Path Integral and Functional Methods in Quantum Field Theory

Prof. Urjit A. Yajnik
Department of Physics
Indian Institute of Technology, Bombay

Lecture - 03
Path Integral Formulation – I

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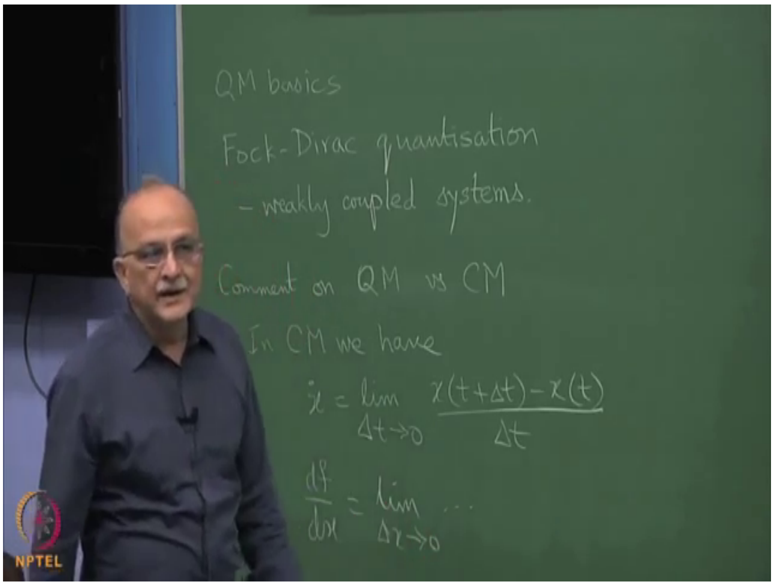

This is for weakly coupled systems.I connection with the quantum basics I want to add something comment on QM versus CM Quantum Mechanics versus Classical Mechanics. There is much debate and doubt and controversy about quantum mechanics being so bizarre and then what is really happening is it you know these words ontology and epistemology is it. Epistemology is how you organized knowledge where as ontology is what exists what is actually out there.

So, how do you know whether this is just organization of how what you can know versus whether there is a absolute reality out there, so the lot of philosophical things going on. So, I just want to point out that there are as many elephants in the living room in the classical theory that people don’t seem to worry about any more and one of them is calculus.

So, in classical mechanics we have the idea of derivative right $\dot{x}=\lim_{\Delta t\to 0}\frac{x(t+\Delta t)-x(t)}{\Delta t}$ and similarly other derivatives $\frac{df}{dx}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$. Did anybody in real life actually take the lim → 0 did there well when they did they found quantum mechanics. Before they reach 0 they began to find quantum mechanics this smallest scales they reached was atomic scale nuclear scale LHC has reached some astronomically smaller scale then that, but nobody has reached 0 right nobody has explored 0.

So, this idea that this limit exist is a fiction from the point of view of physics because no limit has nobody has got a stopwatch which can be calibrated to go to exact 0 the best cesium clocks we have frequencies in whatever. So, you can tune your frequencies correct to parts 10^{18} whatever you want its never 0.

So, this limit was never actually taken in practice, but peoples so formally believe in it that the thing that classical mechanics therefore, made the world predictable and then you could provide x and $\dot{x}$ at one instant of time and then Newton told you how to go to the future did they how precise he did they know the past. It is just that the precision of all their measurements was, so poor that it looked fine.

But in reality classical mechanics is essentially founded on fictional ideas of what is the continuum. The ideas of what is continuum may I have been made very precise and refined, but whether those ideas actually defined the physics continuum we do not know. In fact, what we do know is that if we try to explore at the real is small scales define quantum mechanics. So, better the open minded about what is out there and learn it rather than try to be pedantic about what was learnt from larger scales.

So, I think this is something that most people seem to not get and our cognition has to be expanded to absorb new things rather than trying to force newer things to fit in to the cognition cognitional devices developed earlier ok. So, that is enough of a comment. Now let us go to this particular formulation of path integral.

