Path Integral and Functional Methods in Quantum Field Theory

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Lecture – 31
Topological Vacuum of Yang Mills Theories - I

So, today we do the really fun stuff of Yang-Mills theories. This is what led Sidney Coleman, the great expositor of field theories to say that this 70s brought us unimaginable wonders from like a different planet. So, the origin of this particular topic that we are going to do today has to do with the non-linear nature of Yang Mills theory. But as we saw that non-linearity is introduced in a very specific geometric way and that is what leads to a very deep significance to this theory and leads to these effects that we are going to talk about today.

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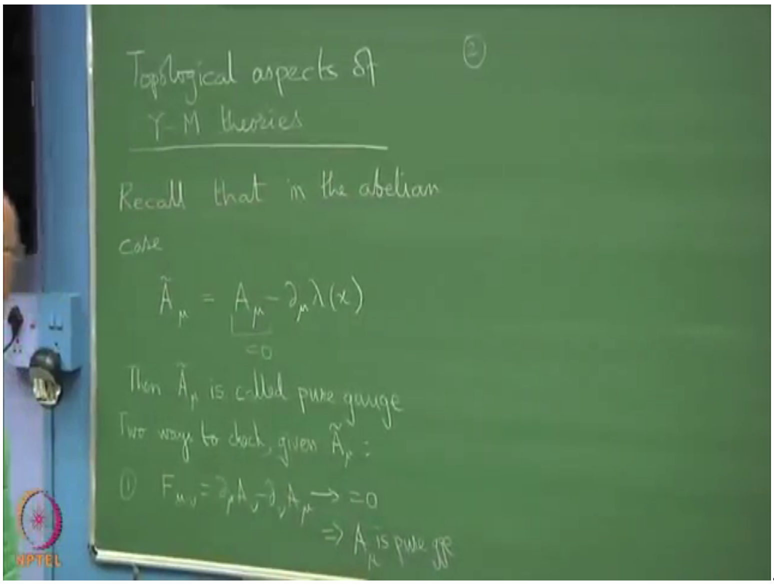

So, let us begin with the idea of gauge transformations. So, recall that in the Abelian case, $\tilde{A}_\mu = A_\mu - \partial_\mu \lambda(x)$. Now if this $A_\mu = 0$ to begin with, then $\tilde{A}_\mu$ is called pure gauge. So, it has no it will not turn on any electric or magnetic fields, but you might see this superfluous somebody hands you an $\tilde{A}_\mu$ and you worry whether it has any gauge fields or not. Well two ways to check.

One is just to take it is antisymmetrise derivative and you should find $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu=0$ implies that A is pure gauge ok. So, if you find this. So, this is the differential method, but there is an integral method. The integral method says try to solve.

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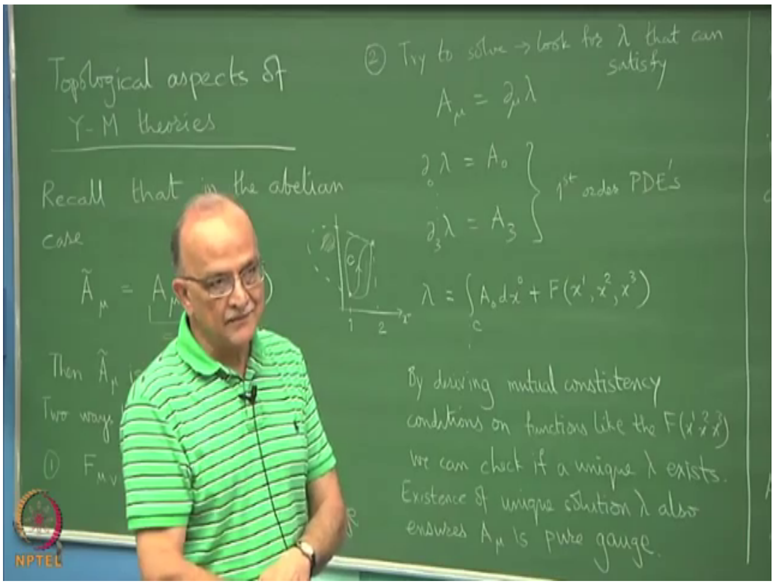

That is look for **λ** that can satisfy this. This then you read the other way around and you ask whether.

