Path Integral and Functional Methods in Quantum Field Theory
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Lecture – 25
Yang Mills Theory – Coupling to Matter

The first thing I wanted to advertise was this book which I think I had seen some time ago, but somehow never noticed it.

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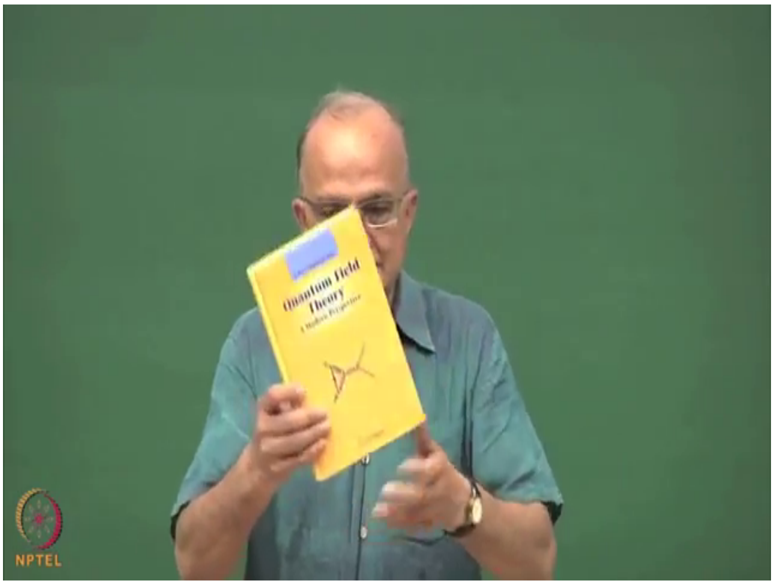

But, now I find that it is an exceptionally well written; it contains a lot of the things that would go with what we call modern field theory.

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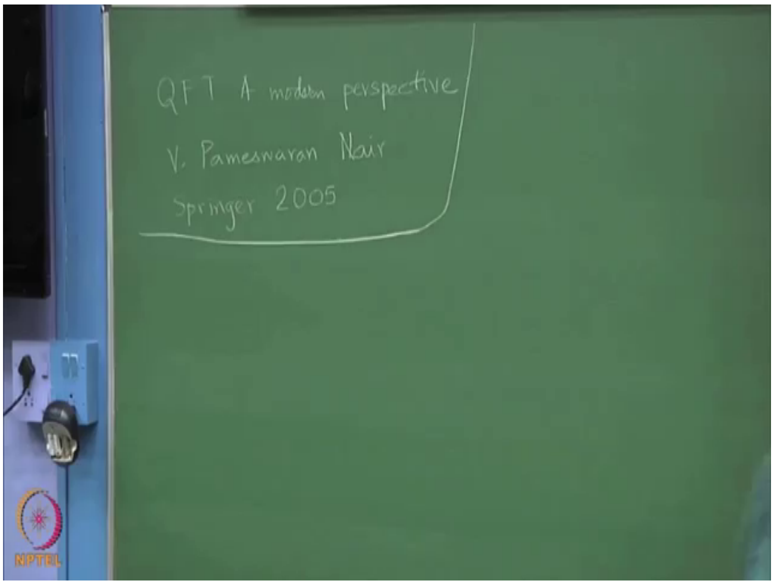

So, it is by V. Parameswaran Nair in Springer, 2005 ok. So, not that much has changed in this level of field theory in 14 years. So, do not have to worry ok. So, I think that is also a good supplementary text and in future I would include some topics from there.

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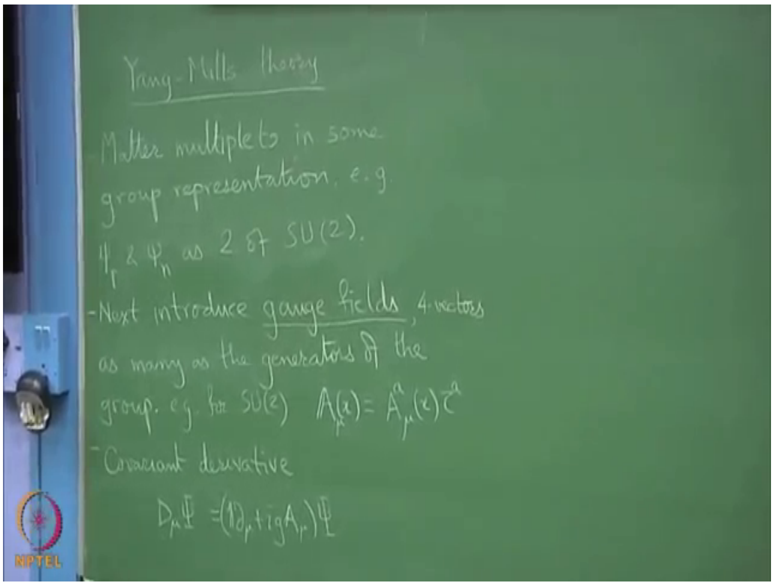

