Path Integral and Functional Methods in Quantum Field Theory

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Lecture – 26
Yang Mills Theory – Physical Content

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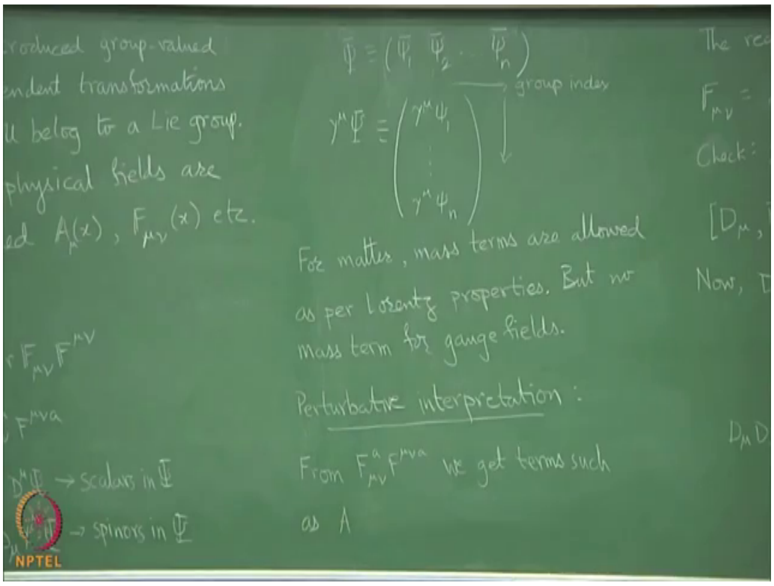

roduced group-valued
endent transformations
ll belong to a Lie group.
physical fields are
ed $A_\mu(x)$, $F_{\mu\nu}(x)$ etc.

$F_{\mu\nu}F^{\mu\nu}$

$F^{\mu\nu a}$

$D^\mu\Psi$ → scalars in Ψ

→ spinors in Ψ

$\bar\Psi = (\bar\Psi_1\ \bar\Psi_2\ \ldots\ \bar\Psi_n)$ → group index

$$\gamma^\mu\Psi = \begin{pmatrix}\gamma^\mu\psi_1\\ \vdots\\ \gamma^\mu\psi_n\end{pmatrix}$$

For matter, mass terms are allowed as per Lorentz properties. But no mass term for gauge fields.

Perturbative interpretation:

From $F^a_{\mu\nu}F^{\mu\nu a}$ we get terms such as A

The rea

$F_{\mu\nu}$ =

Check:

$[D_\mu,$

Now, T

$D_\mu D$

Right, I now want to make some particular remarks, one is that if you try to do perturbation theory using this. So, that is how now this nonlinearity becomes important in the diagrammatics. The perturbation theory shows that the coupling g is asymptotically free as it is called g grows weaker at so, firstly, from the $F^a_{\mu\nu}F^{\mu\nu a}$ terms.

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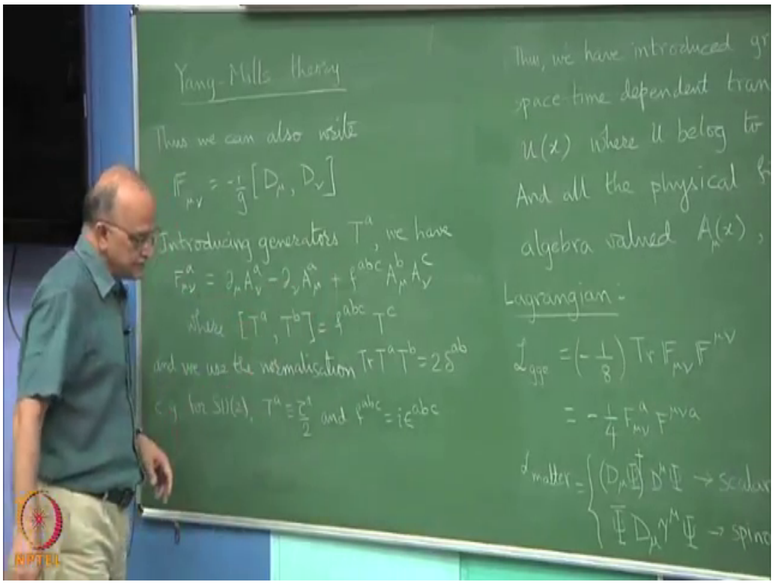