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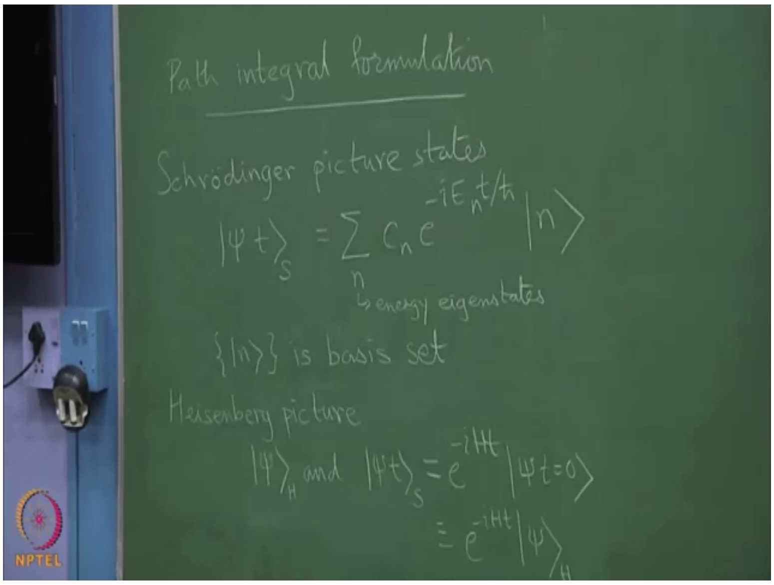

Let us begin with some fixing of from nomenclature and notation, we write Schrodinger picture states in this notation $|\Psi t>_s = \sum c_n e^{-iEt/\hbar}|n>$. So, now, we have another notation these things written like this with some eigenvalues written in them with no time dependence or anything or basis states.

So, we use the same kind of angular bracket notation, but $\{|n>\}$ is basis set. On the other hand Heisenberg picture we write $|\Psi>_H$ the state is not time dependent and the relation is $|\Psi t>_s = e^{-iHt}|\Psi t=0>=e^{-iHt}|\Psi>_H$. So, this is sort of for convenience if you like, but well the time-independent state is the Heisenberg picture state ok. And for Heisenberg picture state we will are the simple statement it will simply the C and $|n>$.

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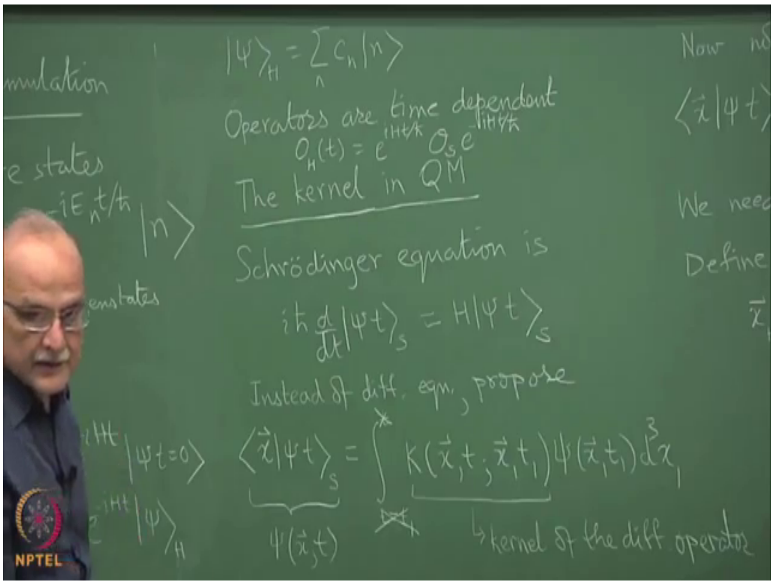

And of course, one should then write the involution for Heisenberg picture and that is given by you write for operators we have right.

But this is just to fix the ideas that we write $|\boldsymbol{\Psi}>_H$, we write $|\boldsymbol{\Psi} t>_s$ and then we will write like this. Now, this is the basic point I mean basic notation next we consider the kernel. So, quantum mechanics time evolution can be written out either as a, so Schrodinger equation of motion $i\hbar \frac{d}{dt} |\boldsymbol{\Psi} t>_s = H|\boldsymbol{\Psi} t>_s$.

Now, there is an alternative way of writing, so instead of a differential equation we propose $\boldsymbol{\Psi}(\vec{x}, t) = \langle \vec{x} | \Psi t \rangle_S = \int K(\vec{x}, t; \vec{x}_1, t_1)\, \Psi(\vec{x}_1, t_1)\, d^3 x_1$

So, this is the idea, so what we have proposed is a kernel. So, is the meaning clear. So, instead of solving the differential equation we want to propose an integral equation in which this will be found directly. So, that of course, requires lot of information because you have to know this two point function, but if you know it then you have the answer.

