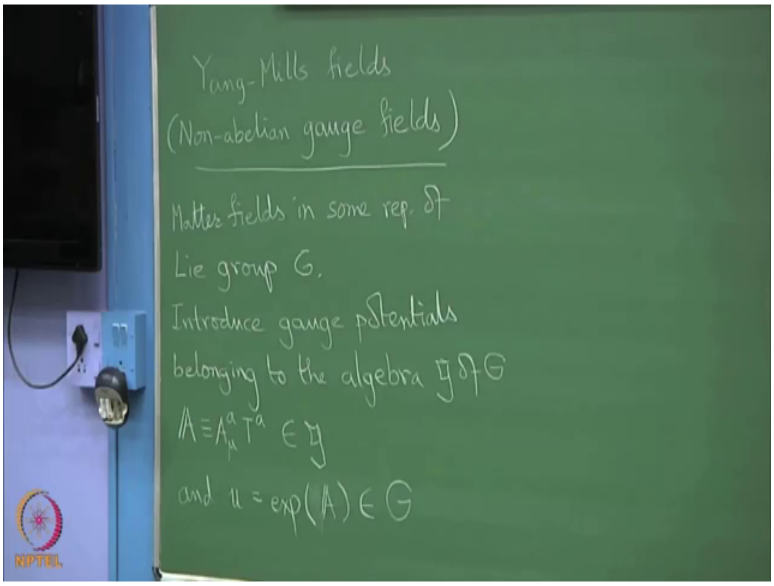
Path Integral and Functional Methods in Quantum Field Theory

Prof. Urjit A. Yajnik
Department of Physics
Indian Institute of Technology, Bombay

Lecture – 27
Yang Mills Theory Constraint Dynamics – I

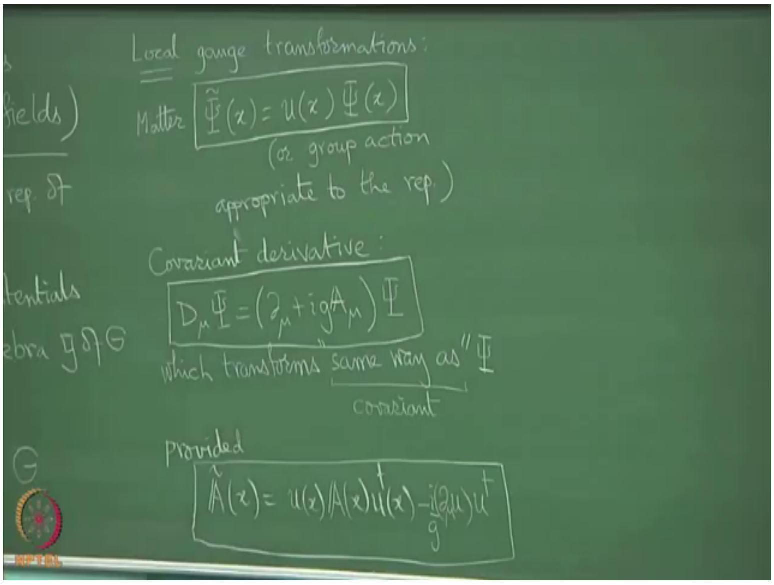
(Refer Slide Time: 00:19)



So, what we learnt about these is that we have matter fields in some representation of group G you know Lie group G. And, we introduce gauge potentials which belong to algebra of $\mathfrak{g}$ often it is written with some kind of Gothic character.

So, $\boldsymbol{A} \equiv A^a_\mu T^a \in \mathfrak{g}$ and $u = \exp[\boldsymbol{A}] \in G$ here because we have already defined **A** belonging to the algebra. So, you exponentiate get the group elements.

(Refer Slide Time: 03:09)



But, then what we say is that the so, called local gauge transformations or that your matter multiplet $\tilde{\psi}(x)=u(x)\psi(x)$, where u is in the appropriate representation ok. So, this is the simplest case is when ψ is just in the fundamental or in the vector, but and then we said that the covariant derivative, which we define to be $D_\mu \psi=(\partial_\mu+ig\,\boldsymbol{A}_\mu)\psi$.

