t_f. So, I am going to write the formula here, but already to give you a picture what we are doing is time slicing.

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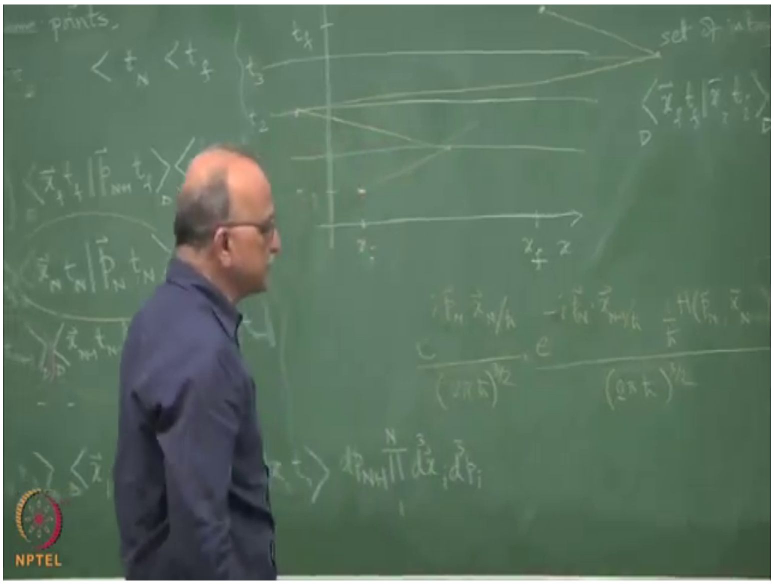

This of course, was then later by Feynman, but the time slicing was done by Dirac, but he did not draw any pictures. So, well in Feynman's where we draw x in this direction and t in that direction and we have some x_i,x_f and t_i,t_f and what we are doing is introducing for convenience, let us put 3 of them. And so, between this point and that point there are these 3 slices time slices; t_1,t_2,t_3, what turns out is that you have to introduce one more q, a one more p then, the q's because we want to write.

So, there is an integral sign which is going to become a gigantic integral sign in the end

$$\langle \vec{x}_f t_f | \vec{x}_i t_i \rangle_D = \int \langle \vec{x}_f t_f | \overrightarrow{p_{N+1}} t_f \rangle_D \langle \overrightarrow{p_{N+1}} t_f | \vec{x}_N t_N \rangle_D \langle \vec{x}_N t_N | \vec{p}_N t_N \rangle_D \langle \vec{p}_N t_N | \overrightarrow{x_{N-1}} t_{N-1} \rangle_D \ldots \langle \vec{x}_1 t_1 | \vec{p}_1 t_1 \rangle_D$$

$$\langle \vec{p}_1 t_1 | \vec{x}_i t_i \rangle_D \, dp_{N+1} \prod_1^n d^3 x_i d^3 p_i$$

right. So, there is of course, this as well, but and everything is Dirac picture a somewhat crucial detail that most people forget to write.

Because these are basis states; these are not wave functions. So, they normally they cannot have time-dependent they are basis, but this is the time-dependent basis. It is actually a sort of motivation for Dirac's later introduction of the interaction picture. You know the interaction picture where you split H_0 and $H^{'}$ and then the basis are all evolved using H_0, the time-dependent basis in which you calculate the S-matrix so, right.

So, this is the main trick and now you can see what is happening because I have two kinds of contributions probably even have a coloured chalk here. So, I have this which is x p over lap at the same time or I have this which is a p x overlap with a time step, but which I have already computed. So, I insert all these factors as c numbers and I assemble a gigantic formula which is all c numbers. So, the transition amplitude is then given this kernel is then give a which any way is a c number because it is an overlap of 2 states reduces to a c number.

So, and yes and I put an integral sign here and to have all these things to be meaningfully you know resolvent of the identity. I have to have product over.

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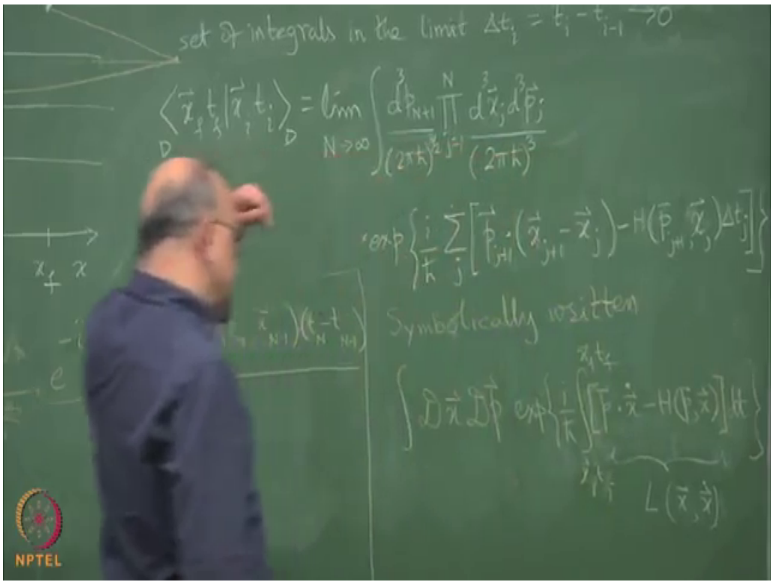

