Path Integral and Functional Methods in Quantum Field Theory
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Lecture – 32
Topological Vacuum of Yang Mills Theories – II

So now that we remember why SU(2) is like S^3, but here you say there is no S^3.

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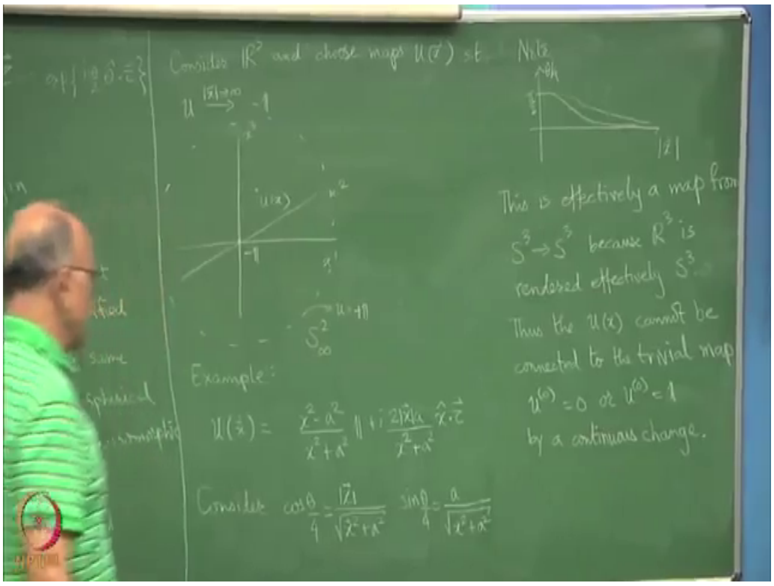

So, what Jackiw and Rebbi propose is to look at $\mathbb{R}^3$ and choose maps u(x) because we are in $\mathbb{R}^3$ such that u→-1 as |x|→∞, you chose only those u(x) such that you would reach the. So, u is a map right from $\mathbb{R}^3$ at every point you have to erect a u(x), but what you do is that when you reach this we sometimes call it as S^2_∞ the sphere at infinity we make sure that the u=-1. So, long as you choose a map like this you have done.

It is a 1-to-1 map, but you have chosen entire S^2_∞ to be mapped into one point of SU(2). You can take a simple example if you like you take a Gaussian +∞ and -∞ are both met to 0 right. So, you can take any function that is localized in $\mathbb{R}^3$ asymptotically it just goes to one point of the real axis. So, these are maps of $\mathbb{R}^3$ into real axis where you regardless of where you start all the infinities mapped into one point.

We are doing the same thing except that instead of real valued map it is a group valued map and in the group space we will all map it into one point. Effectively therefore, the $\mathbb{R}^3$ itself is compactified $\mathbb{R}^3$ itself is converted to S^3 by this kind of argument, the outermost fear is mapped to one point. So, it is a S^3 becomes an S^3. So, we will come back to it again how do you do this, how do you choose such a map, well it is very easy once it is explained, Example choose $u(\vec{x}) = \frac{x^2 - a^2}{x^2 + a^2}\hat{\boldsymbol{I}} + i\frac{2|\vec{x}|a}{x^2 + a^2}\hat{x}.\vec{\tau}$.

We tied up we mixed up space and group space this is the map of it is domain is the values x in $\mathbb{R}^3$ and it is range is SU(2) group, we can easily check that and note that the what we were writing as $\hat{\theta}$ and I was showing you here is literally chosen to be $\hat{x}$ the direction in space you are going out so, it is exactly $\hat{x}$ of the real space.

Now this looks mysterious, but one can check easily that this is an SU(2) element because well you could do $u^+ u = \boldsymbol{I}$ and you will find that it is equal to 1 and but it is a little easier if you notice the clever form of it is given you can think of consider $\cos\frac{\theta}{4} = \frac{|\vec{x}|}{\sqrt{x^2 + a^2}}, \sin\frac{\theta}{4} = \frac{a}{\sqrt{x^2 + a^2}}$ then the puzzle is solved.

