Path Integral and Functional Methods in Quantum Field Theory

Prof. Urjit A.Yajnik
Department of Physics
Indian Institute of Technology, Bombay

Lecture – 29
Gauge Fixing and Faddeev Popov Ghosts- I

So, last time we saw the basic motivations for why we have to treat gauge field theories carefully.

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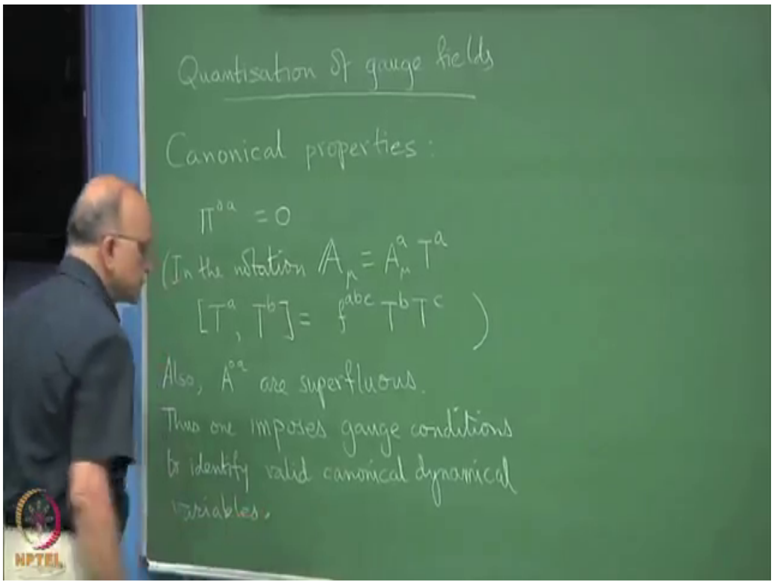

So, as we know the quantization goes through canonical structure and we saw that $\pi^{0a}=0$ for any gauge field. So, right in the notation in the notation that the gauge potentials are written as $\boldsymbol{A}_\mu = A^a_\mu T^a$ and T^a are generators. So, this π^0 just means that for the conjugate to A^0 are all zero, all the internal group index in the adjoint representation. We also saw that in many a sense the $A^0 = 0$ so, called Coulomb gauge.

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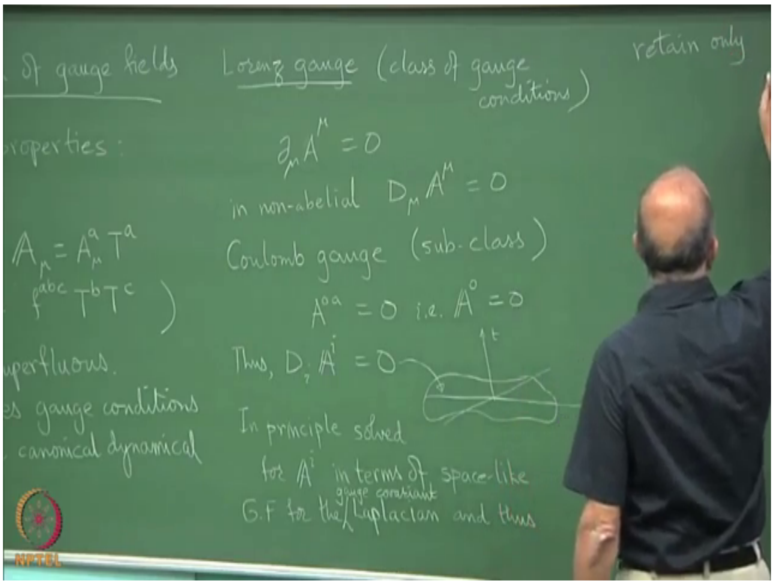

The Lorenz gauge is almost always imposed and, it is a class of gauges, which is simply that $\partial_\mu A^\mu = 0$ and here we would have to say a covariant derivative that is wait. So, in the electromagnetism we said this equal to 0 and in non-Abelian case $D_\mu \boldsymbol{A}^\mu = 0$.

So, this is a covariant gauge and within it one further puts the restriction that $A^0 = 0$ So, the Coulomb gauge which is a subclass further imposes that A^{0a} are all equal to zero equivalently $\boldsymbol{A}^0 = 0$, which leaves behind $D_i \boldsymbol{A}^i = 0$. Now, this we can see is a essentially a space derivative there is no time involved except that it is a differential condition, it involves derivatives of fields it is if they are only space derivatives.

