Path Integral and Functional Methods in Quantum Field Theory

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Lecture – 28
Yang Mills Theory Constraint Dynamics – II

Now, what we are driving towards is quantization of the gauge fields, a quantization of the Yang-Mills system and the canonical structure becomes very important because you remember that we started our course by defining the path integral using the canonical formalism.

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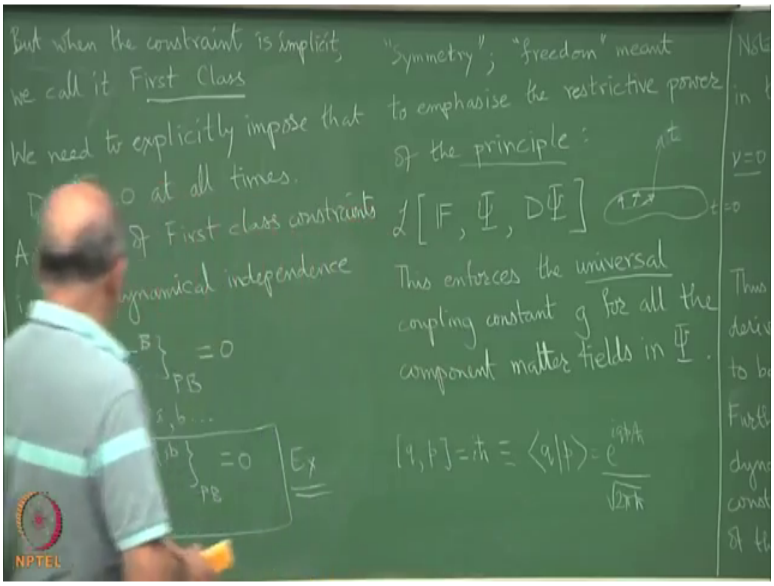

We had the canonical variables and we had the fundamental relation

$$[q,p]=i\hbar\equiv\langle q|p\rangle=\frac{e^{iqp/\hbar}}{\sqrt{2\pi\hbar}}$$

So, canonical structure is very important to us for quantizing and that is why we have to be careful about what the canonical structure of these equations is. And so, towards quantization quite a lot of subtleties arise.

Now, historically turned out that this was very early days and nobody thought that Yang Mills fields will become important; they have proposed in 1956 and throughout the

1960s people were mostly concerned about spontaneous symmetry breaking. They were trying to get pions as you know superconductivity of the strong force and things like that. But, Feynman had begun to worry about this and there was famous quantum gravity person called DeWitt.

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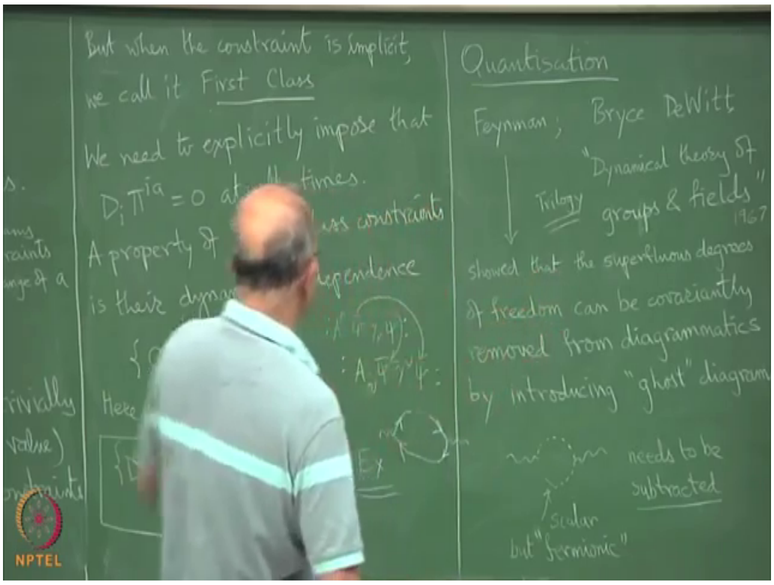

So, Bryce DeWitt is a name not known much now to the ignorant, but to the those who know it is a big name because Bryce DeWitt gave his full force to quantize quantum gravity along the canonical method, but actually more adopting Schwinger's; he was Schwinger's student. So, more adopting Schwinger had his own way of saying how you vary the action. So, it is a functional approach, but not path integral one and then in later life Bryce was a professor at Texas Austin and I have taken courses from him.