So, it is exactly that form $\cos\frac{\theta}{2}, \sin\frac{\theta}{2}$. So, you have to choose the angle $\theta(x) = 4\cos^{-1}\left(\frac{|\vec{x}|}{\sqrt{x^2 + a^2}}\right)$. Now, one can see what this map does all it does this is just one example it is a very easy to it is a nice example, but one did not have to do this. Note that the map is such that at x=0 it just becomes equal to -1 and x = 0 this term is not there.

So, it is the map which is actually opposite it is u (x=0) = -1 and u(x=∞) = +1 and there this will reduce to 0 because it will go as 1/x. So, it will go to 0. So, actually the Jackiw-Rebbi's choice is like this you could put a minus sign in front of it if you do not like it that does not matter ok. So, that is the particular choice in this solution, but note that what this means is that we can try to plot x as a function of and this is of course, modulus x right this x is same as that is what we mean x^2 meaning is clear it is a vector squared.

So, thus note that we constructed at $\sin\theta\approx\theta$ for small θ. So, as x→0 this becomes equal to 1 right and that means, that it is $\frac{\theta}{4}=\frac{\pi}{2}$; that means, that θ is equal to or rather we can just plot the $\theta/4$ function because that is all that enters into that picture.

So, this is equal to $\frac{\pi}{2}$ at the origin and asymptotically this reduces to 0 and this reduces to 1 this becomes 1. So, asymptotically it is going to 0 as a function of $|\vec{x}|$, but this is all you need you did not have to choose the very nice form which is whatever you like to call it Lorentzian or something you could choose anything that you like so, long as it is monotonic ok. And so, long as it starts with $\frac{\pi}{2}$ and goes to 0 it is fine I can construct for you a reverse example we know it take a tan hyperbolic function which starts with 0 and goes to 1.

So, all you do is you multiply you take $\frac{\pi}{2}$ well. So, I give this as an we would think of other functions that can still be written as some kind of transcendental functions and which have that same property. Then you can stuff it in here it will look more complicated, but it will it will not be very different the two will be related by a local gauge transformation.

So, now what we so, at least this one you can see why it is firstly, a nothing, but an SU(2) element is just that we have chosen those values for the θ, except that it is a $\mathbb{R}$ valued radius valued map and every at every point of $\mathbb{R}^3$ it is an SU(2) element and it maps the $\mathbb{R}^3$ into the SU(2) both of which are converted to S^3 because in $\mathbb{R}^3$ we choose S^2_∞ to also all be mapped into one point of u.

Now the interesting fact about this particular map is that we cannot shrink it to 0 all though there are no obstructions in the space. So, remember in the case of Abelian case the gauge group was a 1-valued space λ was one thing that parameterized it. So, if you are simply connect that space you could string it to 0, but the gauge group space here is itself 3 dimensional. So, to shrink it to 0 you need a trivial 3 dimensional space a 3 dimensional space that is exactly $\mathbb{R}^3$ you can shrink any u to 0, but a 3 dimensional space which has effectively secretly become an S^3 you cannot shrink it to 0.

So, you cannot everywhere press this down and earn it out. So, because $\mathbb{R}^3$ is rendered effectively S^3, thus this u cannot be made trivial ok. Trivial map and what trivial map can you have $u^{[0]}=0$ or $u^{[0]}=\boldsymbol{I}$ after all $u^{+}u=\boldsymbol{I}$. So, you can choose some possible constant values of u cannot these two differ for topological reasons we are not modifiable into each other by a continuous change. Well if you make non trivial non continuous changes then you will have defects in the derivative, but derivatives would turn on gauge fields.

So, it would not be very nice to then you have to turn on you have to feed some energy infact one can show that the energy would be very large. So, it cannot be done, I am sorry it can be turned if you turn on if you feed some energy into the system then you can go from one to the other, but otherwise you cannot. This map also has a very interesting property that it is integer powers are also gauge transformations.