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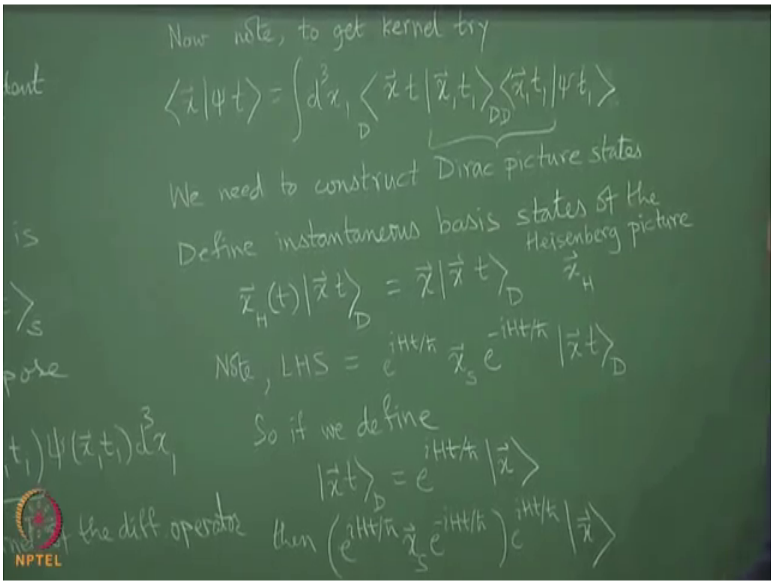

Now, that can be determined as follows we say that integral $\langle \vec{x}|\Psi t\rangle = \int d^3x_1 \langle \vec{x}\, t|\vec{x}_1 t_1\rangle_D \langle \vec{x}_1 t_1|\Psi t_1\rangle_D$ .We introduce this time-dependent basis. What we are going to say is that we need to, so the thing is to get the kernel try something like this try inserting a complete set of states right. So, this would add to one, so it is like taking this directly an overlap between the final and the initial. The point is that this needs a careful construction of this Dirac picture states.

So, we need the idea is that we make them instantaneous eigenstates of the Heisenberg picture operators by saying that we defined. So, earlier as said that things that I am nothing else in the exception eigenvalues are basis, but now we are going to make the basis time-dependent and it is time-dependent in the sense that it will return exactly the same eigenvalue regardless of the time at which I measure it provided I use the Heisenberg picture time-dependent operator and I may as well write down now the, so it is this is if you want Schrodinger picture operator. So, that is the relation between the Schrodinger picture and Heisenberg picture.

So, the meaning of this is that action of x on this written some instantaneous eigenstate and we can write this out as because this is time-dependent. So, if we define $|\vec{x}\ t\rangle_D = e^{iHt/\hbar}|\vec{x}\rangle$, then this will work. So, $(e^{iHt/\hbar}\ \vec{x}_S\ e^{-iHt/\hbar})e^{iHt/\hbar}|\vec{x}\rangle = \vec{x}\ e^{iHt/\hbar}|\vec{x}\rangle = \vec{x}\ |\vec{x}\ t\rangle_D$.

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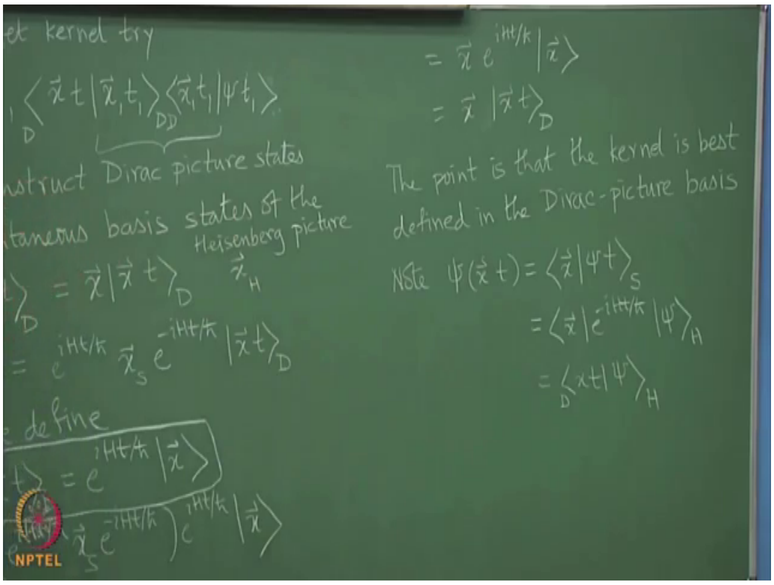

So, this is a bit of technicality and if you are reading this for the first time you are find that time struggling over technical point well it is technical unfortunately most books don’t tell you that the Feynman kernel is a calculation between Dirac picture state and Dirac picture state this as just write in state and out state, but the in state and out state are actually is this Dirac picture ok.