So, Bryce DeWitt wrote a trilogy. So, he has a book called Dynamical Theory of Groups and Fields, in which he meant to attack this problem of systems that have this kind of redundancy and as we know general theory of relativity has the huge redundancy due to reparametrization invariance and the covariant derivative of differential geometry. So, Bryce like a juggernaut just cleaned out the whole field of doing what was required. Feynman was somehow interested in this in the Yang and Mills proposal.

And, he observed that from a very physical point of view that the gauge conditions shows rather that there are superfluous degrees of freedom can be covariantly removed by introducing ghost. Ghost diagram is one that has scalar loop it could have anything outside, but it has a scalar, but the diagram has overall minus sign ok. So, it has to be subtracted. So, if you draw the diagram you do not get any minus sign, but you say. So, you say that diagrams like this have to be subtracted. So, you can interpret it by saying that well, you know in Wick's theorem that when you have closed loops of fermions you get a minus sign because of the anticommuting property.

So, you would have had ψ-ψ from this vertex. So, you know how the Wick theorem will go is that you have $:A^{\mu}\bar{\psi}\gamma_{\mu}\psi:$ and $:A_{\nu}\bar{\psi}\gamma^{\nu}\psi:$ and you have to link this ψ to this $\bar{\psi}$ and this $\bar{\psi}$ to this ψ. But, in doing so, so that would give you this diagram. So, one A_{μ} coming in producing ψ and $\bar{\psi}$ you know one arrow going this way one and another one like this and then you have to. So, this is one vertex, this is the other vertex where this is the product of the two.

So, you join up these lines that joining up in Wick's prescription is coupling this $\bar{\psi}$ with this ψ and this ψ with this $\bar{\psi}$, but that will require going taking this ψ across one $\bar{\psi}$. So, there is sorry this is normal ordered. So, there is no question of going across that, but it entails same negative sign from having to take the ψ across to this side ok. So, then it becomes propagator. So, that minus sign normally comes if you have the fermion statistics there, but here it is difficult to see how minus sign would come.

So, people said, well, but and it is required to be scalar there are no indices on it. So, people said, well, all you do is that you claim that it is a scalar, but a fermion; but it never appears in the asymptotic state. So, it is not going to violate our spin-statistics theorem. Spin-statistics theorem that scalar cannot be a fermion; scalar has to be a boson. All integers spin and 0 spin have to be bosons, but this violates normal that condition. However, it is not going to be seen outside it is only a trick in the diagrammatic calculation. So, that's what Feynman observed and he probably did not say it was fermionic loop he just said these need to be subtracted.

Bryce DeWitt in his own juggernaut way just came to the point and say you need to put this and you need to put this and he never realized that he had done the same thing, but outing Lebedev Institute Russia, two Russians were worrying about this and they had read Feynman's paper.

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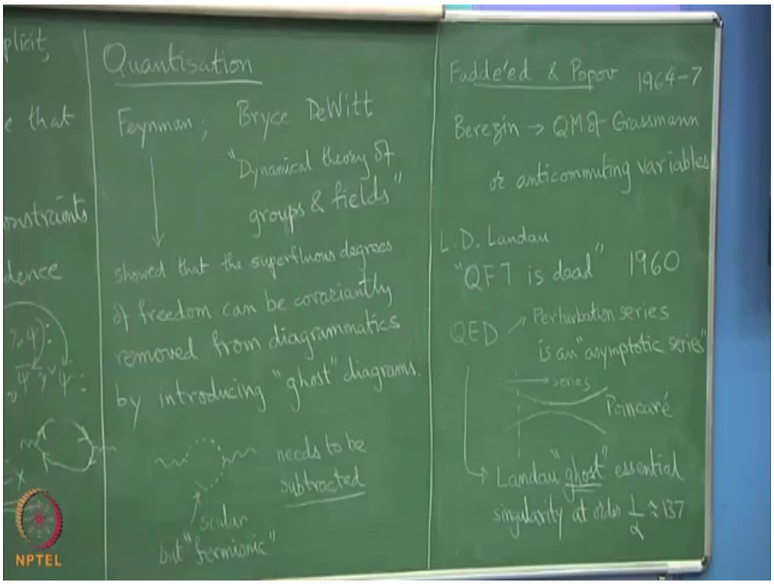