Yang-Mills theory

Thus we can also write

$F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]$

Introducing generators T^a, we have

$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$

where $[T^a, T^b] = f^{abc} T^c$

and we use the normalisation $\mathrm{Tr}\, T^a T^b = 2\delta^{ab}$

e.g. for SU(2), $T^a = \frac{\tau^a}{2}$ and $f^{abc} = i\epsilon^{abc}$

Thus, we have introduced gr
space-time dependent tran
u(x) where u belog to
And all the physical f
algebra valued $A_\mu(x)$,

Lagrangian:-

$\mathcal{L}_{gge} = \left(-\frac{1}{8}\right) \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}$

$= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$

$\mathcal{L}_{matter} = \begin{cases} (D_\mu\Phi)^\dagger D^\mu\Phi \to \text{Scalar} \\ \bar{\Psi} D_\mu \gamma^\mu \Psi \to \text{spino} \end{cases}$

Such as $g\,\partial_\mu A^a_\nu f^{abc} A^{\mu b} A^{\nu c}$.

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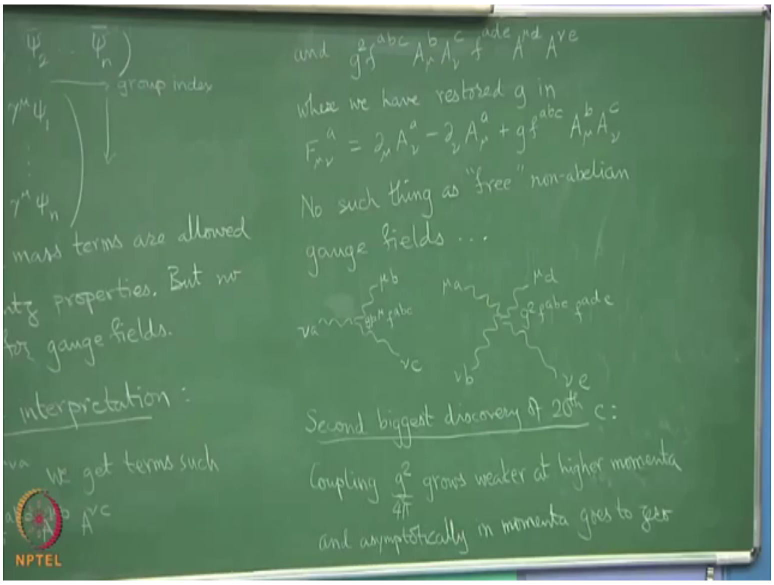

$\bar{\psi}_2 \ldots \bar{\psi}_n)$

group index

$\gamma^\mu \psi_1$

$\gamma^\mu \psi_n$

mass terms are allowed
tz properties. But no
for gauge fields.

interpretation :

va we get terms such
$A^{\nu c}$

and $g^2 f^{abc} A^b_\mu A^c_\nu f^{ade} A^{\mu d} A^{\nu e}$

where we have restored g in

$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$

No such thing as "free" non-abelian gauge fields . . .

Second biggest discovery of 20th c:

Coupling $\frac{g^2}{4\pi}$ grows weaker at higher momenta and asymptotically in momenta goes to zero

And so, and there is $g^2 f^{abc} A^b_\mu A^c_\nu f^{ade} A^{\mu d} A^{\nu e}$ So, if you want to do perturbation theory then you have to introduce this g.