Now, let us continue with gauge fields. So, we have some matter multiplets. This is as I said the original quite a bit of a revelation, that you can actually think of nucleon as essentially two spin projections of the same “particle”. This is the innovation due to Heisenberg and so, matter multiplet means in some group representation. For example, proton and neutron as 2 of SU(2) and this is what we will deal with. This is the simplest case, you can deal with higher representations by simple generalization of group theory rules how the representation transforms.

So, then we say gauge fields for example, for SU(2) so, introduced gauge fields as many as the generators of the group. As you know SU(2) has representation 2, it has 3, it has 5 whichever representation that is the number of gauge fields needed is always the same, it is just the number of generators. So, for example, for SU(2) we need $A_\mu(x) = A_\mu^a(x)\tau^a$ where the $\boldsymbol{\tau}^a$ are generators of A. So, there are as many four vector fields as there are as many as the generators of the group. Then we introduced covariant derivative.

Now, this has to do with how the ψ transforms under gauge group. So, here it is because $\psi \rightarrow u\psi$, but a higher representation say higher tensor representation because ψ is fundamental.

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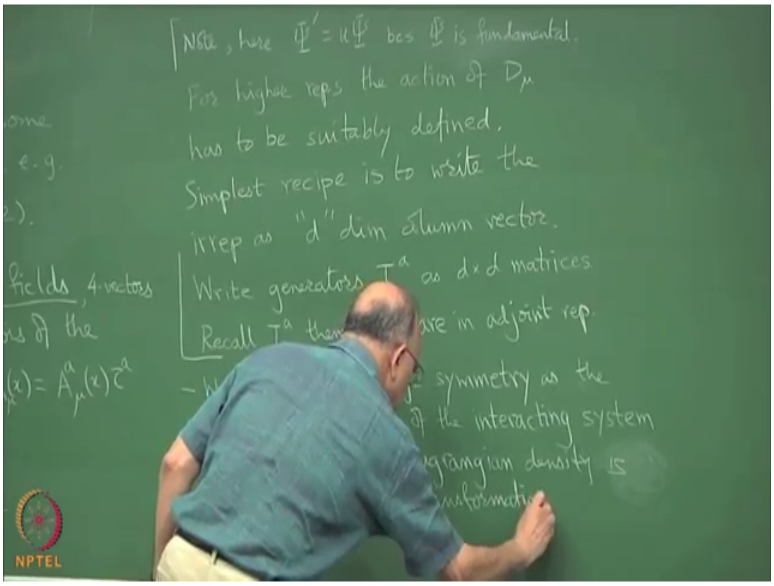

For higher representations the action of D_μ has to be suitably defined. The simplest mathematically the simplest way is that, the generators belong to the adjoint representation and you can always write adjoint representation in the basis that is of any dimensions which is irreducible representation of the group. So, you have to write. So, write the irrep as d-dimensional column vector and write generators T^a as d by d matrices ok. This is more easily said them done, but this is at least mathematically a correct way of saying it.

More generally people use you know tensorial representation that is they write the higher reps, actually as a cross product of the fundamental you know like you write third rank anti symmetric made out of the lowest ψ and so on. Then you have to remember to apply the T correctly to that anti-symmetric representation. But I think this is the simplest recipe for a d-dimensional irrep write a d by d representation of the generators of so, since T^a ok.

So, you can do this for any representation of the group, but this recipe words therefore, writing a column vector and hitting it on the left trick works. The fourth thing then is this was 1 2 3 and the 4th thing is that with the introduction of covariant derivative, we now note that we building the as the dynamical symmetry by insisting that the Lagrangian enjoys is invariant under transformations.