So, these are first order differential equations. So, $\partial_0\lambda=A_0,\ldots,\partial_3\lambda=A_3$ are just a set of partial differential equations which are first order. So, you can even attempt to solve them by this well known method that you say therefore,

$$\lambda=\int_C A_0\,dx^0+F\left(x^1,x^2,x^3\right)$$

right and so on you do for the second one you write it is integral of this, but with another and then you again take that. So, by deriving a consistency conditions on added you know in constants of integration such as $F\left(x^1,x^2,x^3\right)$ we can check if the integral if a unique **λ** exists.

So, we want to they want **λ** of course, to be sensible. So, the whole idea is look for a unique **λ** you know genuine spacetime function which is not multiple valued or etcetera.

So, thus looking for now so, we have understand this mutual considerate you take derivative now you take the A to derivative of this λ. So, there will be $\frac{\partial F}{\partial x^2}$, then you check that it is consistent with the dx^2 equation of that and so on.

So, now reason why I am writing out all this rather easy points is that there are cases in which this may not. So firstly, if you are if method 1 has worked then method 2 is guaranteed to work because you know that it actually is secretly $\partial_\mu \lambda$ and then the curl of this would be 0 because curl of gradient is 0. And if the curl is 0, then the path integral of A is uniquely defined right.

Essentially there is a line integral of A. So, the line integral of A is uniquely defined. So, if that integral can be done uniquely in the whole plane then you get a unique λ. So, it should be independent of the path along which you do this integral. So, that is ensured of course, if it is like this now there are cases in which this may not be. So, obvious is that.

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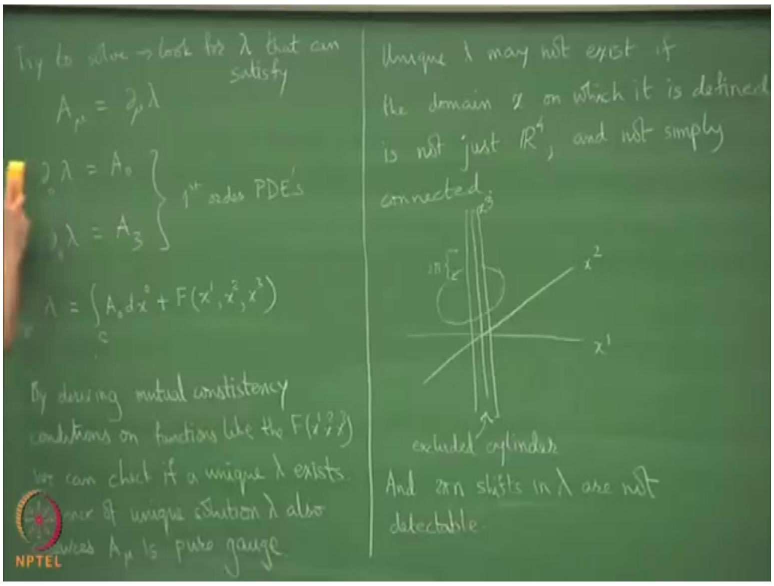

Unique λ may not exist in the sense of. So, the point is two is an integral method and one is differential method and two is the stronger method because it actually checks the whole plane this may be locally true.

But when you check the whole plane it may not come out to be true. So, in some way so, unique method may not unique λ may not exist if the space if the domain on which it is defined is not just $\mathbb{R}^4$, but something with holes in it.

So, for example, if you take the and we can only draw 3D I mean we can visualize 3 axis x^1, x^2 , x^3, but suppose I exclude a cylinder along the x^3 axis; I removed the x^3 axis. So, I exclude this. So, I make a very non trivial space. So, that the loops around this are not shrinkable, the domain x is excluding this.

Then you can get away with λs that are not actually unique, but it is a differ by 2π and then it is effect you may not detect. So firstly, it is not simply connected and say, $2\pi n$ shifts in λ are not detectable. So, the lesson here was that because the integral of A along any path. So, this is C let us say and you are well integrating only in the x^0 plane, but this is x^1 - x^2 plane.