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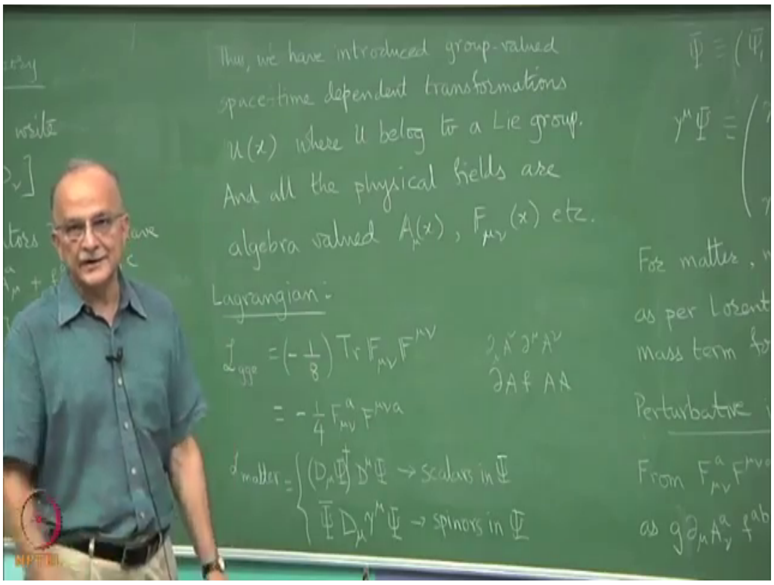

Thus, we have introduced group-valued
space-time dependent transformations
U(x) where U belog to a Lie group.
And all the physical fields are
algebra valued $A_\mu(x)$, $F_{\mu\nu}(x)$ etc.

Lagrangian:-

$$\mathcal{L}_{gge} = \left(-\frac{1}{8}\right) \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} \qquad \partial_\mu A^\nu \partial^\mu A^\nu$$

$$= -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} \qquad \partial A f A A$$

$$\mathcal{L}_{matter} = \begin{cases} (D_\mu \Psi)^\dagger D^\mu \Psi \rightarrow \text{scalars in } \Psi \\ \bar{\Psi} D_\mu \gamma^\mu \Psi \rightarrow \text{spinors in } \Psi \end{cases}$$

$\bar{\Psi} \equiv (\Psi_1$

$\gamma^\mu \Psi =$

For matter,
as per Lorent
mass term fo

Perturbative

From $F^a_{\mu\nu} F^{\mu\nu}$
as $g \partial_\mu A^a_\nu f^{ab}$

Because what even the so called free field theory or the simplest possible theory you can possibly write for the field strengths or for this group algebra valued potentials A_μ, the kinetic terms come packaged which the interaction terms. You cannot; so, by kinetic term we mean things that are just square of derivative right, till $\dot{x}^2$ so, but along with $\partial_\mu A^\nu \partial^\mu A_\nu$ you will automatically also get stuck with stuff like this $\partial A f A A$ which is all non-linear in A.

The only Possible gauge invariant Lagrangian you can write is packaged with interactions, but fully determined, this is not left to imagination it has to be exactly g and it has to be exactly g^2. So, you do not have one derivative and 2 derivative with someone coefficient and 4 is with some other coefficient this coefficient is exactly g^2 and the and whatever the value is that f^{abc} give you. So, it is a big package deal and a little unusual from our conventional approach of taking free Lagrangian's and adding interactions to them. It is a package deal in which interactions if you so, if you want to think of the A_μ as the elementary potentials which you quantize by saying $[A, \pi_A] \approx i\hbar$, then those quanta are automatically forced to be interacting you do not have a theory of free quanta free non-Abelian quanta.

However if you start, if you assume applicability of perturbation theory and treat these as diagrams ok. So, we can treat this as a diagram with 3 gluons where one of them because there is a derivative there is a p_μ on it and g will be the strength ok. So, some you have to put some a and μ sorry and the f^{abc} and then there is a 4 gluon diagram with that g^2.

Now, if you use these as elements of perturbation theory then you find that the coupling g gets renormalized in a way that it grows weaker at higher momenta or scattering amplitudes. So, this is the second biggest discovery of and asymptotically goes to 0 asymptotically momenta goes to 0.