So, Fadde'ev and Popov gave a completely field theoretic interpretation because they had a colleague called Berezin who for his own mysterious reasons because he was a Russian had already worked out how to deal with field theory of or quantum mechanics of Grassmann variables. Well, I know why he had worked it out.

So, Berezin's observation was that if you are going to do path integral quantization of fermions; fermions have anticommuting property. So, classical limit also they must be anticommuting, but if it is. So, then they cannot be ordinary numbers. So, they have to be what are called Grassmann numbers which anticommute their product is not I mean you cannot express them as numbers and those variables have to be anticommuting.

So, just around 1962, 1963 he worked this out Feynman is also around 1961 – 62 and then Fadde'ev and Popov realized that what they need to say is that they use Berezin's method, but they violates spin-statistics theorem then they can actually get Feynman's ghost diagrams in a systematic way from the path integral. So, that is what we are going to do next.

Yeah, oh I just wanted to tell this story that in America nobody was worrying about this because they were not looking at the Yang-Mills field theory, but in Russia it was stronger nobody was supposed to look at field theory at all because Landau had declared in 1960 that field theory is dead ok, QFT is dead. In a major conference he said this based on his analysis of quantum electrodynamics.

So, there were two problems first of all you would like to know the good news that all the perturbation theory that you have been using is wrong from mathematics point of view because it can be proved that the perturbation expansion you make is only asymptotically convergent or they are actually called as asymptotically divergent series. So, first thing is that perturbation series is an asymptotic series they just call it asymptotic series this means that the terms grow smaller I just draw a diagram like this. So, series goes like this keeps growing smaller then the terms begin to diverge again ok.

Now, this somehow was already noticed by Poincare much earlier and so he had warned people that you should cut off your calculation here; while it is still converging cut it off and interpret that as the answer do not go to the bitter end of the series. So, that was one problem that anyway perturbation series people had doubts about. But, Landau further showed for QED that there is Landau ghost; now this is the different ghost not that ghost which is an essential singularity at order $1/\boldsymbol{\alpha}$.

So, if you do QED to 137th order then there is a essential problem. So, he said that it is not even benign like what Poincare said. So, this is Poincare's description, but that there is an essential singularity and therefore, this does not make sense. I am quoting this because it is important to remember in history of science that great men can be when they are wrong they very badly wrong. So, there was of course, no reason to stop studying quantum field theory, but because Landau declared it nobody in Russia was working in quantum field theory.

So, Fadde'ev recalls many years later about 5 year later he has given some set of lectures I think in TIFR actually they are on the video that it was like censorship. So, they are secretly brought out a preprint, but they did not show it to anyone and then the preprint migrated to Europe secretly in somebody's suitcase and when it reached the west everybody was enlightened and then said hey, because by then it was late 60s, 1967 or so that, but I think. So, 4 to 7 they first wrote this thing in 1964 the preprint migrated out they never published it, but by late 60s Yang-Mills theories has applicable to the weak interaction had come back invoked.

So, it was Ben Lee I think B W Lee, one of the pioneers of perturbation theory for spontaneously broken electroweak theory. He brought it out as a Fermi lab preprint he reprint he translated and produced it as a Fermi lab preprint and after that by 1967 they actually published it as a physics letters paper in the typical Russian style where each pages like fifty pages of calculation. So, they have just wrote up the note. Their whole lecture notes they wrote in 4 pages and that appeared as Physics Letters B 1967.

And, I think DeWitt's book a is actually this books date is also 1967, but he had already a set of papers in physical review 3 papers. So, it is called the trilogy to all the older quantum gravity people upto year 2000, DeWitt's trilogy was the bible of why quantum gravity does not work or if you had to do it how you would do it and it is very useful document because if you actually do calculate he has done all the calculation which you cannot imagine doing.