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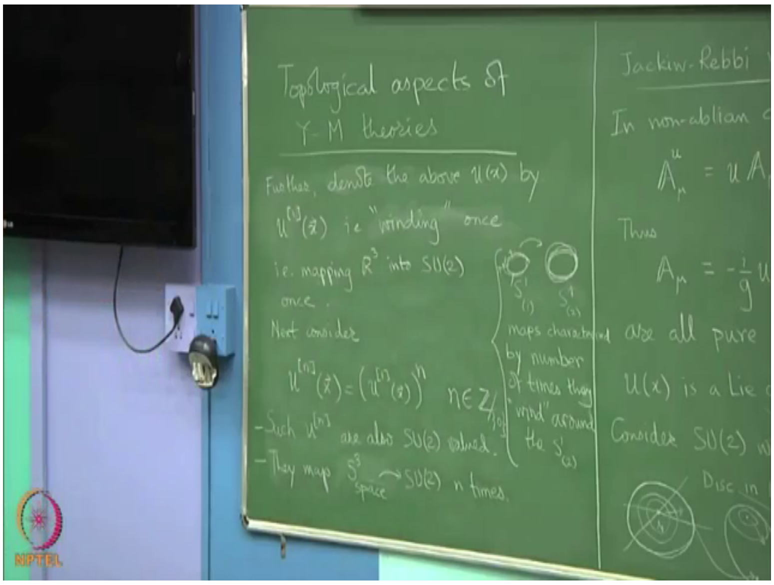

Denote the above u(x) by $u^{[1]}(\vec{x})$, i e it is called winding in memory of if you take a circle if you take one circle and another circle number 2 and if you go through $2\boldsymbol{\pi}$ here and suppose here you go through $4\boldsymbol{\pi}$ ok. So, this is called number of windings. So, map is characterized by around the circle around the $S^1_{(2)}$. So, that term winding number is a standard in topology borrowed from the 1 dimensional example. So, this one winds once because it maps $\mathbb{R}^3$ into SU(2) once. So, it is of course, 1-to-1, but it is not just that calculus requirement that is important it is a test topologically it covers the space once ok.

But now consider $u^{[n]}(\vec{x})=(u^{[1]}(\vec{x}))^n$ where, n ∈ $\mathbb{Z}/\{0\}$. Then we know what happens if I have a unitary matrix it is nth power is also unitary is all we have to do.

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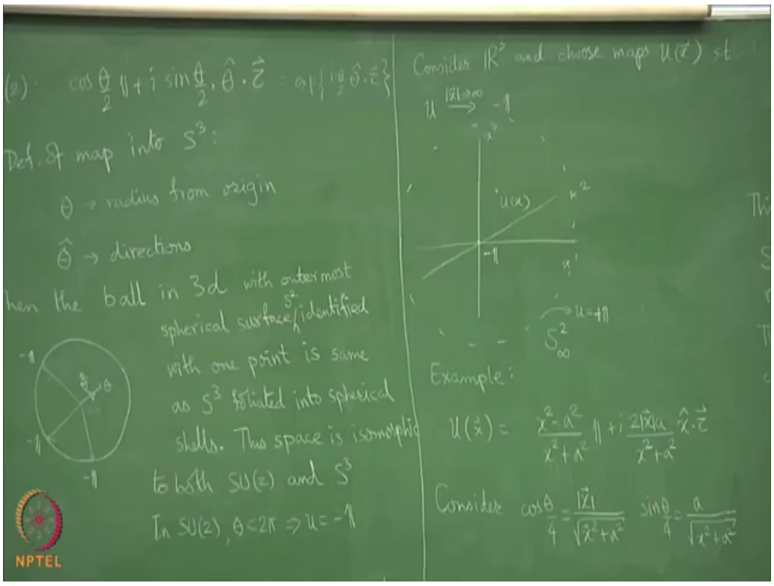

So, $u^{[n]}(\vec{x})$ is also unitary matrix, but now you see what will happen, what will happen to the n we can see over here this is nothing, but $\exp[i\frac{\theta}{2}\hat{\theta}.\vec{\tau}]$. If you raise to power n what will happen n will just go and sit multiply the θ right so, here the θ was a function of x.