Suppose this is x^0 and you are integrating from point 1 to point 2 and this is the C. So, the point was that whether you took this path or this path or that path did not matter because it turns out unique was that regardless of what path you took it should give the same answer in the end.

That is not guaranteed provided there are some regions of space that are excluded, then you cannot go around that or if you go around you may get a different answer and then you cannot complain.

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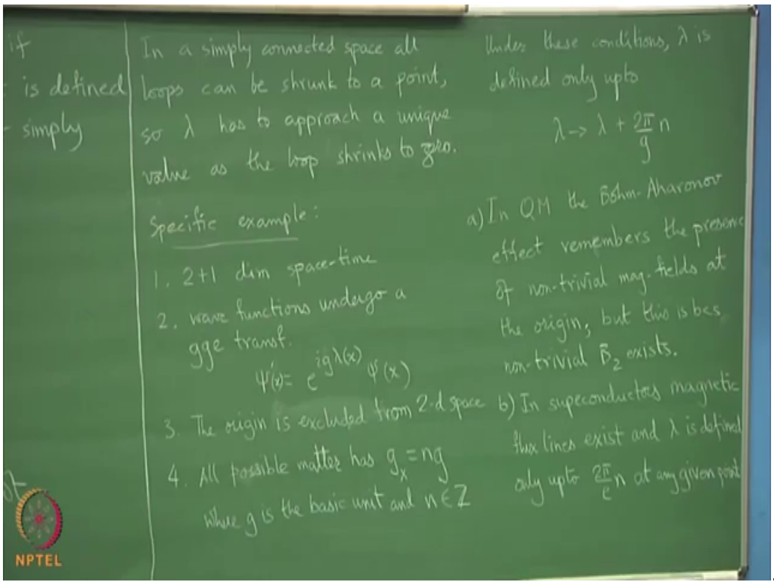

But if the domain is simply connected then you can shrink all loops to 0 and then single valuedness would require that λ cannot have any discontinuity. In a simply connected space, all loops can be shrunk to a point. So, λ cannot have any approach a unique value as the loop shrinks to 0, but in a non simply connected space you may be able to circumvent test number 2.

Because. So, to be specific let us only deal with 2+1 dimensional space. So, that we do not have to draw a cylinder and 2 we have charges whose matter wave functions undergo gauge transformation $\psi'(x)=e^{ig\lambda(x)}\psi(x)$ and then 3 we exclude the origin. Origin is excluded. We also need a fourth condition which is that all charges have the same charge.

So, there are no wave function all the possible matter has $g_x = ng$ where g is the basic unit and n is integer. In that case, you can get away with a non unique λ by allowing λ to change by 2π as you wind around this loop and if you change it by $\lambda \to \lambda + \frac{2\pi}{g} n$.

So, under these assumptions under these conditions plus $\frac{2\pi}{g} n$ and therefore, λ is multi valued, but you will never have a way of telling unless you have observables that can check this that you have gone around a loop it turns out to be true in quantum mechanics because you have so called Bohm-Aharonov effect. So, the Bohm-Aharonov effect is in quantum mechanics remembers the presence of A_μ even at the of nontrivial magnetic fields at the origin.

So, the point is that the Bohm-Aharonov effect will be able to measure whether there was a, but note that that happens because you actually have a non trivial magnetic field so. But this is because non trivial B_z exists; if it did not then, you could get away with shifting by $2\pi n$ and you would not know.

So, the example of that is in superconductors magnetic flux lines exist and λ is defined only up to $\frac{2\pi}{e}n$ at any point. So, the summary of all this is that there are things lurking around in this gauge invariance business that have to do with the global properties of the space on which you are living and their connectivity. So, the overall moral of all this is that.