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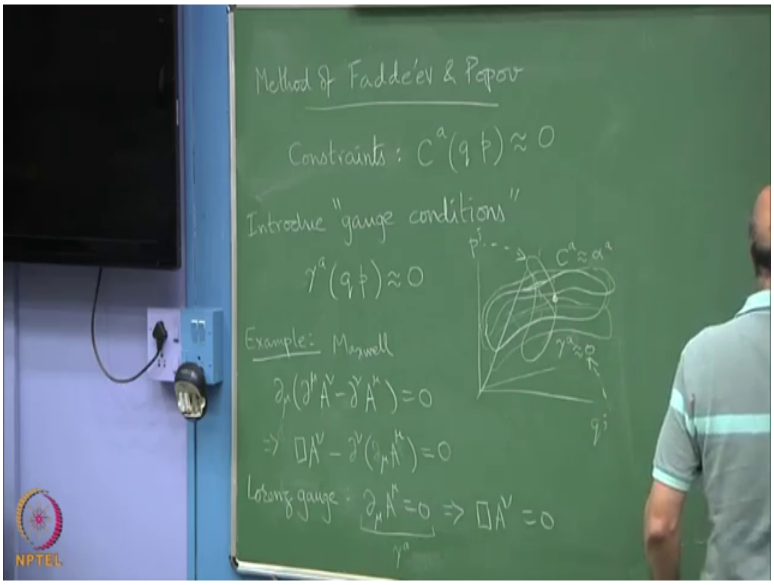

Now, therefore, Fadde'ev and Popov came out with a very elegant way of doing things they were of course, in this mould of Dirac. So, they said that corresponding to a gauge constraint you also put gauge conditions on the phase space $C^a(q, p) \approx 0$ and we write like this because this is this is the formal equality ok. It is not a not really a function. It is some expression on the space of q's and p's which is set to 0 and so, introduce sorry I am sorry that is the second line. So, constraint the first line is constraints that is our these things. So, we already wrote C for it.

So, we have constraints C. So, we introduce gauge conditions. This nomenclature also has lot of disputes, but it has states. So, I will not bore you with that. So, we introduced gauge conditions $\gamma^a(q, p) \approx 0$ such that they are complementary to this C. They choose a particular point out of this. So, the problem with this C is that if you draw some n-dimensional phase space ok. So, that is a q_i and then p_i and lots of dimensions and you have a constraint surface $C^a \approx 0$ right in this big space you are supposed to restricted to that and the point is that if you actually try to do path integral and kept cutting it at lot of places you are doing redundant integration. If you are doing integration with in this then you are doing redundant integration because they are all equivalent. You should cut it only once.

So, the gauge condition $\boldsymbol{\gamma}$ are supposed to be something that cut them in judicious way. So, $\boldsymbol{\gamma}$ should fix a particular trajectory out of it. For the timing I will draw I mean I do not know how to draw it, but. So, it will choose a particular point out of these. So, that you are not integrating over superfluous degree. So, the fact that superfluous integration would happen perpendicular to it because $C^a \approx 0$ is the physical surface. So, you would be integrating within that and going outside of it is actually not required.

So, you want to fix it. So, that you choose make a particular choice of unique variables out of the constraint ones and that is done by erecting a complimentary surface $\boldsymbol{\gamma}$; only thing is I do not want any overlap at all. I just want one point out of it right. So, the correct thing to show would be to draw a stack of these 0, 1, I mean any number any fixed number would be fine. Any one of those is a valid surface and you want to select one out of it that is the correct thing to show which the little hard to draw ok. So, $C^a \approx \alpha^a$ some constants. So, any one of them is fine because if you set it equal to some constant it is ok. So, you put this gauge conditions which make a choice out of this stack of surfaces.

And, an example is we know take Maxwell. We know the equation $\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$ or some constant or some external current. This reads $\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$. So, Lorenz not Lorentz observes that if you set $\partial_\mu A^\mu = 0$, then we get only $\Box A^\nu = 0$, which has just d'Alambertian independent equations for each variable. So, this it is no longer tied up with derivatives of the other components and they become all free. So, this is the gauge this is our gauge condition γ^a the condition is this.