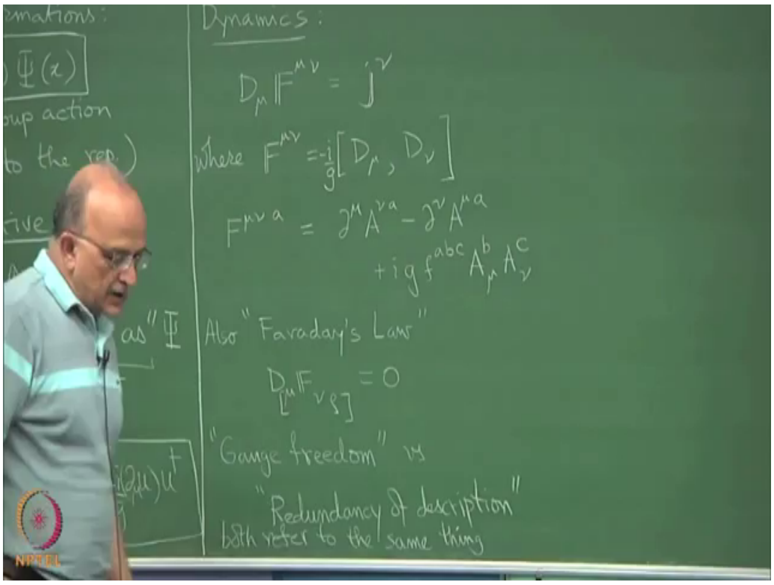
And, this is where have to worry about the where the g goes. Probably from here itself I think I have to normalize the a correctly we will just see. So, there is the geometric point of view where you do not worry about this coupling g, but in physics we do not need this g. So, better put it. This is good because this is certainly very physical.

Now, this has the property that same way as ψ itself, the same way as is what is meant by covariant. So, it transforms the same way as ψ, provided

$$\tilde{\boldsymbol{A}}_\mu(x)=u(x)\boldsymbol{A}(x)u^+(x)-\frac{i}{g}(\partial_\mu u)u^+$$

So, I do like this particular definition of covariant derivative with the i and g there that makes everything physical and Hermitian. So, those three things go together, it is a packaged deal that if ψ goes like this and if $\tilde{A}$ a goes like this, then this derivative will be covariant it will transform exactly like application of u to it.

(Refer Slide Time: 08:39)



Now, we come to the equations of motion which were so, we learn that $D_\mu \boldsymbol{F}^{\mu\nu} = \boldsymbol{j}^\nu$

where $\boldsymbol{F}^{\mu\nu} = -\frac{i}{g}[D^\mu, D^\nu]$, but it has the detailed expression

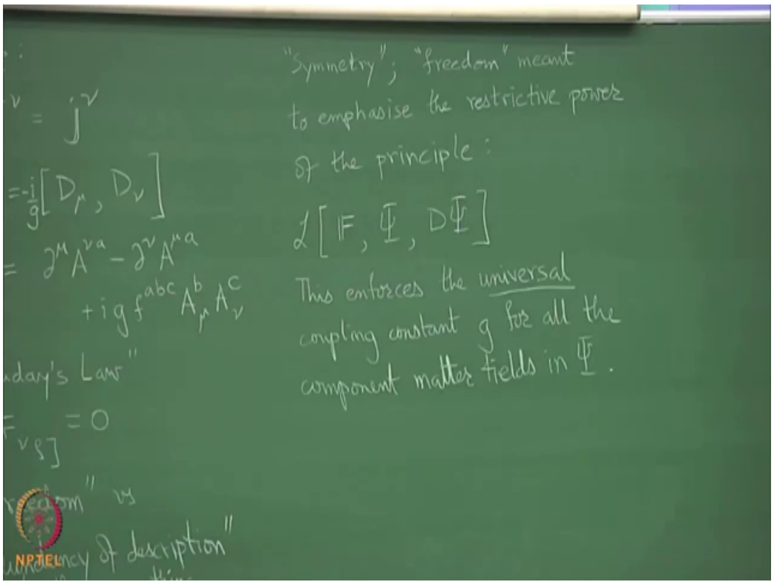
$$F^{\mu\nu a} = \partial^\mu A^{\nu a} - \partial^\nu A^{\mu a} + igf^{abc} A^b_\mu A^c_\nu$$

. So, the other part is Faraday's law which is not going to be all that important right now, which is this where you permute this cyclically is equal to 0, that is sort of trivially satisfied because of the way F is defined. So, that is not a new input.