So, the first biggest discovery is to discover that gauge symmetry describes all the interactions, but the second biggest discovery is that the couplings actually go to 0 asymptotically.

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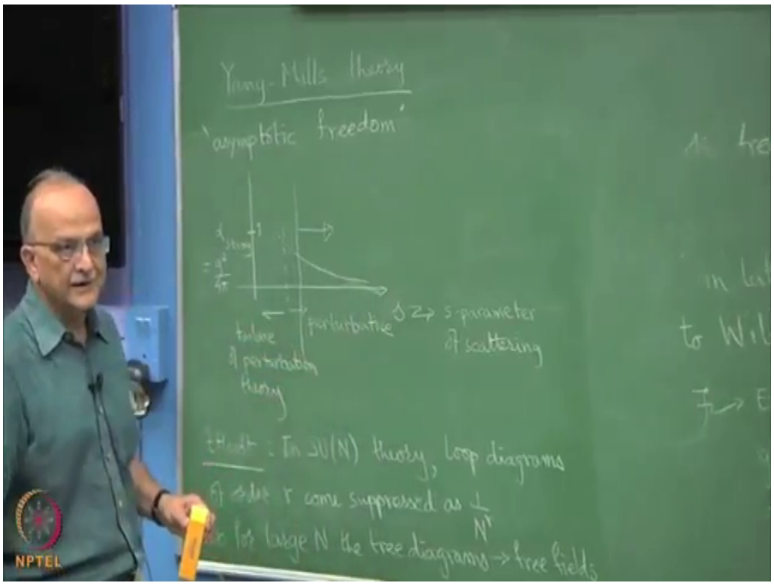

So, this led to finding the very mysterious term asymptotic freedom asymptotically the gauge field theory is a free field theory. So, no such thing is the little too strong if you can leave asymptotically at infinite momentum then you have a free field theory, but then you have to live asymptotically until you here. And, but there is another clever thing that was discovered by t'Hooft that so, wait in parts. So, we draw this by saying that $\alpha_{strong} = \frac{g^2}{4\pi}$ for SU(3) not SU(2) it runs like this. So, it is so, put 1 over here and it is something small and of course, going to 0. So, square in S you know S is the s-parameter of scattering is a function of s is it goes to 0.

But then you can only put dotted lines here, because here perturbation theory begins to fail, if it goes towards one then the perturbation theory fails. So, we do not really know what happens here what once it approaches one we do not know whether it becomes one. In fact, it becomes meaningless because now you cannot separate you. So, the main conceptual problem here is not that it is a just that it is a strongly coupled theory, it is not as if you had the same excitations A_μ which are now fighting with each other more strongly.

It is that you do not even now what the degrees of freedom are, if this theory is asymptotically free then you can see a free quantum streaming out and you can say oh this spin-1 particle or helicity-1 particle is what in the interaction region interacts, but when it is here you cannot even see what the ingredient of the theory is.

So, really whether one can then separately quantize the A’s and then think of interaction among them all that fails, you remember in the beginning of the semester we went through this Fock-Dirac quantization where we said you can ice identify that there is a bosons or fermions and then if there are bosons the states get can be labeled simply by the number of bosons. All that logic was applicable to the free theory or very weakly interacting theory. So, that the interactions were local, but by and large the system was non interacting, that entire picture fails completely and we do not actually know how to well. So, quantizing the system here is a dicey affair ok.

Some light on this was thrown by a t’Hooft’s clever observation that if you use large N. So, the in that you take SU(N) group. So, the N of SU(N) you make it large then it turns out that that N enters somewhere here I should have written it in that form. So, the coupling itself becomes showing the diagrams all any loop of with any graph of N loops has a 1/N in front of it therefore, in the limit of large N all the loops become sub dominant and only tree diagrams survive ok.