So, it is at a particular time. So, if you actually think of taking derivatives as taking difference of fields at nearby points, it is just a condition on values of fields on a particular time like surface right. So, if t is going this way we have some domain in the space like part and we are just saying that this has to be imposed here on a particular time like slice.

So, you could in principle invert this and find out and solve for this and put it permanently in the dynamics and not have to worry about it later. So, in principle can be solved not that you will do it in practice well actually it is done also I should not say.

So, in, but I do not want to get into too much of the detailed formalism can be solved for A^i in terms of space like Green's function for the Laplacian this is of course, a gauge covariant Laplacian.

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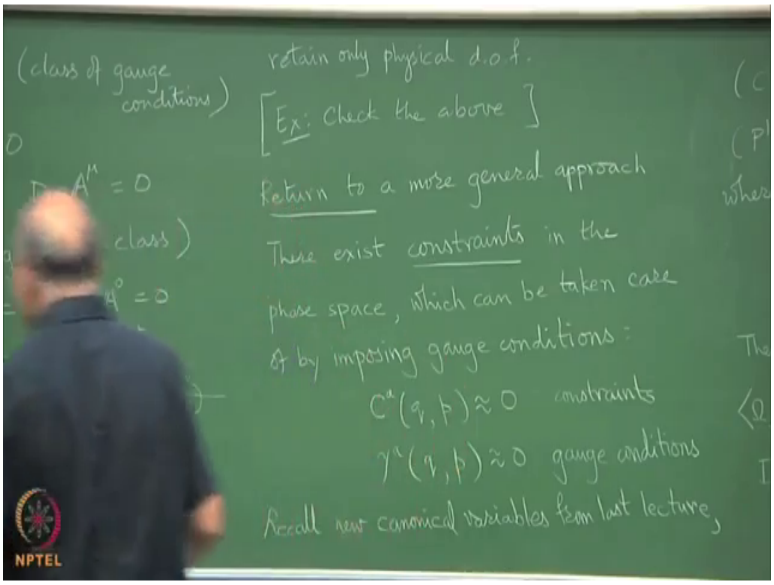

And, thus retain only a physical degrees of freedom. So, I will give this as exercise that is one way. But, it is a bit cumbersome and ultimately it has to do with very specific gauge. So, the general approach is to recall what we were doing last time. So, what is the general lesson, we had that there are some coordinates that can be superfluous further there can be also space like differential conditions on the fields.

So, we note that there are constraints in the phase space, which can be taken care of by imposing gauge conditions. So, we say that $C^a(q,p)\approx 0$ are constraints and some $\gamma^a(q,p)\approx 0$ are gauge conditions. The requirement that the imposition γ^a takes care of the condition C^a and leaves behind only canonical independent set. So, we went through this last time.

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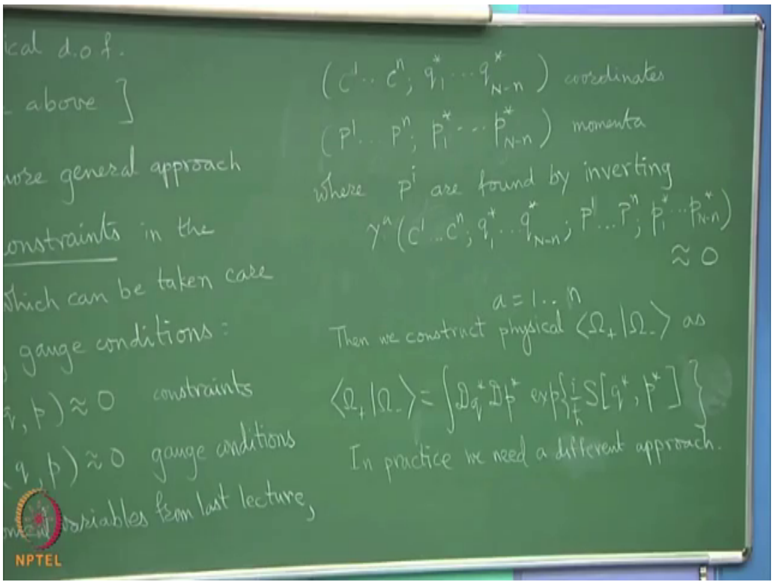