Because, our we are treating the gauge fields as the fundamental thing the gauge potentials as the fundamental dynamical degrees of freedom. So, the thing is that for the purpose of doing all the important manipulations, it is useful to use the gauge potentials. However, the gauge potentials have a huge ambiguity in them, which people call the gauge degree of freedom, but at the same time; it is actually lot of people call it redundancy of the description ok.

So, both describe the same thing. We call it freedom because we want to make virtue of it because it allows us to make a prescription that all of physics is determined by.

(Refer Slide Time: 12:48)



So, the symmetry, freedom etcetera are meant to emphasis the restrictive power, that l can power of the principle, which is Lagrangian depends on the dynamical variables as, $L[\boldsymbol{F}, \psi, D\psi]$.

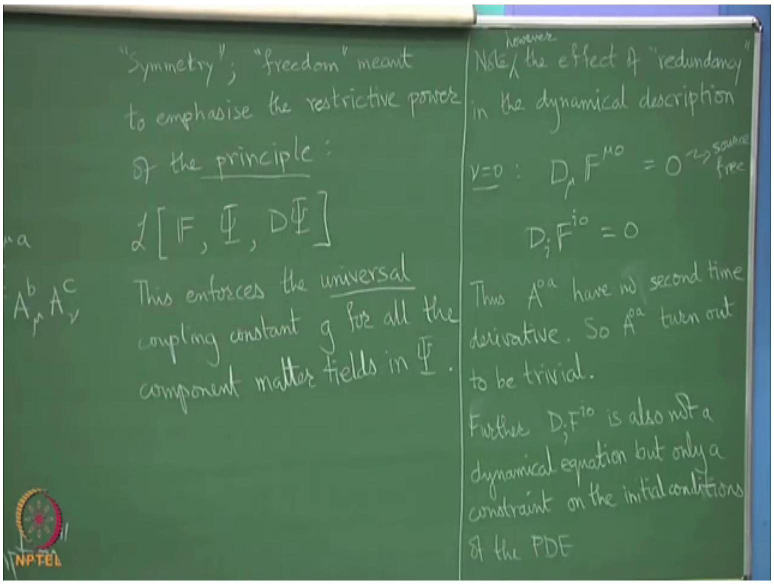
These are the only things that it can consist of. So, the fact so, that is the power of the gauge the covariant derivative; only the covariant derivative can occur not ordinary derivative. And, that brings up the most important physical thing that somebody asked last time, which is that remember the ψ is a multiplet, it is either a vector or a matrix in some representation of the group G. For all of those there is only one coupling constant small g.

So, this is called universality. So, whether you have neutrino which has a mass 0 as far as week interactions are concerned or you have the top quark, which has mass 176 GeV. They are coupling to the gauge goes on $W^{\pm}$ and Z is exactly the same is the same number ok. So, the entire generation of fermions all have exactly same g.

So, that is the power of this principle and that is when people try to think of positive words like it is a symmetry, it is a freedom in the gauge fields and so on.

But, if we now begin to do the dynamics then we see the nuisance that this creates. And that is where we observe that it is actually a big redundancy in the description. So, we begin with that now.

(Refer Slide Time: 16:20)



So, note; however, in the dynamical description. Firstly, so, the thing is that it allows you to define the Lagrangian very elegantly, but once you are defined the Lagrangian. So, all kinds of superfluous terms that you would be worried whether to include or not are gone; not only that lot of the terms are exactly same coefficients.