So, for large N only that tree diagrams are dominant which is the free field theory, well not exactly free, but because the tree diagrams are still there, but for quantum field theory tree is supposed to be the free part of it, because people think of it as the essentially the classical thing. There is no real quantum mechanics it is just that things can break up and recombine, but there are no, in a tree diagram there is no internal momentum to be integrated over right, because the energy momentum conservation fixes all the momenta in the so, all you have is a matrix element.

So, tree diagram essentially is free fields although things can break up and recombine, but there are no quantum corrections. So, SU(N) theory is for large and you get free fields. In fact, I think only the planar diagram survive or something like this. So, this was a interesting simplification, the reason I am telling you this is that although we have a differential equation you know from that $F_{\mu\nu}F^{\mu\nu}$ Lagrangian we can write equations of motion which I did not write I will come back to it ok.

So, we can write the equations of motion for F, but it is meaningless to solve them because there is nothing called a classical Yang-Mills field it is meaningless there is no state of the system. So, for photons we have plenty of classical looking states. In fact, they are fully quantum, but we can think of free streaming photons, we can count individual photons, we have you know children attach meters and major potentials, you cannot do any such thing to gauge fields, they actually do not have a classical existence at all.

Anybody was trying to solve a differential equation find an exact solution of it and try to think that he has actually found something physical is mistaken, because it has no such interpretation there is no such thing. So, you might try to say $A_\mu = \langle\psi|A_\mu(x)|\psi\rangle$? right in which I find this as an expectation value.

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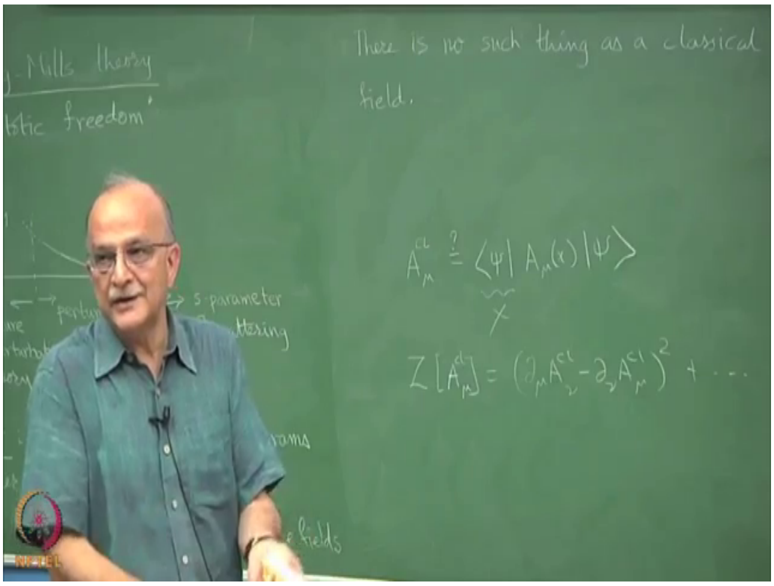

There is no such thing as a classical field.

$$A_\mu^{cl} \overset{?}{=} \langle\underset{\times}{\psi}| A_\mu(x) |\psi\rangle$$

$$Z[A_\mu^{cl}] = (\partial_\mu A_\nu^{cl} - \partial_\nu A_\mu^{cl})^2 + \dots$$

There are no such states available in which you can take expectation value of your field and then say oh this is my classical value, in other words we cannot write any effective potential which is you know whatever, there is no such thing Z[A] classical it has no meaning. In nowhere that you can derive such an effective potential out of the full path integral the functional of the theory.

However classical solutions do play a very interesting role and so, I will come back to it later. So, first thing is there is no such thing as free non-Abelian gauge fields and the second thing is that there is in fact, no classical gauge field, because the only gauge invariant Lagrangian you can write for it comes packaged with interactions and if you put in the interaction then it is strongly coupled and there is nothing like that is classical. You can say oh, but I am going to observe it at only very high momenta, very good luck because you have to first get one to give it to a give it high momentum, but you can never get one out because. So, in reality we only find them.