So, recall from last time things like what did we say C^a new canonical variables, which are you treat the $(C^1, \ldots, C^n; q_1^*, \ldots, q_{N-n}^*)$ as the coordinates. And, some $(P^1, \ldots, P^n; p_1^*, \ldots, p_{N-n}^*)$ as the conjugate momenta, where P^i are found by inverting γ^a of written in this language $\gamma^a(C^1, \ldots, C^n; q_1^*, \ldots, q_{N-n}^*; P^1, \ldots, P^n; p_1^*, \ldots, p_{N-n}^*)$.

So, you have to invert these and find from them the P^1 to P^n. There will of course, be a =1,..,n ok, there will be as many conditions as there are constraints and that will be that many constraint here, not to be confused with the a of the gauge fields, but they are going to become the same in the end this is more general discussion right. So, this is sort of formal we do not know how exactly you are going to do it, but we assume that it is doable.

Then we know that then the true we can construct physical vacuum to vacuum amplitude as a path integral only over the true variables well

$$\langle \Omega_+ | \Omega_- \rangle = \int Dq^* Dp^* \exp[\frac{i}{\hbar} S[q^*, p^*]]$$

So, that would be the vacuum to vacuum amplitude. So, you would modify it appropriately by introducing sources and so on. So, but this does not all I mean because the detail is not my point what we are going to do is in practice this is not what we do. So, what we do is right.

So, I think we were writing the external current language only for ϕ version without the πs, π having been integrated out. So, that is why I got a concerned about the notation, but here we want to retain both ϕ and π.

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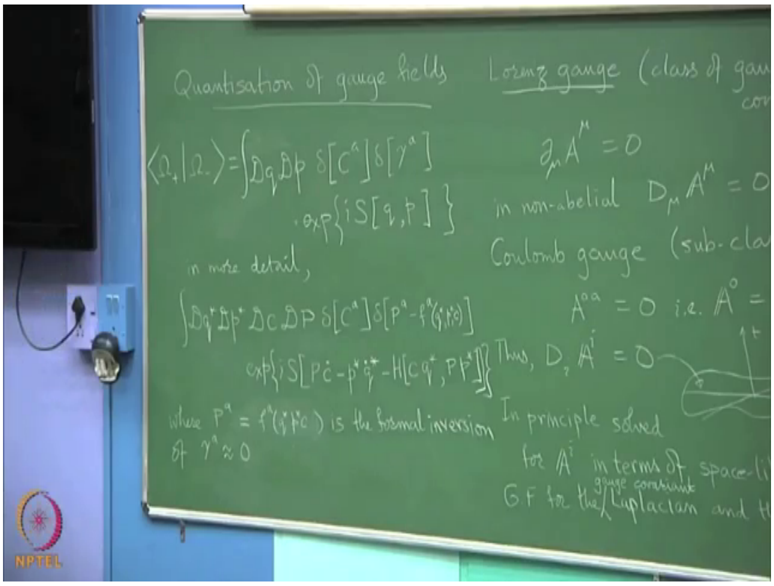

So, what we do is we write. So, let me return to ϕ,π language or well we can continue to write this. So, we write

$$\langle\Omega_+|\Omega_-\rangle=\int Dq\,Dp\,\delta[C^a]\delta[\gamma^a]\exp[\frac{i}{\hbar}S[q,p]]$$

Now, there is a simple argument that says that the imposition of γ^a is as good as same. So, yeah so, we in fact, write this is

$$\int Dq^*Dp^*DC\,DP\,\delta[C^a]\delta[P^a-f^a(q^*,p^*,C)]\exp[iS[P\dot{C}-p^*\dot{q}^*-H[Cq^*,Pp^*]]] \quad .$$

So, where $P^a=f^a(q^*,p^*,C)$ can be thought of as the formal inversion of the conditions $\gamma^a=0$ I think this is enough to write. And so, we are imposing that here. The thing to remember now was that in order that this transformation does actually disentangle the constrained and the unconstrained things correctly is that the $\{C^a,\gamma^a\}_{PB}\neq 0$. So, in a sense they remain they would after due rearrangement of the γs into $P^a = f^a$. The f^a would become conjugate to this C^a.