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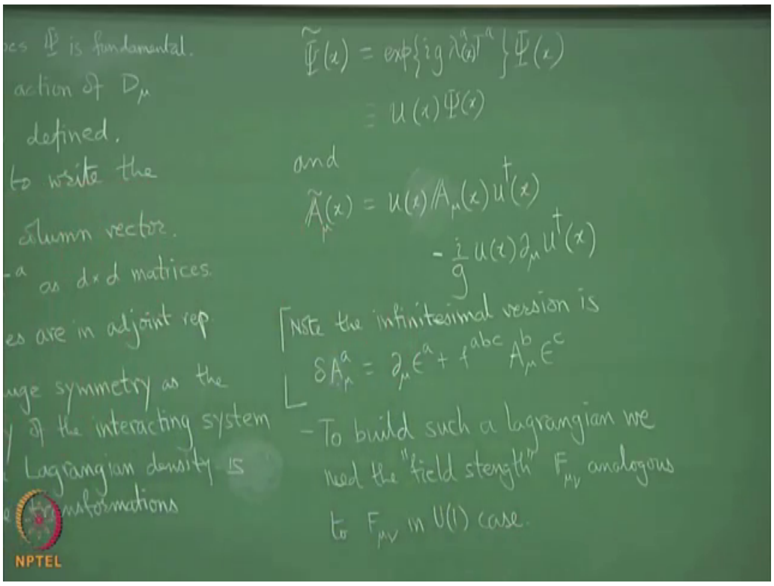

Can you someone just check what it was?

$$\tilde{A}_\mu(x) = u(x) A_\mu(x) u^+(x) - \frac{i}{g} u(x) \partial_\mu u^+(x)$$

So, the above requirements then Intel that the Lagrangian is made up entirely of gauge invariant combinations of ψ its covariant derivative and a field strength tensor to be made up out of the A_μ .

Here I just want to quickly comment that note the infinitesimal version; version is

$$\delta A_\mu^a = \partial_\mu \epsilon^a + f^{abc} A_\mu^b \epsilon^c$$

We next need a field strength corresponding to this A_μ , and the simplest choice turns out to be to write $D^\mu A_\mu$ just put covariant derivatives.

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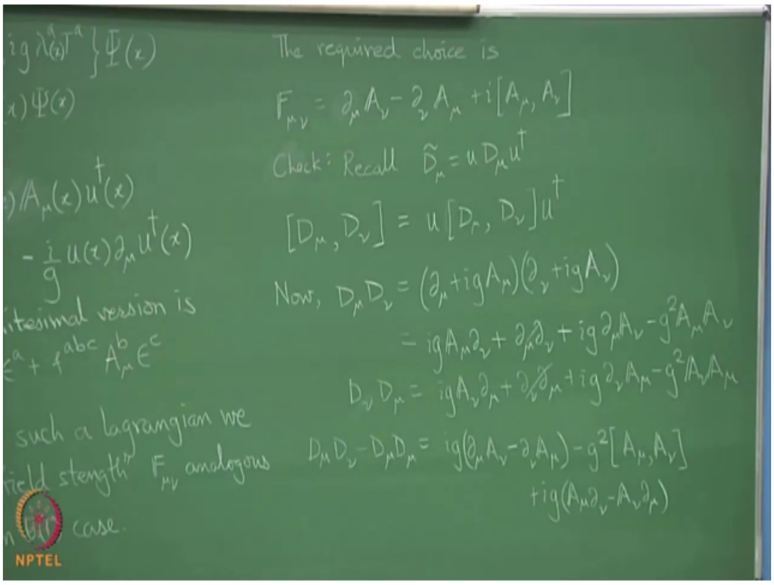

So, the correct choice turns out to be, I am just trying to check the signs

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

note that this can also be written as. So, we try to check we can see that D^μ we already know is a covariant quantity it will transform like $\tilde{D}_\mu = u D_\mu u^+$.

So, if you take $[D_\mu, D_\nu]$; you can see that when I insert $uu^\dagger uu^\dagger$ I will simply get $u[D_\mu, D_\nu]u^+$, but now this D^μ already contains A_μ as well as derivatives. So, it contains

$$D_\mu D_\nu = ig A_\mu \partial_\nu + \partial_\mu \partial_\nu + ig \partial_\mu A_\nu - g^2 A_\mu A_\nu$$

and let us do the same thing in reverse here.