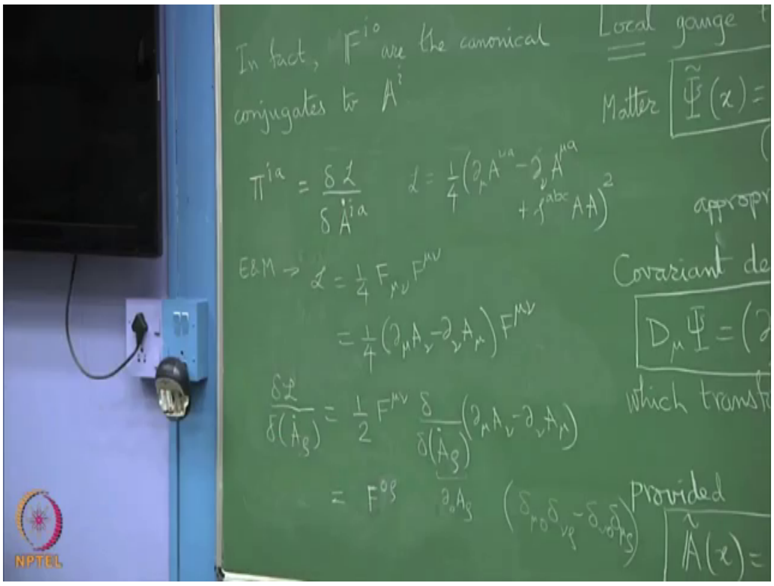
So, it is a very powerful restriction, but now you come to the dynamics and begin to look at it the first thing is that the zeroth component of this equation $\boldsymbol{\nu}=0$ requires that so, that $D_\mu \boldsymbol{F}^{\mu 0}=0$, but you or whatever the current is. So, it does not matter if it is non-zero.

So, source free case just to keep thing simple, but the assertions we are making about freedom versus non freedom are all the same, because this is some given current. So, what this means is because F is anti-symmetric; if this index is already fixed to 0, this $\boldsymbol{\mu}$ can only be space indices. So, this is actually the same as $D_i \boldsymbol{F}^{i 0}=0$. Now, they says essentially an identity. So, it contains no dynamical information.

All the dynamical information was supposed to be here, but now you notice that the A^{0a} never has a second time derivative. In fact, there is no time derivative here at all it is all space derivatives. So, so A^0 now this equation is nothing, but Gauss law ok, because you know that F^{i0} are the electric field components. So, this is just says $\nabla . \vec{E}=0$, div E constrainted to be whatever source you have. So, this is not a dynamical equation constraint on the initial conditions of the PDE, you have a partial differential equation to solve.

So, you want to set up fields and it is derivatives on the initial surface, what they say that you cannot do the spatial derivatives arbitrarily. The x dependence if so, you are going to be having have time evolution starting from some surface that are t=0. Let us say, but the you have to set up E fields here and you also have to setup there you setup the full E field and their time derivative, but the gradient of the E field in this also is constraint it is not the arbitrary.

(Refer Slide Time: 21:59)



Now, let us look at this from a canonical point of view, because $\pi^{ia} = \dfrac{\partial L}{\partial \dot{A}^{ia}}$

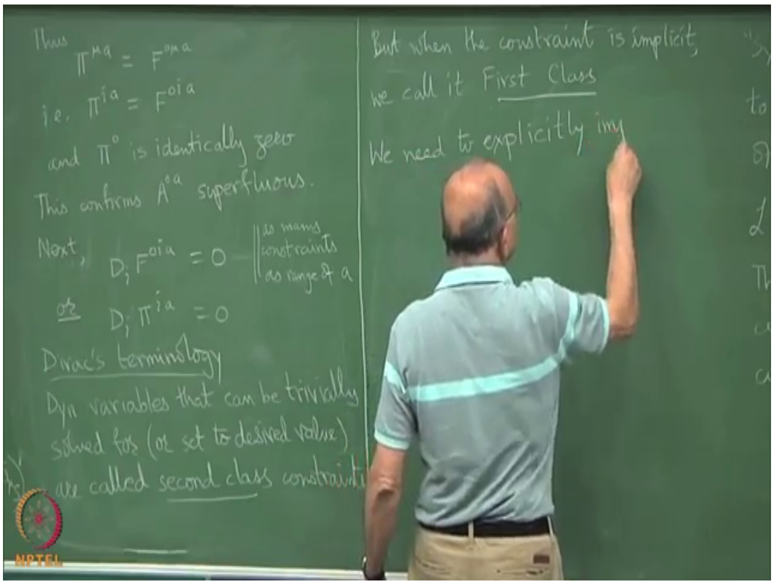
But, then $L = \dfrac{1}{4}\left(\partial^{\mu} A^{\nu a} - \partial^{\nu} A^{\mu a} + f^{abc} \boldsymbol{AA}\right)$. So, if you try to vary this you have to search where the time derivatives are.