So, now instead of starting with $\frac{\pi}{2}$ it will start times $n\frac{\pi}{2}$ and cross through $(n-1)\frac{\pi}{2},(n-2)\frac{\pi}{2},\ldots$ and then go to 0, because the θ has got multiplied by n that is all that happens. So, you have to here it will look complicated it is no longer simple, but it is cosine of nθ/4 that will be become that. So, the θ just becomes and θ got scaled up. So, the θ becomes n times whatever it was before, but it is still an SU(2) element, but we can now see that what this does is take the S^1 example if I wind around this once and this winds once that is one map. So, the trivial map is I wind around the whole circle I stay put at origin that is the trivial map.

Then I wind around once, I wind around once, but suppose as I reach $\boldsymbol{\pi}$ I already reached $2\boldsymbol{\pi}$ here then as I go from $\boldsymbol{\pi}$ to $2\boldsymbol{\pi}$ I wind once again that is double winding map. Then I reach 1/3 of the circle I cover the whole circle here, then I covered next 2/3 I covered the whole circle last 1/3 I covered the whole that is the winding number 3 map.

That is exactly what happens here, you will start from this -1 and hit +1 at not 1/3 now, but somewhere and that is why if you are raised to power 3 and then you will start and hit -1 again you will hit +1 again and finally, you will reach +1 after having crossed +1 two times in between and then the third time at the infinity. So, it is a triple winding map that 3 windings cannot be shrunk to 2 winding and 2 winding cannot be shrunk into 1 winding. So, there are maps indexed by the integer $\mathbb{Z}$ ok.

So, they map $\mathbb{R}^3$ or the compactified $\mathbb{R}^3$. So, let us just say S^3_{space} in to SU(2) and number of times, the meaning of -1 is just winding the opposite way. So, in the case of circle as you go from 0 to $2\boldsymbol{\pi}$ here you start in the opposite direction that is all it means in the present case it will mean starting from $-\frac{\pi}{2}$ and going there and etcetera. So, they are anti-winding maps in they are also independent, because the this direction cannot be converted into that direction ok.

So, they map also n times and so, each one of these is an independent is a gauge transformation is a set. So, this set of gauge transformations indexed by n cannot be mapped into each other cannot be smoothly deformed into each other.

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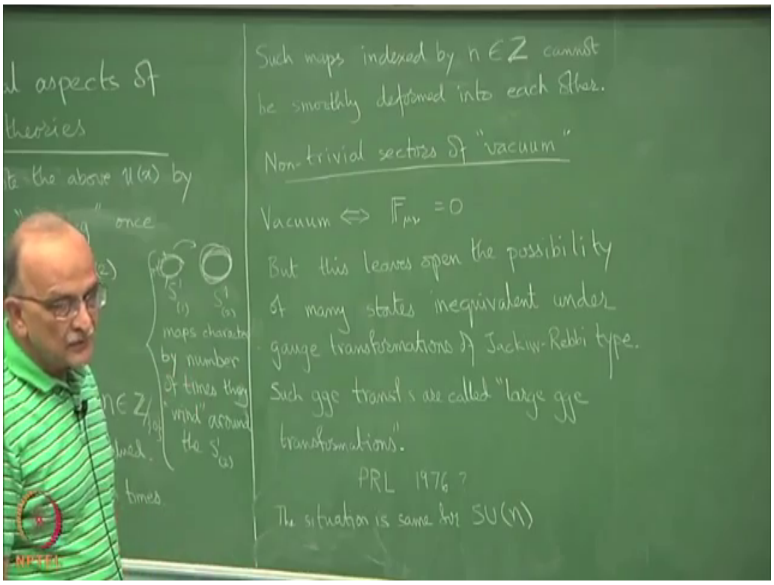

And I now include all $\mathbb{Z}$ because the n = 0 case would be our $u^{[0]}$ and this belongs to n = 0. So, such maps indexed by n cannot be smoothly deformed into each other, we have a peculiar situation that the situation with no electromagnetic gauge fields is what we called vacuum ok.