So, now we go back here and then we can see what is the meaning of this kernel the kernel is actually, so shall I just quickly repeat what we have done. We have introduced a time-dependent basis $|\vec{x}\ t>_D = e^{iHt/\hbar}|\vec{x}>$.

And now we will see the use, so we just check that, that is correct in the sense of you know this intervening operator and all does not affect any thing we exactly regain this statement and then our argument is that the kernel has to be define in terms of the Dirac picture basis. So, note that, $\Psi(\vec{x},t) = \langle\vec{x}|\Psi t\rangle_S = {}_D\langle xt|\Psi\rangle_H$.

So, the wave function the usual wave function is either Schrodinger picture state projected on the usual basis or it is the Heisenberg picture state projected on to time-dependent basis ok.

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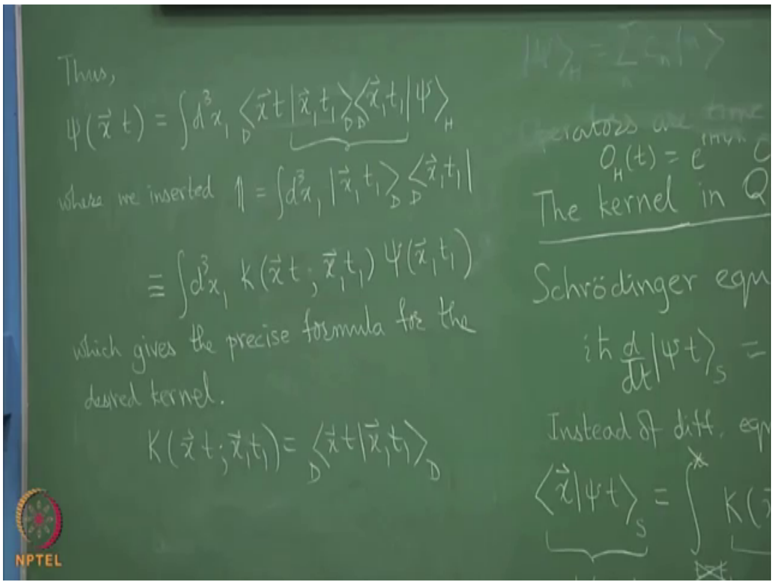

And so now, we can read off what this kernel is $\Psi(\vec{x},t)=\int_D \langle xt|x_1 t_1\rangle_D \langle x_1 t_1|\Psi\rangle_H$ and all we have done is inserted the identity in between. So, now we got the precise mathematical tool which passes master as kernel. So, then it will just look as if, so side the Dirac basis overlapped with Heisenberg picture operator is same as ordinary basis overlapped on Schrodinger picture which is just the wave function.

So, that is the precise meaning of the kernel provided both these are in the Dirac picture point one, so end of you know preparatory remarks one. Now, we go to the next part which is that Dirac's original motivation for the path integral.

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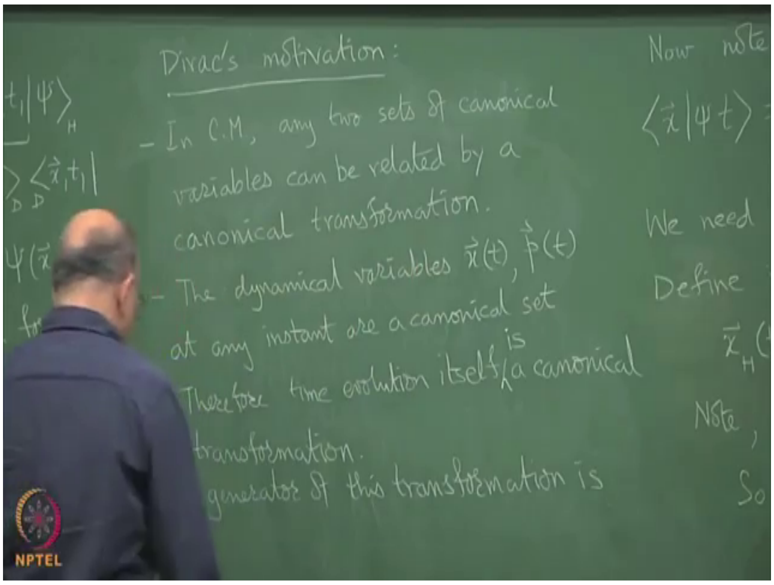

And here again there is a danger of lapsing into very technical discussion, so I will not do it because it is given in Dirac’s book I can forward to you the notes and what we will do is that we will agree to some basic a statement that the reasoning goes like this that in classical mechanics, any two set of canonical variables can be related by canonical transformation. By the way the word canonical actually comes Goldstein’s remarks from the word Kaanon means law you know, so kind of legal the formal the ok.