Of course, that also does not fix everything because as we know there are two different things. So, in this if you now set A^0 to be trivially 0, then you will get that divergence A would have to be 0 which is that A has to be purely a curl like field, but this is the meaning of putting this. Now, in the canonical language we can then think of the $\boldsymbol{\gamma}$'s as some kind of conjugates to the gauge conditions.

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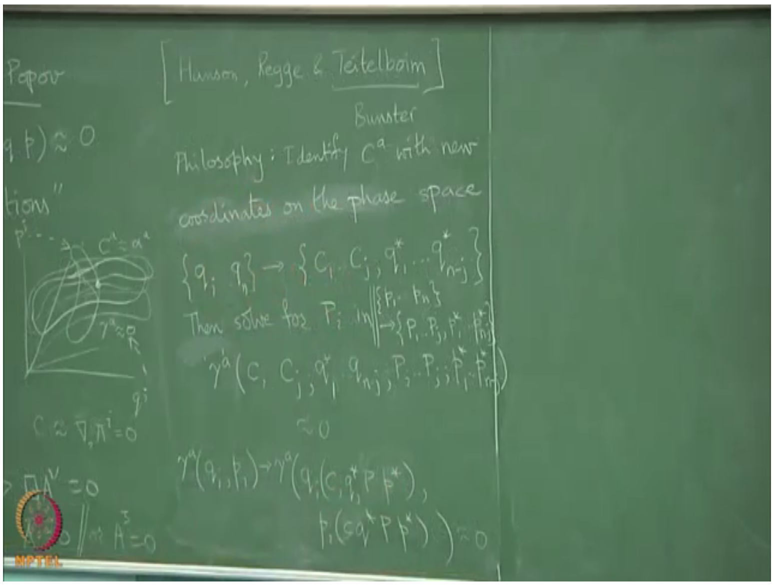

So, there is a ancient book called Hanson, Regge and Teitelboim. So, I am going to explain a few things out of that book. This is a very arcane book, but somebody has brought out a new version now we were taught by Teitelboim and Teitelboim has now a new name he is called Bunster ok. Teitelboim is very clever Chilean physicist; he was working on quantum gravity according to Bryce DeWitt program, but using functional methods path integral and John Wheeler thought that he was going to crack the problem of quantum gravity. Anyway, Teitelboim went back from Texas to Chile and started a new institute and later he changed his name to Bunster.

Whatever that is coming back to this the philosophy can be stated like this. So, the philosophy is that identify C^a with the with new configuration space variables, new coordinates some subset right. So, $\{q_i, \ldots, q_n\} \rightarrow \{C_1, \ldots, C_j; q_1^*, \ldots, q_n^*\}$, the starred one will be free. So, you some what do this transformation on the phase space and then you try to. So, yeah then you try to treat the $\boldsymbol{\gamma}$s. $\boldsymbol{\gamma}$s themselves will not be conjugate to the C, but you arrange it so that you find conjugate.

So, then solve for P_i in $\gamma^a(C_i, ..., C_j; q_1^*, ..., q_{n-j}^*; P_i, ..., P_j; P_1^*, .., P_{n-j}^*)$. So, now you claim that so, your γ^a were on the phase space. So, they were functions of q_i and p_i. This can be written as $\gamma^a(q_i, p_i) \rightarrow \gamma^a(q_i(C, q^*, P, p^*), p_i(C, q^*, P, p^*))$. So, you make a this is all formal nobody is telling you that you can actually do it. The claim is that you can first you adopt some coordinate transformation such that the ones that are unencumbered or group as q^* and ones that have any problem with them you just put the C_1 ,…., C_j whatever constraints you have treat them as coordinates on this phase space now. So, it is an nontrivial statements. So, actually not so, I should say coordinates on the phase space.

And, then formally propose that p_1, p_2, p_n similarly becomes that I should have written as $\{p_1, ..., p_n\} \rightarrow \{P_i, ..., P_j; p_1^*, ..., p_{n-j}^*\}$. Now, this is a purely formal statement for the time being you do not know what the hell the capital P's are, but you do know the conjugates to the unencumbered one. So, you list them and put he has some symbols P. Now, you substitute this over here of course, what to substitute you do not know. So, you are going to just write them as functions of q to begin with, but let me just write out the formal procedure then you can think about what is actually doable and what is a nearly a statement of existence of such transformations.