So, now, observe or we say so, by vacuum we mean no electric magnetic fields it should be 0, but you can have pure gauge transformations many states in equivalent under gauge transformations of above type, such gauge transformations are called large gauge transformations to distinguish them from small modifications of the $u^{[0]}$. So, all modifications that are topological equivalent to the trivial one we call them small like you perturbed around u = 0, but where you took the trouble of going all the way to infinity it is a lot of work to go to infinity and then soldered everything correctly then we call it a large gauge transformation.

But now this causes a problem to quantum theory, because what do we mean by vacuum in quantum theory because although these are all corresponding to $F_{\mu\nu}=0$, you could have done a large gauge transformation and the state where state may not remain the same it might change differ by a phase, because there is no way for you to without turning on electromagnetic fields compare those two states, you do not know what is the relative phase between them ok. So, what does the vacuum of the quantum theory look like and actually Jackiw, Rebbi do a careful like this is a very beautiful Physical Review Letters paper from which year I forget.

But if you just look for Jackiw, Rebbi what do they say vacuum structure of gauge field theories or something like that 1973 or no 1976 may be. There is another thing you might think that this is all specific to SU(2), but it is certainly true of all SU(n) groups because if you have an SU(n) group all you do is take an SU(2) subgroup of it and groups are homogeneous species they are simply connected and homogeneous. So, you just take a shrink to an SU(2). So, all you have to do is keep mapping from your $\mathbb{R}^3$ into some subgroup of that SU(n) you get the same answer so, the answer does not change.

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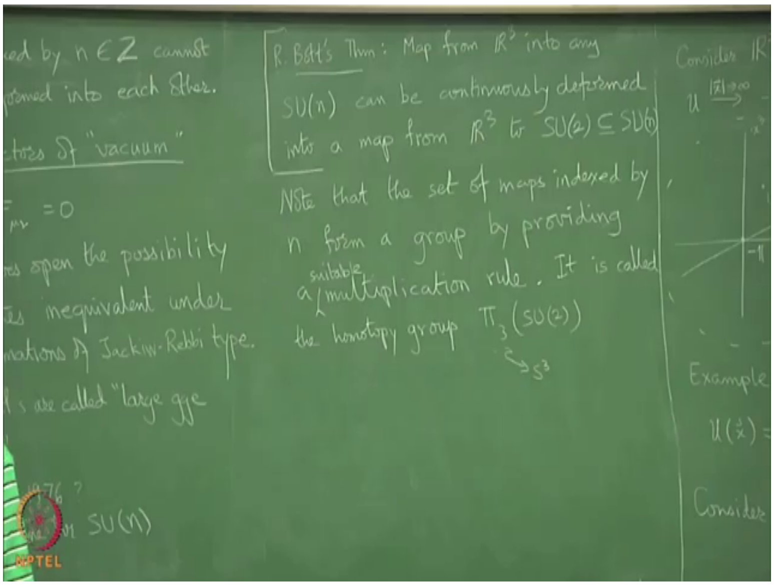

In fact I think there is a theorem of Bott Raul Bott map from any map from $\mathbb{R}^3$ into any now this is the part I do not remember SU(n) or any connected Lie group, but the weaker thing is certainly true which is the sub group of the SU(n) ok, by the way this counting that we did with winding number and all that is also called the homotopy group the maps.