So, by a canonical transformation and I hope that you are study this transformations with small q and big Q and you get from one to the other by differentiating the generate, so there is usually a generator associated with it, but the dynamical variables at any time right. Therefore, time evaluation itself should write at any instant of time there canonical there satisfy the canonical brackets, Poisson’s brackets.

So, therefore, time evolution itself is a canonical transformation and comes the technicality what is the generator of this transformation and the answer is that it is the Hamilton’s principle function which is defined in Goldstein’s book as.

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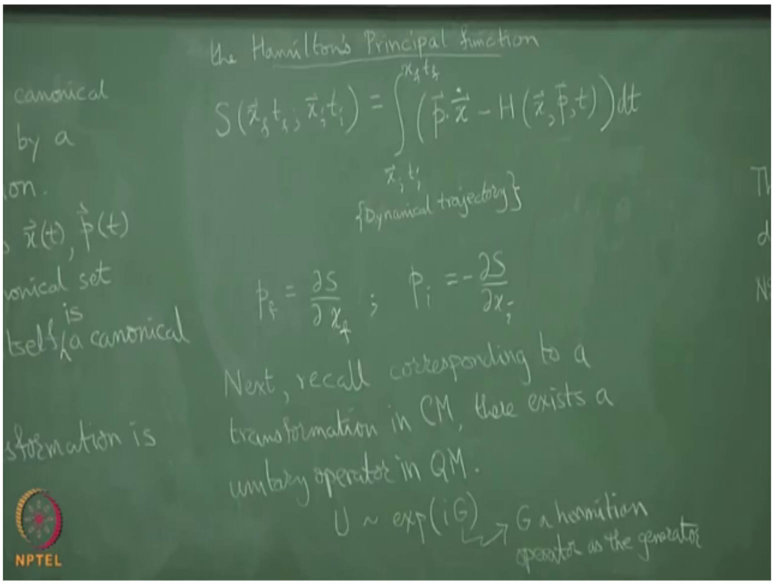

So, this is a canonical you remember there are four kinds of canonical transformation old qs new ps, old ps new qs or old q and new q. So, this is of the type old q to new q.

This is the generator, so $S(\vec{x}_f t_f;\vec{x}_i t_i)=\int_{\vec{x}_i t_i}^{\vec{x}_f t_f}(\vec{p}.\dot{\vec{x}}-H(\vec{x},\vec{p},t))dt$. where the time integration is on the classical trajectory or one should say I am because everything is classical mechanics.

So, the integral, so this is a appended to the integral sign, you might say what have you learnt because if you already know the dynamical trajectory then what is that we solved, but this is a formal statement about what is the meaning of the canonical transformation what is it generator and what it consists of. So, in the language of canonical transformation the time evolution is a canonical transformation generated by this particular generator. So, that $p_f=\frac{\partial S}{\partial x_f}, p_i=-\frac{\partial S}{\partial x_i}$, I did not write I have notes in which have use q and p here I started writing x and p. So, this is how the old and new this is how the generator works it is a function of the old momentum old coordinates and new coordinates and therefore, the new momenta old momenta are derived from it by this.

So, but you have to do the integration, so do not mistake this for the action principle or the action the action is a functional this is not a functional it is a two point function of real numbers real arguments which are only the end points and the p x H everything is function of time said that it is lying on the dynamical trajectory and you have done the integral along the dynamical trajectory, but from the two between the two desired end points. Then the object that you assemble of course multivariate calculus function not a functional that function is called Hamilton's principle function.

So, all this was formalised by Jacobi and I think the word canonical also goes to Jacobi, so this theory of transformations. So, coming back to Dirac's motivation he says look for every transformation of basis that you can doing classical mechanics there exists a unitary operator which implements it in quantum mechanics ok. So, next recall postulate number which no third only third, so we just said liner operators second one was observables are Hermitian operators and third was transformations are implemented by unitary operators.

So, corresponding to a transformation in classical mechanics there exists a unitary operator in QM which will implement the same thing and the operator in QM U=exp(iG), where G would be a Hermitian generator. At the unitary operator will be given by exponential of a i times Hermitian operator which we usually refer to as a generator, here we found that for the classical system the generator is this.

So, if we exponentiate that we should get a unitary operator that implement this, but there is one major problem the it is a dynamical evolution and these quantities at different times do not commute. So, we cannot just exponentiate the whole integral, what we can be sure of is that in the infinitesimal limit it would work.