So, let us look at E and M first, $L = \dfrac{1}{4} F^{\mu\nu} F_{\mu\nu} = \dfrac{1}{4}\left(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}\right) F^{\mu\nu}$. So, if we vary

$\dfrac{\partial L}{\partial \dot{A}_{\rho}} = \dfrac{1}{2} F^{\mu\nu} \dfrac{\delta}{\delta \dot{A}_{\rho}} \left(\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}\right) = F^{0\rho}$ So, that is what the answer will look and the first term gives exactly this and the second term is actually the same after you exchange the indices here and this is here. So, and now you see that this does not change if you go to the Yang-Mills case, because in the Yang-Mills case this term is not does not contain derivatives of A. So, it is not going to contribute.

So, you vary F squared so, you are left with this so, you add here f^{abc} if you like, but times the whole $F^{\mu\nu a}$ and then the variation then enforces these in decease. So, in the Yang Mills case, it will be $F^{0\rho a}$, that is the only difference. So, we find that the canonical conjugates to the gauge potentials are the $\boldsymbol{F}^{0\mu}$ components, but; that means, that A^0 has no canonical conjugate.

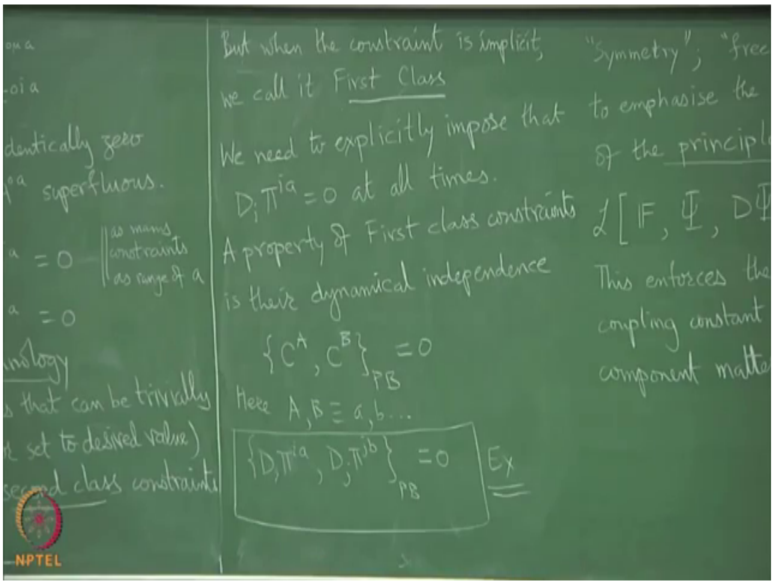
(Refer Slide Time: 28:51)



$\pi^{\mu a}=F^{0\mu a}$, but that automatically means that $\pi^{ia}=F^{0ia}$. So, that is why the A^0 is completely redundant, it has no dynamics really its time derivatives do not appear, but now we get the additional constraint. We have the constraint that $D_i F^{0ia}=0$.

So, there are as many equations as range of a or which I meant to say really $D_i \pi^{ia}=0$. So, in the canonical dynamics the even the space like momenta are not really completely on constraint. So, Dirac the great formalist try to put this in a elegant language and then he classified that things of the type A^0 should be called second class constraint some part of British value system is here, what is really superfluous, we call second class.

(Refer Slide Time: 32:57)



On the other hand when the constraint is implicit, we call it first class. In particular you have to check that in the course of dynamics you do not violet it. So, it is not only a matter of setting initial condition.

But you have to maintain that at all times. That because it is so, that is why we call it a constraint it has to be carried along and this Dirac calls first class. Here we have so, a property of first class constraints is that dynamical independence namely $\{C^A, C^B\}_{PB} = 0$, because in the and which is true of this. Because you can think of each of these as a, it is a bit unfortunate I should call it something else, this is a more general statement. So, here A,B ≡ a,b,c,....

And, we can say that it holds because canonical bracket of any particular $\boldsymbol{\pi}$ with any other $\boldsymbol{\pi}$ is non-zero. So, in the pass on bracket you pull the derivatives out the derivative is a well is a linear operation if it or the ordinary derivative in the case of or this ok. So, it is actually not all that trivial. In the case of Yang-Mills of course, it is not a linear operation, but one can check that it will work out ok. So, this is a non-trivial exercise, because, you will have to use Jacobi identity of the f^{abc} s to make sure. So, $\{D_i \pi^{ia}, D_j \pi^{jb}\}_{PB} = 0$ so, this is an exercise so, for electromagnetism it is obvious, but there is only one constraint so, nothing to check.