So, this set of maps so, this is the end of the Raul Bott's statement note that the set of maps indexed by n form a group by providing a multiplication rule we will not going to detail, but you can guess what that rule is used as continue into the next venue compress the first once you are done it 3 times. Then you say oh well I have to do 2 times more you go 2 mores you know, you are 3 winding then you want to do 2 winding will do 2 more windings to it. So, you can define such a rule that would be called multiplications or by providing a multiplication rule and it is known as the homotopy group. $\pi_3(SU(2))$, 3 because we are mapping from S^3 mapping S^3 into the space ok, now I wanted to talk about the vacuum structure let me see if I can do it in 10 minutes.

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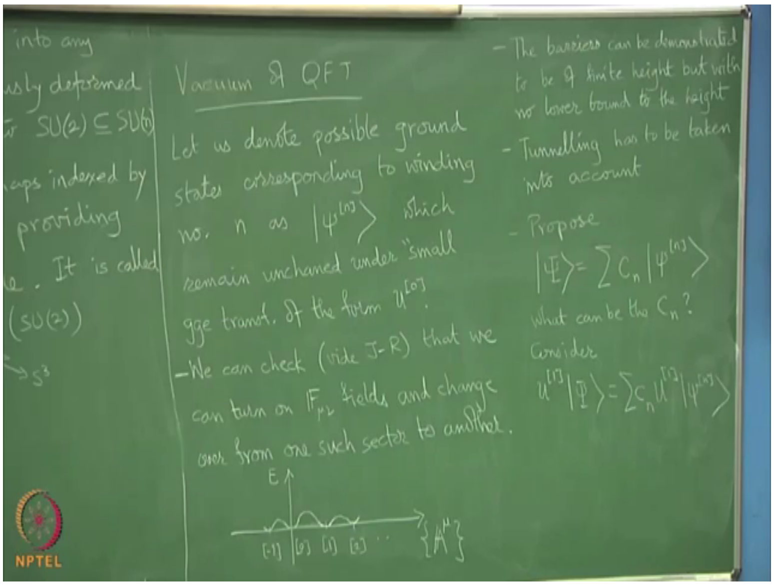

So, let us see the effect of the vacuum of quantum field theory. So, let, so, remember that small gauge transformations are all allowed, but the sector would remain the same, all gauge transformations set that the u asymptotically approaches $u^{[0]}$ or constant, such gauge transformations will leave you in that sector winding sector and it will not change the winding number at infinity. So, all small gauge transformations do not matter, but let us denote possible ground states well this is certainly ground states with so, corresponding to winding number n as $|\psi^{[n]}>$.

So, within which treated all treated equivalent unchanged under small gauge transformations this there is one thing here we can, we might then consider the possibility that the true vacuum should be a linear super possession of all of these ok. So, there is one thing here we can check that and you can see the Jackiw-Rebbi paper that we can turn on $F_{\mu\nu}$ fields and change over from one such sector to another ok. Now why is this important it is important because it shows that the various sectors of this vacuum are not like completely independent, you can actually turn on some energetic activity and when the activity dies down you may find yourself in a different winding sector without having to do something violent and dangerously violent ok.

So, often it is shown like this that I have here the grand class of this is the functional axis all possible gauge field configurations if you like I will put this as well. And here we draw energy total energy of the system you calculate the Yang Mills energy $\left(|\vec{E}|^2+|\vec{B}|^2\right)$. Then we already know this is the sector 0 and we know that there is a sector 1 and there is a sector 2 and so on and of course, there is a sector -1 so, on the other side.

What we do not know is that when you get there you have zero value of energy, what Jackiw and Rebbi show is that infact there is a finite barrier, but this barrier height cannot be defined, you have to turn on some electromagnetic fields to go from one to the other, but there is no lower limit on the how much electromagnetic energy you have to turn on. The reason is that the gauge theory has no scale in it is one of the thing is classical theory at least you know that it is just there is only that dimensionless charge g then no where there is a dimensionless number in that whole Lagrangian.