So, that requirement was that the $\{C^a, \gamma^a\}_{PB} \neq 0$.

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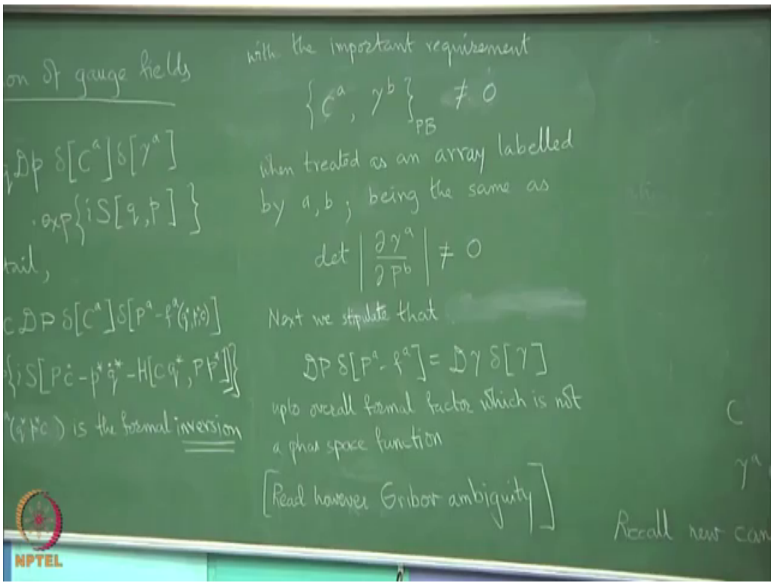

So, that the remainder treated as an array labeled by a and b or in other words the determinant of this should not be equal to 0. So, being the same as determinants of well you can work out this Poisson bracket it is same as $det\left|\frac{\partial \gamma^a}{\partial P^b}\right| \neq 0$ what is happening to me yeah right. So, that for the invertibility the inversion requires that this has to this determinant has to be non-zero.

And, which ties in with our need that they become canonically conjugate to each that that some combinations of them can be made into mutually canonical. We, now argue that the

$$DP\,\delta[P^a - f^a] = D\gamma\,\delta[\gamma]$$

we stipulate that. That this that because the P s are found from just inverting the statements of our γ's. So, this would be true up to this is in the functional space, it would be true up to sum over all constants right which are not important.

So, you can take it as a stipulation where we assume that the γ's can be normalized correctly or sum over all norms so, that this can work out correctly ok. So, up to the point is that whatever that adjustable factor is not dependent on the phase space ok.

So, let me write it here. I should tell you though that all this is all this is a way of justifying to oneself what one is doing and some of these things may have strange flaws these arguments are slick, but they may have flaws. And, in fact, there is this thing called Gribov ambiguity which you should try to read on your own. So, they may not always be quite correct, but turns out that. So, people do identify them then fix them later and so on, but for the time being we assume that this works the Gribov problem by the way has never been completely resolved, but people just live with it yeah.

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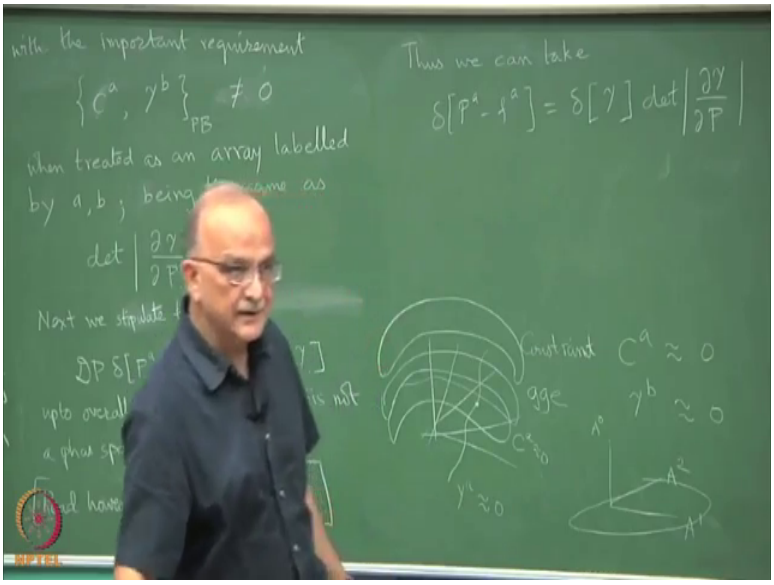

So, this is if we assume the stipulation, then we claim that this

$$\delta[P^a - f^a] = \delta[\gamma]\, det\left|\frac{\partial \gamma}{\partial P}\right|$$

the Jacobian by transferring the measure here. So, we can equivalently think of this was some measure time some distribution. So, equivalently we take this to be the Jacobian of the transformation.