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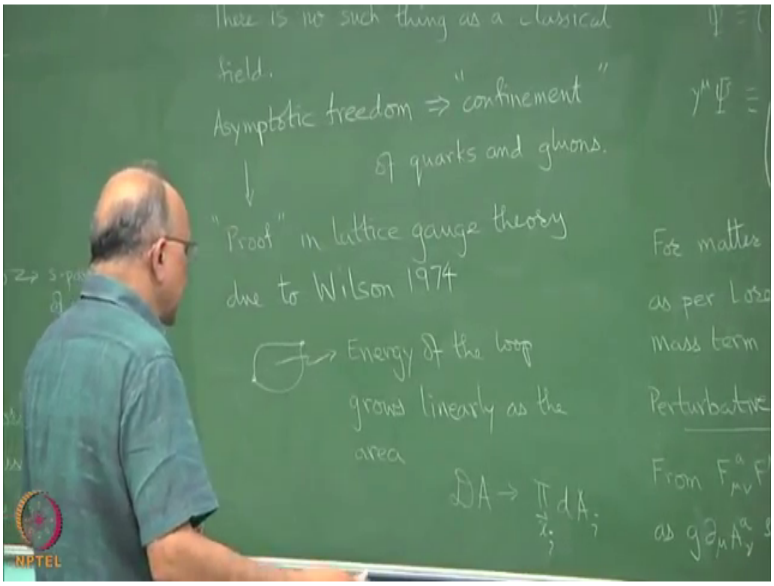

So, this asymptotic freedom implies confinement, which is a funny word because then you at least presume that you know what is confined, but you do not know what is really confined well there is a quarks and gluons.

Neither quarks nor gluons can be pulled out indefinitely, because if you do it then you are going to infrared limit, if the point is to separate two things out and to see them separately you will have to go to large distances, but large distances usually mean low wavelengths, which means the infrared, this is the s-parameter. So, at large collision energies you can think of it as weak, but when you are trying to pull it apart you are in this region and then you can never really pull it apart ok.

So, this required a proof by lattice gauge theory. So, the only proof we know; what Wilson did was to introduce the lattices primarily a regulator you introduced some separation you to not to avoid the ultraviolet parts. So, you have integrated ultraviolet parts you just have points in space and what he proved was that any two quarks the force between them grows as the area bounded by the flux lines that connect them so, alright. So, the statement is that the area of this energy of this grows linearly as the area. So, if you try to take contributions of all the possible loops then you get infinite answer.

So, this indicative proof that send you have to you can think of some quark sitting on this loop. So, I have already put the quarks, but he just proves it for gluons at between the gauge fields. any gauge loop it is energy grows as the linearly as the area, but nobody has the full picture in the same language. So, the problem with lattice is a very peculiar one. So, once lattice gauge theory was invented people actually started doing numerical calculations, because now you take the functional integral and instead of doing the DA you simply start doing product over.

So, here you remember you had to do product over all space time points, but now you can do it over the lattice points x_i, right you can do this. So, people put it on a numerically on a lattice and then try to calculate. They get various answers, but eventually they have to extrapolate them to lattice spacing going to 0 to recover the continuum. What happens then is a very interesting thing, because lattice is the real lattices in real life they themselves show phase transitions as you change the lattice spacing depending on what interaction you have put between the near binding interval between the members of the lattice.

There are phase transitions so, you do not know whether you are actually approaching a lattice phase transition or you have reach the continuum limit ok. So, conceptually there are some issues with lattice gauge theory. Now so, it is an unproven mathematical fact, but it is a well established empirical fact that the effective coupling grows weaker at high. So, this now in LHC of course, which is such a high energy they are mostly weakly coupled quarks and things that they did in early days can now be applied very freely by extrapolation, the energy dependent running is exactly as expected and so on.

So, there is a lot of confirmation of QCD in this regime, we know that it is exactly like this we also have never pulled out any free quark or gluons. So, the confinement hypothesis looks correct, whether it mathematically follows from this particular Yang-Mills symmetry is not proved, but it would be too surprising if there was something uglier that gave the same answer, because this is very elegant and has all been ingredients needed to explain it. The last thing I can do is we get to this topological part and the way to motivate it is that well.