So, we will how to write. So, we assume that these are these are at the same time. So, I get e raised to the x p overlapped. So, let me write the final answer,

$$\langle \vec{x}_f t_f | \vec{x}_i t_i \rangle_D = \lim_{N\to\infty} \int \frac{dp_{N+1}}{(2\pi\hbar)^{3/2}} \prod_{j=1}^{N} \frac{d^3x_j d^3p_j}{(2\pi\hbar)^3} \exp\left[\frac{i}{\hbar}\sum_j \left[\overrightarrow{p_{j+1}} . (\overrightarrow{x_{j+1}} - \vec{x}_j) - H(\overrightarrow{p_{j+1}}, \vec{x}_j)\Delta t_j\right]\right]$$

So, if you take this orange and this one then you will find that the this x p overlap just gives you

$$\frac{e^{i\overrightarrow{p_N}.\overrightarrow{x_N}/\hbar}}{(2\pi\hbar)^{3/2}} \frac{e^{-i\overrightarrow{p_N}.\overrightarrow{x_{N-1}}/\hbar - \frac{i}{\hbar}H(\overrightarrow{p_N}, \overrightarrow{x_{N-1}})(t_N - t_{N-1})}}{(2\pi\hbar)^{3/2}}$$

So, this is the unit of the whole long formula which gets repetated and so, we can reassemble it in this form which is very nice and suggestive because all I have to do is put Δt below this and take Δt out and then it looks like $p\dot{x} - H$ ok.

So, then we stop quarreling about all this one extra p and all this limit and all this. And so, well it is exponent yes because now we have all these intermediate integrations over dx as well as dp's and if we just focus on x and ignore p for the time being, then what we are saying is I go from here to here. This is my infinitesimal propagation, but then I have to integrate our all possible t_1's ok

Then I could go to some other value of x at time t_2, then some third value of x at time t_3 and then will end up there, but I have to integrate over this entire thing because that is what this complete set of states means right where we introduced it as | $\vec{p}_N t_N$ $>_D$. So, we have to integrate over the entire spectrum of eigenvalues of x at every instant of time which is what Feynman called sum over paths right. If you introduce enough things and it is amounting to and the values of x range over the entire spectrum of x eigenvalues all the you may just be going x_i to x_f. So, if I send the ball from here to there it is going to also explore America Andromeda galaxy and go there.

So, we will stop here today and this is essentially the time slice path integral formula we just write it symbolically like this and it is very elegant because it just says e raised to i over. So, what of people think that this is the well quite correctly this is the action the usual action is the functional of p and x.

But remember that it's origin is in the Hamilton principal function which is an integral over time and this thing is symbolic and if you ever run in to legal problems of some left over things you can always fall back on it is technical definition which is this. So, there any faults then go back to this and try to repair it ok, this is the fine print behind the symbolic integration over paths. Now Feynman didn't struggle with all this Feynman was much cleverer than us. I did not I do not think you probably have a read about this.

So, he just latched down to the answer and said yeah and of course, this is the Lagrangian in when you this is same as $\vec{p}\cdot\dot{\vec{x}} - H(\vec{p},\vec{x}) = L(\vec{x},\dot{\vec{x}})$, I mean the way he explains it simply is that this details some is simply the statement that the amplitude to go from here to here is made up of a product of amplitudes to go from here, but at every point it is a complete set if state.

So, you have to integrate was in intermediate set of eigenvalues with just make sense and then, you don't have to struggle over all this, but at the same time I want to show you that there is a very precise derivation which makes this formula exactly as good or as bad as the Hamiltonian as the Heisenberg formulation which is based on canonical variables.

So, if you read Feynman then you will not see any of this he will simply say, oh it is this and then you will only said look, but if you are going from here to here you could be going from here and then going there, but then you may introduce more and then it will look like this. So, but why it would be the action, he does not take the responsibility of answering, what he does is that he differentiates this expression and shows, there it is satisfies Schrodinger equation.

So, it is the kernel of the Schrodinger operator and so, you have the answer. So, that is Feynman's we are doing it which we will effectively do next time. So, he is an cannot get the proportionality constant right. So, then he fix is it in some way. So, if you read Feynman and Hibbs then all those clever tricks are given which are cleverer, but mysterious because you do not know why they work why the reason; why they work under the bonnet; it is this all this is the wiring ok. So, we will do probably one more turn off this.