So, due to the I should probably start putting some dashes here. So, we can check that this is true the barriers can be demonstrated to exist to be finite, but with no lower bound any one map that you will construct will have a finite energy, but there is nothing intrinsic in the theory that says if you do it of this scale then it can be made so, there is nothing to compare with.

So, the this barrier exist, but it has no lower bound, but it exist ok. But now what is more interesting is that the, whatever the barrier is this only finite height, it is not large, it is not infinite or large infact, it can be arbitrarily small, we know in quantum mechanics that then systems tunnel into each other. If you leave the if you start by saying you can say oh forget all these junk I will sit at u = 0 how do you challenge me. Classically you will be fine because doing any going to anything else would require turning on the colour electromagnetic fields, I say I will never turn them on in my world then you are safe, but in quantum mechanics you are not safe because the system measures tunnel therefore, you have to consider.

So, the tunneling has to be taken into account. So, we propose a $|\Psi\rangle = \sum C_n |\psi^{[n]}\rangle$, but how do we determine these C_n ok? Now the point is that so, what are the C_n 's, you may say they can be arbitrary which is true at first sight, but there is an interesting point. If we now $u^{[1]} |\Psi\rangle = \sum C_n \, u^{[1]} |\psi^{[n]}\rangle$ here, $u^{[1]} |\psi^{[n]}\rangle = e^{i\theta_n} |\psi^{[n+1]}\rangle$.

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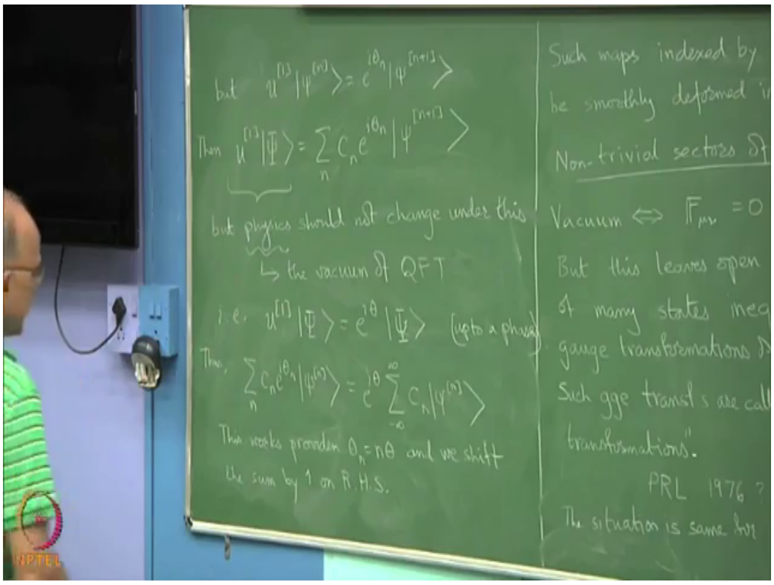

Let me leave some space right and up to some phase if I jump from one vacuum if I somehow went from this vacuum to the next vacuum I might accumulate some phase. Then it will become $u^{[1]} |\Psi\rangle = \sum_{n=-\infty}^{\infty} C_n e^{i\theta_n} |\psi^{[n+1]}\rangle$, but this is a bit awkward because physically this state is same as the original state hitting by one large gauge transformation should not change physics that is by physics we mean the vacuum.

So, the two things this and the original have to be the same up to one over all phase, they have to be same up to a phase right $u^{[1]} |\Psi\rangle = e^{i\theta} |\Psi\rangle$. Now how do you see that this and that will have exactly same phase. Therefore, the two things are provided $\theta_n = n\theta$.

So, this is possible only if $\theta_n = n\theta$ then we can slide the summation by 1 unit take out one θ and everything becomes the same. So,

$$\sum_n C_n e^{i\theta_n} |\psi^{[n]}\rangle = e^{i\theta} \sum_{-\infty}^{\infty} C_n |\psi^{[n]}\rangle$$

.