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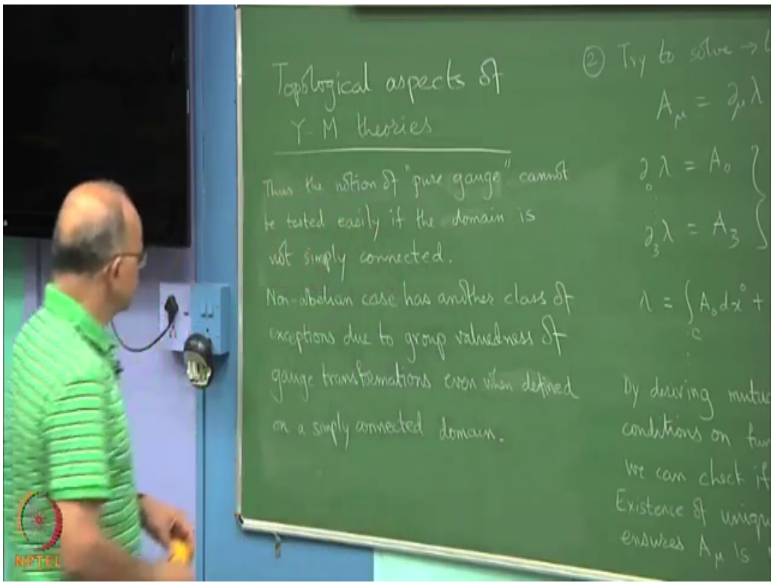

Thus the notion of pure gauge cannot be is not cannot be tested easily if the domain is not simply connected.

Now when we go to the Yang Mills case, even when the space is simply connected because the gauge group is non trivial we can again have non trivial gauge fields. So, non-Abelian case another class of exceptions due to group valuedness gauge transformations even on a simply connected domain, even when dealing with the simply connected; so, that is the preamble and now we go to the non-Abelian case.

So, in the superconductivity case the flux lines then the magnetic flux is quantized because the discontinuity because the $\int A.dl$ which would measure the magnetic flux it can only change in units of $\frac{2\pi}{e}n$. So, the flux is quantized. Often times people say that super conductors display quantum mechanics in a bulk system, but I think that is not true. So, it has more to do with you may say that it shows that a quantum field A does have a classical limit and then obeys classical gauge invariance that is what it does show, but the observable is really not really microscopic ok.

So, I mean the reasoning does not really involve quantum mechanics it involves just classical arguments and connectivity and simply connected argument. So, long as the A_μ field has a classical limit which is true for Abelian fields this is not a quantum mechanical result although there is a quantization ok. So, it is just a classical result, but there are other reasons why you may say it is true, but not this one. So, what we will do next is something quite interesting and let me write the title as the Jackiw-Rebbi vacua.

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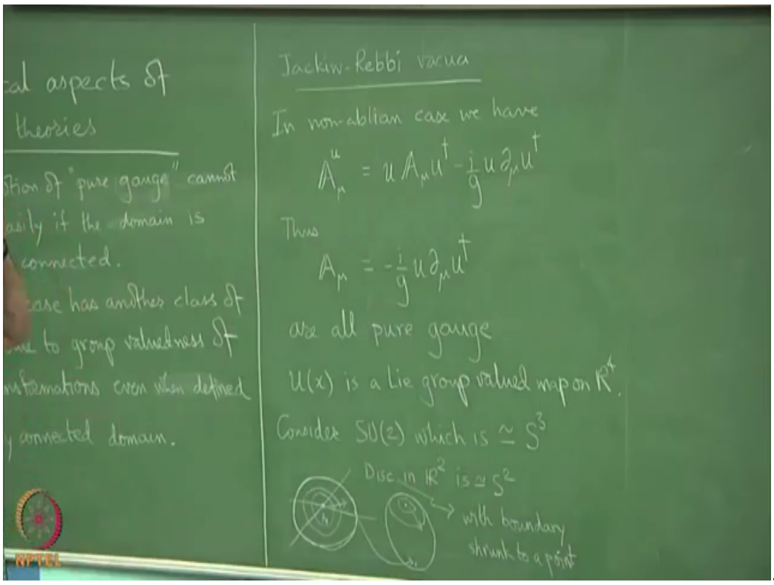

So, now we go to the non-Abelian case and recall that we have that

$$\boldsymbol{A}_{\mu}^{\boldsymbol{u}} = u\,\boldsymbol{A}_{\mu} u^{+} - \frac{i}{g} u\,\partial_{\mu} u^{+}$$

Therefore, configurations of the form $\boldsymbol{A}_{\mu}^{\boldsymbol{u}} = -\frac{i}{g} u\,\partial_{\mu} u^{+}$ are all pure gauge. So, here I should go and correct thus the notion of pure gauge can be tested. So, pure let us remember that by pure gauge we mean a single valued lambda ok, right, but here things are going to be more settled.