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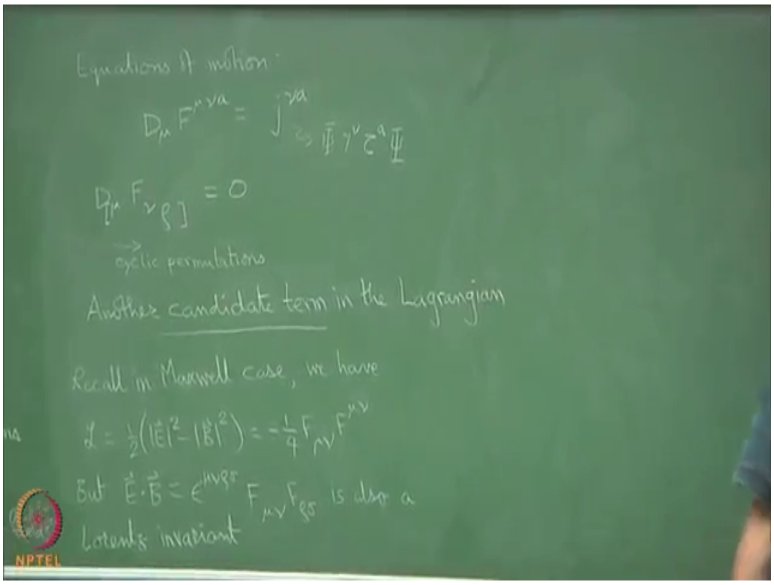

Equations of motion:

$$D_\mu F^{\mu\nu a} = j^{\nu a} \hookrightarrow \bar{\psi} \gamma^\nu \tau^a \psi$$

$$D_{[\mu} F_{\nu\rho]} = 0$$

$\xrightarrow{}$ cyclic permutations

Another candidate term in the Lagrangian

Recall in Maxwell case, we have

$$\mathcal{L} = \frac{1}{2}\left(|\vec{E}|^2 - |\vec{B}|^2\right) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

But $\vec{E}\cdot\vec{B} = \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ is also a Lorentz invariant

So, before we go on just the equations of motion are that no surprise in guessing $D_\mu F^{\mu\nu a} = j^{\nu a}$, j^ν is constructed out of the quark. So, it is $\bar{\psi}\gamma^\nu\tau^a\psi$ right. So, it is so, this is all like this know right. So, that is what the currents are and the other associated equation is the Faraday's law and deep and div B equal to 0 law is simply written by $D_\mu F_{\nu\rho}$ + cyclic permutations of $\mu,\nu,\rho = 0$.

So, this is exactly for the same as in electromagnetism. So, structurally it looks very similar, but as I told you there are subtleties because there will be $A^\mu A^\nu$ terms here, in the f there will be small $f^{abc} A^{\mu b} A^{\nu c}$. And so, there is no point trying to put solve this equation, but we will see some clever solutions that do exist. One other point is that the Lagrangian also contains another candidate term.

So, recall in Maxwell case, we had $L=\frac{1}{2}(|\vec{E}|^2-|\vec{B}|^2)=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, but we also have $\vec{E}.\vec{B}=\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ is also a Lorentz invariant. This we do not put in the Lagrangian by arguing that $Tr\,\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}=4\partial_\rho\epsilon^{\mu\nu\rho\sigma}Tr\left(A_\sigma\partial_\mu A_\nu+\frac{2}{3}i\,A_\sigma A_\mu A_\nu\right)$ this is not parity invariant. If you as you remember in the poor man's language $\vec{E}$ is a true vector and $\vec{B}$ is a pseudo vector. So, if you do a space inversion this term will change sign ok. So, you don't want to include such terms in the Lagrangian. In the more sophisticated language here you what you do know is that the **ϵ** tensor is invariant under the flip of all the 4 $F_{\mu\nu}$, it is the area it is the volume element of that number of space time dimensions the volume element does not change sign.