So, $D_\mu D_\nu - D_\nu D_\mu = ig(\partial_\mu A_\nu - \partial_\nu A_\mu) - g^2[A_\mu, A_\nu] + ig(A_\mu \partial_\nu - A_\nu \partial_\mu)$ and then remains the action of the derivatives on the next things. So, in this $F_{\mu\nu}$ your definition we have to right.

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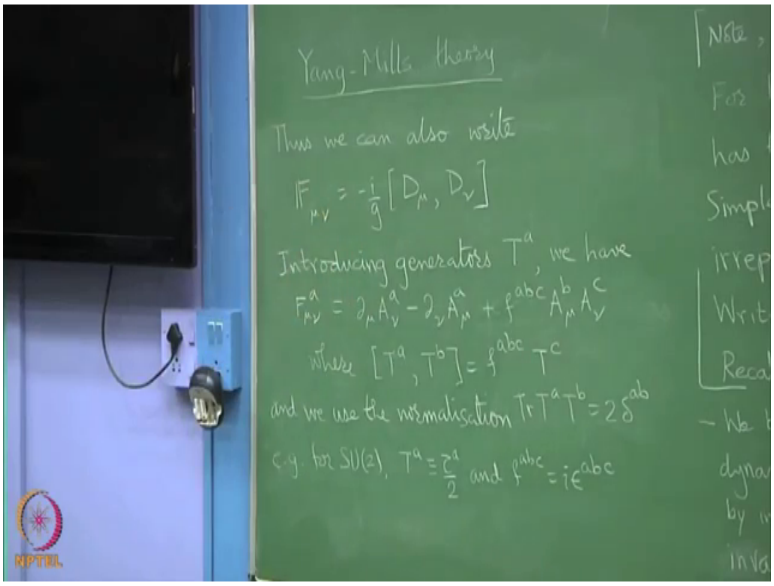

So, we write this as $F_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu]$. Now we can write it out in a more convenient form introducing and you do not have to write this, that is just another way of thinking about it, but it is equal to $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu$ where $[T^a, T^b] = f^{abc} T^c$, f^{abc} are the structure constants and we use the normalization T for T's $Tr\, T^a T^b = 2\delta^{ab}$.

This 2 was introduced in early literature because of SU(2). So, yeah so, but there so, this is the normalization we can see that trace of Ts are of course, 0. So, write like this in the case of SU(2), the structure constants are $i\epsilon^{abc}$. Now, what we have therefore, constructed is a spacetime dependent gauge transformations, I mean which is what we mean usually by non-Abelian or Yang Mills transformations compared to global transformations right where u belongs to the Lie group and all the fields.

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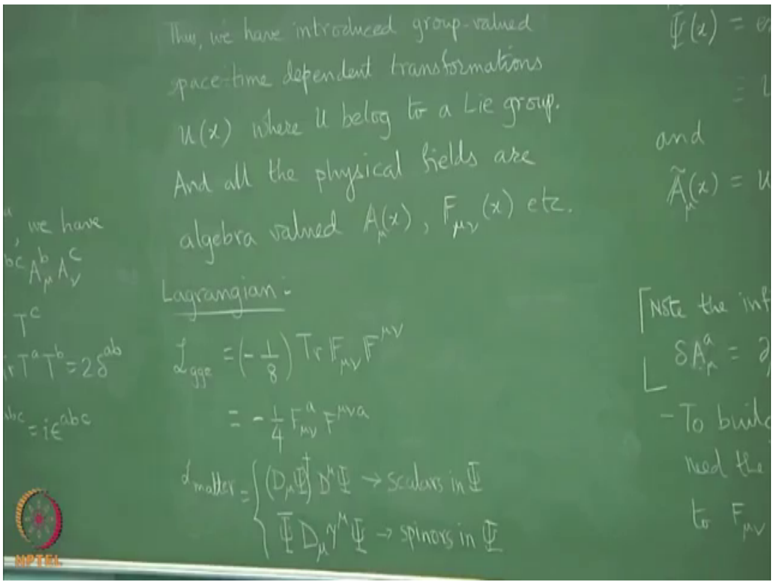

So, Lie group means that uu^{-1} is also in the group and all the physical fields are algebra valued right. Now as I have been saying the greatest news of the century is that, this mathematics of this to describe all the fundamental forces. So, the Lagrangian for the system for gauge fields $L_{gge} = (-\frac{1}{8}) Tr\, F_{\mu\nu} F^{\mu\nu}$ this we would generalize from Maxwell's right because we have $F_{\mu\nu} F^{\mu\nu}$.