So, this logic is correct right that I have $C^a(q, p) \approx 0$ emerging from analyzing the equations and I realize some of it is not dynamical. So, I say that means, oh I have to put this constraint and then to say that. So, the gauge condition is something independent of it. So, this is the picture I try to draw last time and failed I do not know whether I will succeed this time.

So, there is some constraint surface and what we want is that the genuine path integral does not end up traversing this surface, because if you in integral, because these are all equivalent only one of them is a representative member. So, you somehow want to impose a gauge condition that picks out a trajectory that cuts this only once. So, at least in this way where all the fashionably drawn many axes restricted to 3 dimensional. So, it would have to be the codimension of the C^a.

So, this is the condition $\gamma^a(q,p) \approx 0$. So, if the condition γ^a is such that it cuts this C^a only once, but does cut it which is what makes the determinant non-zero, then they are not independent. Then it will be so, this is the geometric picture. Now, that you have the picture you do this clever set of arguments to convert these into canonical variables mutually canonical, by finding f^a which invert the γ^a and then declare them to be the conjugates of this q's yes. So, the identical copies are perpendicular to this. So, there are lot of these which are the redundant ones.

So, that equal to you could have set instead of 0 you could have set 5 or something which would all be equivalent. So, let us think of specifically we set so, if I had the a must let us just say there is A_0 , A_1 , A_2 , A_3 . And, we set $A_0 = 0$ it means that I am restricted to remain in the A_1 - A_2 plane. To do that I well in this case of course, it is trivial, but in principle what it requires to do is to impose the gauge condition that sorry this because this is not a constraint this already is the gauge condition.

We have to say divergence A equal to 0, gauge transformations take you within a particular set of that is what I meant here, that gauge transformations would leave you with in this. I think this is correct picture, but we can discuss it later ok. Because right now I mean the another flow of thought so, but I just want to answer at least algebraically without the picture which I right now do not remember how it works, but that we are trying to convert the existence of constraints and the gauge conditions we place to repair them.

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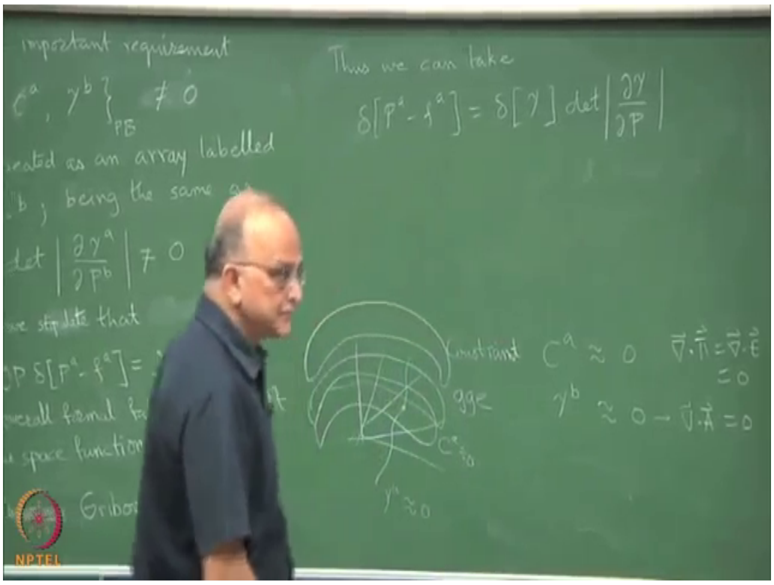

We are so, the gauge condition was $\nabla.\vec{\pi}=\nabla.\vec{E}=0$ and to take care of that the gauge condition was $\nabla.\vec{A}=0$ within the Lorenz gauge. So, these are the two things that are proposed to be made mutually canonical ok.