So, this happens provided $\theta_n = n\theta$ therefore, now we find that actually it is not so. So, what value of θ, well the answer is it could be any θ any one possible value of course, 0 to 2π, because now it is become phase, but two vacua that have different value of θ would now be different. So, what has happened is that, we had a series of vacua series of vacua which was a discrete sum we traded that discrete sum for a continuous indexing θ, but over a finite interval 0 to 2π.

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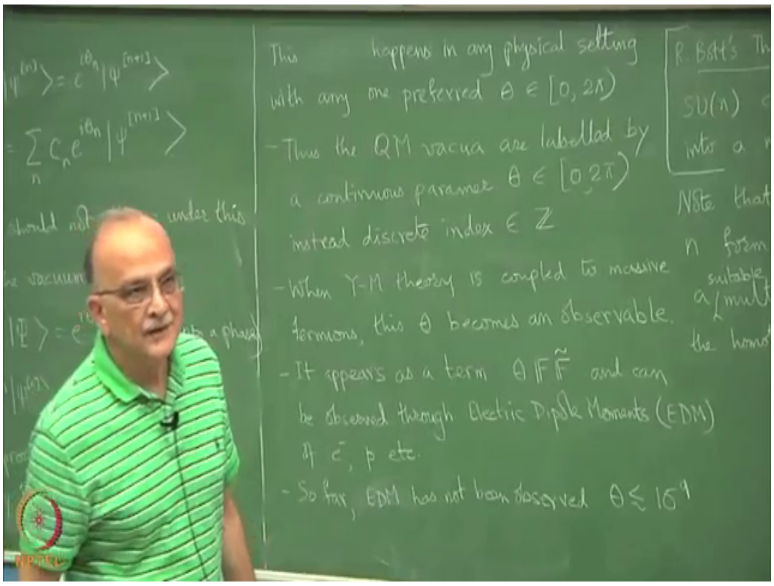

So, the physical states are listed by. So, this happens in any physical setting with any one preferred θ belonging to 0 to 2π right, thus the QM vacua are labelled by a continuous parameter θ of this range instead of discrete index belonging to $\mathbb{Z}$. And that is where we can stop the story for the time being, but the point is that further you may say that who cares you know some θ is there it is actually none observable, but if the theory is coupled to fermions which are massive.

Then you can show that the θ actually becomes observable right for if it was like this you say there is I live in world with one value of θ and I will set it equal to 0 nobody can argue against that like, because you can choose a phase of wave function, but when Yang Mills theory is coupled to massive fermions this θ becomes an observable.

Not only that it appears as a term $\theta F\widetilde{F}$ by some series of tricks you can transfer the, but then you are forced to have a $F\widetilde{F}$ term which you are avoided at first ok. We argued that $F\widetilde{F}$ violates CP, because the $\widetilde{F}$ is parity pseudo tensor, but now you are forced to have the pseudo tensor term you cannot avoid it and it actually becomes observable in induced moments.

It becomes parity violating through Electric Dipole Moments I wrote it in capitals because it is a very popular term of electron neutron at the electron proton etcetera. So, now, the double mystery in nature no such theta is found in nature θ is found to be 0. So, far EDM has not been observed $\theta \leqslant 10^{-9}$, it is really tiny if it exists it is really tiny. So, this is like the puzzle of mister nobody I wish you what is that lumeric does a lumeric that Weinberg writes in his review on cosmological constant.

So, you are in a cyclical trouble from all your clever arguments you found that there should be some θ that it should be observable, but in practice you find that it is value in nature is exactly 0 or close to. So, now, it causes your problem why if you are allowed to have any possible vacuum in this range you have exactly 0 value in nature. So, this is called that strong CP puzzle of QCD or the θ-vacua puzzle of QCD. So, it manifest itself in electronic theory in a different way because there it is spontaneous broken and then you do have a barrier height because the Higg's has a vacuum expectation value which sets it is height, but in QCD itself in pure, QCD because the quarks are massive θ should be nonzero, but it is found to be exactly 0 and we will stop with that.