So, it will not change sign whereas, the other things will change sign together. So, it is the same reason that so, it is a pseudo scalar it is not a genuine scalar it is a scalar, but it is pseudo scalar. Therefore, there is a determinant that comes in the flipping. So, we may not want to include such a term, also in the Abelian case it turns out to be a total derivative. So, you can throw away the term by saying well it is something to do with things at infinity, but in the non-Abelian case that term on infinity also matters.

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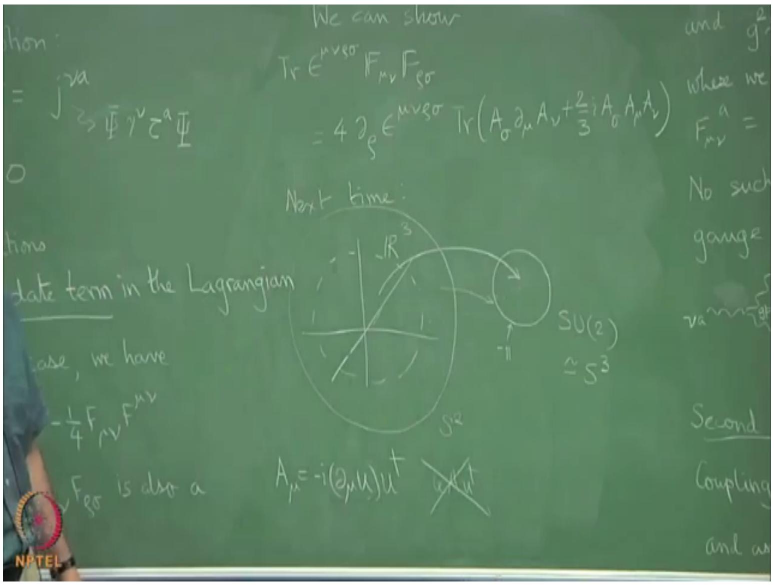

So, it turns out that and actually you can choose gauge field configurations that are pure gauge, but they are laid out in space time in such a way that they cannot be contracted to 0, because you have a u as a function of x. So, even though the function, even if also we will see next time we have space $\mathbb{R}^3$ or we could have $\mathbb{R}^4$ as well as spacetime, from this we are mapping into this SU(2) space right you make some map from here to there.

Now, this is a compact as we had argued last time it is like a compact ball which is an S^3, if you take this $\mathbb{R}^3$ and treat all of it is boundary S^2 as one point. So, if you treat S^2 as one point then this $\mathbb{R}^3$ also becomes a 3-sphere for the same argument we had here remember you start from the origin and you come out the outermost shall you identify with one point because it is just -1 so outer most surface is equal to minus identity of SU(2), for the same reason if you identify the outermost in other words you map the outermost SU(2) exactly into this -1, then structurally this has become topologically same as this.

And then the number of ways you can map and $S^3 \to S^3$ is classified by integers if you start mapping you reach halfway point here you map cover the whole of this space. So, suppose some circles sphere here you map into -1 and then continue going out such that you begin to go inward. So, that when you reach out for at infinity you have mapped back to 1. So, you start with one go there and map back there, that map is distinct from when you start from here and end only here the entire thing maps here and then you can do play this game many times, you can reach -1, then reach 1 then -1, then 1 and so on.

So, you can map this into this in many number of ways which are essentially indexed by integers with positive and negative winding number. So, because of that fact even if you give me a pure u. So, suppose I construct a gauge field $A_\mu = -i(\partial_\mu u)u^+$ right this is a pure gauge field because the uAu^+ part is 0 you did not start with any gauge field to begin with, suppose you construct a gauge field that is entirely of this form.

So, it is a gauge transform of nothing, but this you could be a non trivial map, in that case you are stuck with a non trivial A_μ which although it is pure gauge it has no physical field strengths, it is not identical to it is not the same as the vacuum you started with earlier. So, these kinds of things arise in gauge fields which we will do next time.