So, these are; so, if the original **A** was 0 then clearly the transform one is this and we drop this u and just say that if I am handed a configuration which is constructed out of a Lie group valued map now we come to the so called Jackiw-Rebbi observation let us simplify to the case where we will deal only with the spacelike $\mathbb{R}^3$ and we will look at time-independent case. So, the origin of the argument is that take SU(2) example. Topologically this is same as S^3 . Everybody remembers why you can construct a sequence of 2 spheres that go from North Pole of the S^3 to south pole of S^3 and which are isomorphic to.

So, which is and if you want I will quickly recapitulate the argument by going one dimension lower which is that disk in $\mathbb{R}^2$ is isomorphic to S^2 . So, since a disk in $\mathbb{R}^2$, what is a disk? It is a circular region including it is boundary. So, that is disk and that is that can be shown to be equivalent to S^2 provided you identify the outermost circle with the south pole of S^2. So, what you do is you put the North Pole at the center and then consider a sequence of circles going outward and at the same time on the S^2 you start with North Pole and start drawing these circles when you reach the south pole.

So, when you reach the boundary of the disk you map it into the South Pole. So, this is. So, it is not really a disk. So, it is disk with boundary shrunk to a point. You can think of it in reverse. You take a sphere, tear it at S^2 and then flatten it out and just remember that where you tore although it became many points now you should think of it as one point.

So, then you can visualize a 2 sphere which is intrinsically usually we embed it in 3D space can be visualized by an insect living in the disk. All he has to remember is the rule that the boundary is identified with a point and for the same reason SU(2) has which has the form.

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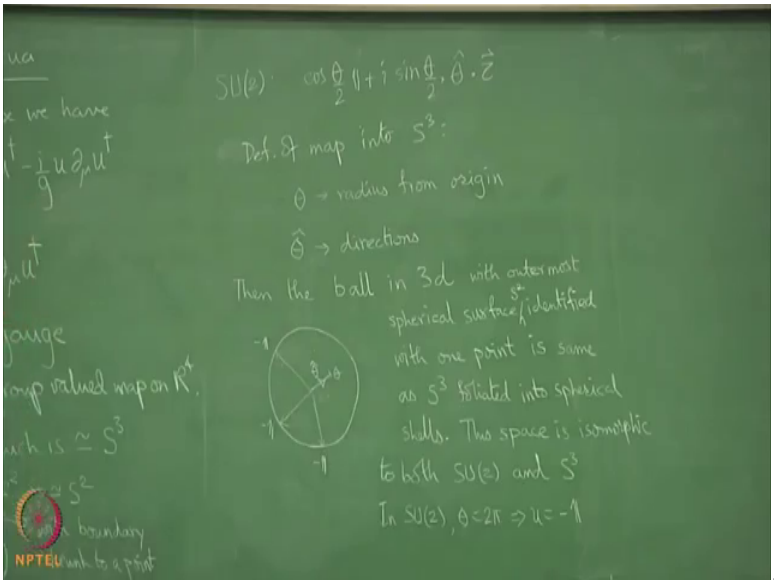

$$\cos\frac{\theta}{2}\hat{\boldsymbol{I}}+i\sin\frac{\theta}{2}.\hat{\theta}\vec{\tau}$$

You can do the same map.

So, treat θ as definition of map. θ is the radius from origin and $\hat{\theta}$ directions. Then you can see that the ball in 3d which is essentially the map by these two right the value of the radius θ and the direction $\hat{\theta}$, they together map out all the points in this ball then the ball in 3d with outermost point identified with spherical surface identified with one point is same as S^3 foliated as shells and both and this construction is a isomorphic to both SU(2) as well as to S^3.

S^3 because if I continued to if I continue to travel outward, but at some point shrunk the outermost circle to a point it maps a homogeneous space of dimension 2. If I have a sequence of 2 spheres and I take the outermost S^2 and shrink it to a point. So, outermost spherical surface S^2 identified with one point, then it is S^3. In the case of SU(2), it is the point $\theta=2\pi\Rightarrow u=-\hat{\boldsymbol{I}}$.