So, only the space like ones have and we need minus sign because they come with minus sign because of this signature and we put a 1/4 to make it one half in that case because there is a doubling here, here we have to put a 1/8 because there is an extra 2 coming from the trace of the $T^a\, T^b$ which are inside this, additionally I remember seeing a $\frac{1}{g^2}$ in front of these. So, that happens if. So, you will also see in textbooks yeah. So, in this F mu nu definition sometimes this g is not put in that case you have to supply g here to recover the physical $F_{\mu\nu}$ s right and so, this is also equivalently equal to $-\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$ where summation over a is understood.

So, this correctly generalizes the Lagrangian from the Abelian case and for the matter it has to be $L_{matter} = (D_\mu \psi)^{+} (D^\mu \psi)$ because it is a if you think the that dagger is a matrix as well as Hermitian conjugate. So, you have to put a dagger. So, that the whole thing remains Hermitian there is.

So, here ψ is of course, a scalar field for scalar field ψ it will be like this or by that I mean yeah scalar and should be simply simply $\bar{\psi} D_\mu \gamma^\mu \psi$ for spinor or spinors in ψ, let me put it like this scalars in ψ. So, this ψ is the multiplet say matter multiplet. If you have several spacetime scalars then you will have this, this is like the Klein Gordon $(D_\mu \phi)^* (D^\mu \phi)$ except that now you have to put covariant derivative acting on the ϕ. So, this is how the Higgs kinetic energy will look like Higgs field which is a doublet under SU(2) its kinetic energy looks exactly like this whereas all the fermions it will be like the Dirac case where now the bar has to apply to each component.

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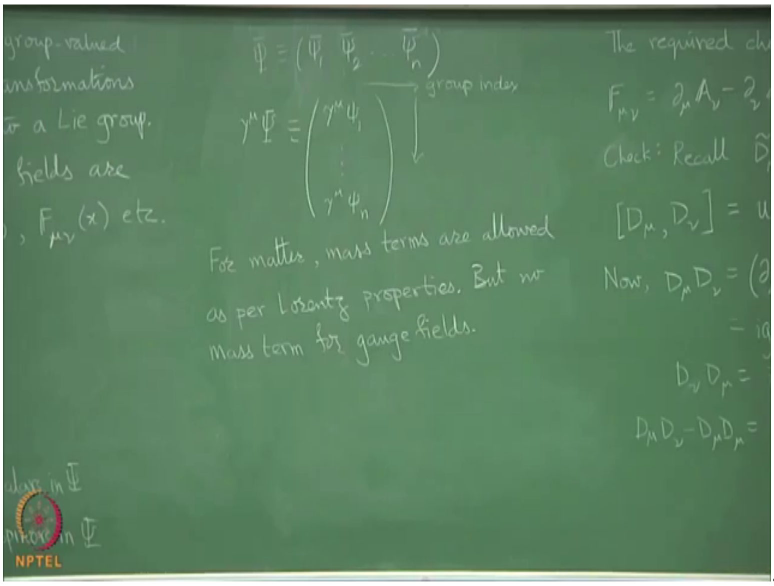

And the γ^μ is of course, attaching itself to each. So, it is a hybrid notation there is a group theory some if group theory contraction and there is a gamma 4 contraction. So, there is a gamma 4 for each of the row and column. I think it is best to write it as I am sorry because it is a bar you probably want to write like this.

This is clear right it is $\bar{\psi}$. So, this should be simply written as $\gamma^\mu \psi = \begin{bmatrix} \gamma^\mu \psi_1 \\ \dots \\ \gamma^\mu \psi_n \end{bmatrix}$ let the γ^μ stick to the ψ then you will not go wrong. So, this is the group index up to left to right or up to down is group index and then individual it will be a Dirac spinor right. So, this is the easiest way to remember how to write it and of course, you can write mass term, you know as per Lorentz properties, but no mass term for gauge fields.

However there is a big mischief because the gauge fields are now highly non-linear, the U(1) gauge theory was essentially a linear theory, but here the A_μ is coupled to themselves because of this structure of $F_{\mu\nu}$ where it has F contains two terms are linear in A, but this term is quadratic in A. So, the gauge field is non-linear in the field strength is non-linear in the potentials and this creates a very complicated geometric structure, which when you are lucky can be delineated by some very clever geometric arguments, but if you are not lucky then you just have to suffer it.