So, there is a geometric picture of what the constraints and the gauge conditions are and then there is the canonical picture where you try to move from the geometric picture to the canonical picture. And, in the process you claim that imposing the delta function for these formal P's that were meant to invert γ^a . So, the great advantage of this is that we actually get rid of the P's completely in the end.

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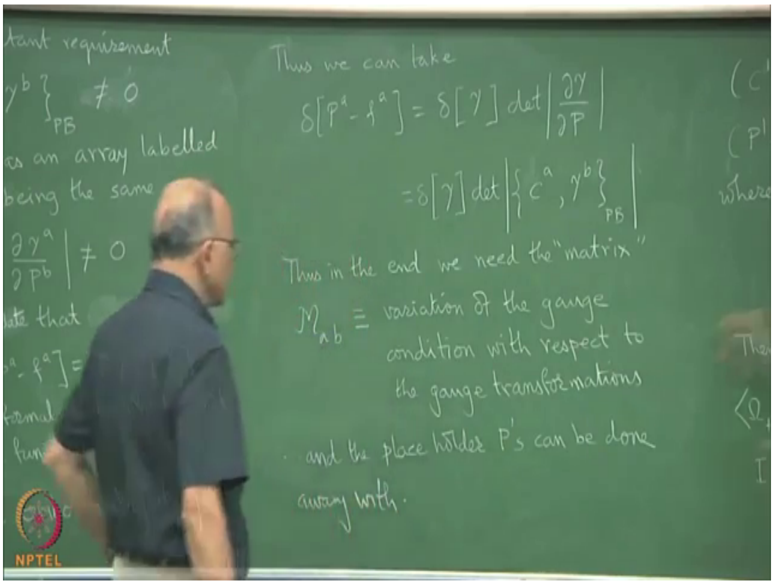

So, we now answer that this is same as $\delta[\gamma] det\left|\{C^a, \gamma^b\}_{PB}\right|$ and we thus get rid of the superfluous P's which were placeholders to think through this thing ok, we need the "matrix" M_{ab} which is variation of the gauge condition with respect to the gauge transformation, that is what this boils down to.

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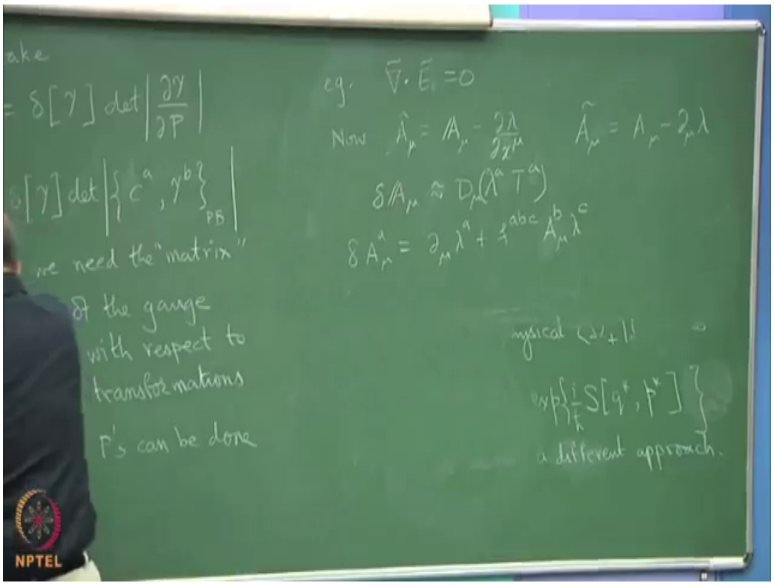

So, for example, what is the time yeah for example, we had the gauge condition $\nabla.\vec{E}=0$ right and, but under gauge transformation

$$\widetilde{A}_\mu = A_\mu - \partial_\mu \lambda$$

So, right so $\widetilde{A}_\mu = A_\mu - \frac{\partial \lambda}{\partial x^\mu}$, and for the non-Abelian case it is $\delta \boldsymbol{A}_\mu \approx D_\mu (\lambda^a T^a)$. So, it has $\delta A_\mu^a = \partial_\mu \lambda^a + f^{abc} A_\mu^b \lambda^c$. So, even the infinitesimal one has the imprint of non-Abelian gauge transformation actually this is why this Gribov ambiguity arises.