But, the point is it conceptually clarifies to us what is going on. So, if you do this wherever in the old phase space you replace all the small q and p by the condition C and the unencumbered q^* s some formal P's and p^* s and similarly old p's like this and that set that is equal to 0. There will be as many of them this as there are this fictitious p's. So, you solve these to find p's in terms of γ. You invert this to determine the capital P in terms of the old q and p.

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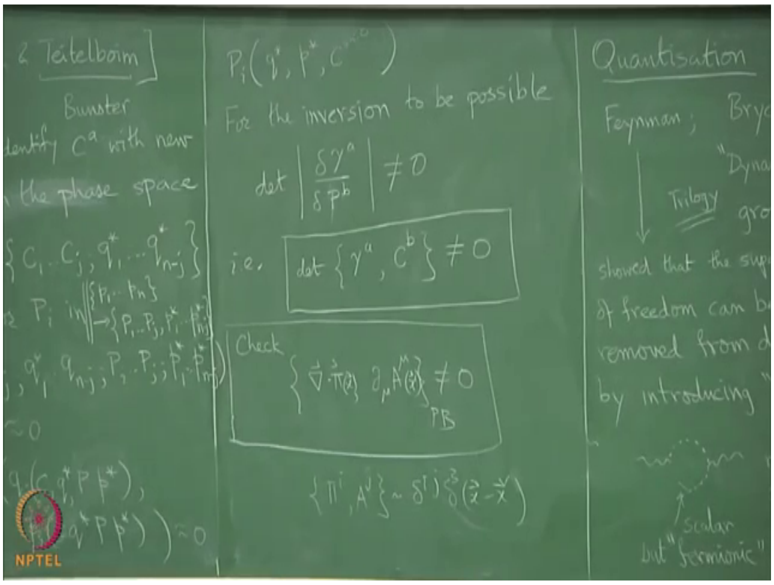

So, you solve for capital P in terms of q^* , p^* , C that is all that is really because you are going to invert it and C is that anyway equal 0. So, but you will have some formal expression for the capital P in terms of q^* , p^* and the C's and now you. So, in order for this inversion to be possible, it is necessary that γ and the old p are conjugate to each and the p's conjugate to each other they are new P's.

So, we need that. So, we are coming to a somewhat crucial condition that comes out of all this rather formal mumbo jumbo which is that very. So, the formal statement is that this set of equations should not be equal to 0. But, this we can now cleverly state as $det\left|\frac{\delta\gamma^0}{\delta P^b}\right|\neq 0$ right way we need this because, we are going to find the solution you need to there has to be independence of the variables γ and there have to be as many γs as p's and they cannot be coinciding with each other, otherwise you will have a 0. If there any 0 in the determinant then you will not be able to invert.

So, you required this, but then you look at it and that is note that this is nothing, but the $det\{\gamma^a, C^b\}\neq 0$. This is the real requirement on γs. So, now you see that you do not have to worry about this fictitious P's after all, but this is how one thinks to arrive at that answer and we can see that this C and this γ will have a nonzero Poisson bracket $\{\nabla.\vec{\pi}(\vec{x}), \partial_\mu A^\mu(\vec{x}')\}_{PB}\neq 0$.

So, you can do it is actually visible trivially, but anyway you can do it. So, right because there is no A^0 in this and the only dot there is A^0, other things are all canonical sorry I am going the reverse answer, but 1 minute. So, what mean is the A's and **π**'s of course, $\{\pi^i(x), A^j(x')\} \approx \delta^{ij}\delta^3(\vec{x}-\vec{x}')$. So, you will find that there are non-trivial Poisson brackets left over and the it is not collapsing to 0.

So, that selects for you the physical phase space. Now, next time we will complete the Fadde'ev-Popov trick where we will see that this determinant is what is going to be an important feature not that determined itself after one more reinterpretation of the determinant it will enter in the final answer ok.