Path Integral and Functional Methods in Quantum Field Theory
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Lecture – 24
Gauge Invariance - Non - abelian case

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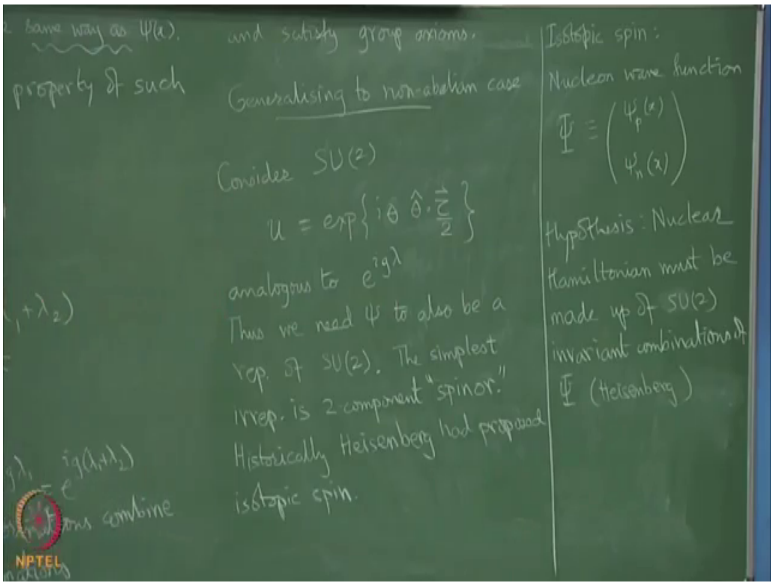

So, in this Nucleon wave function $\Psi=\begin{bmatrix}\psi_p(x)\\ \psi_n(x)\end{bmatrix}$, this is the hypothesis due to Heisenberg. And this basically assumed that nuclear forces do not care whether you have a neutron or a proton and they are independent of charge, the fact that proton has some charge does not matter to the nuclear force. Now, if that is true then the two things together should look like the same thing for nuclear forces and this is quite a far reaching hypothesis.

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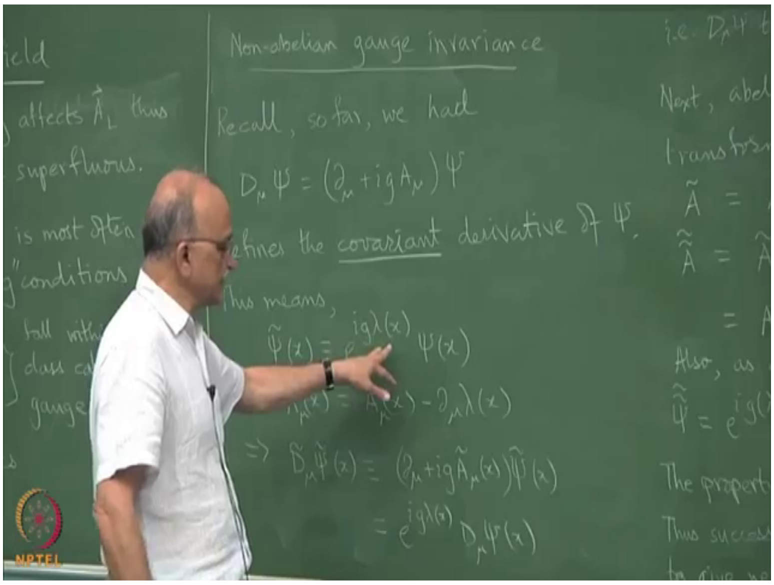

So, what Yang and Mills did was to say that therefore just as electromagnetism involves rescaling, you know the classical gauge invariance of electromagnetism where you shift A^{μ} requires that you should simultaneously shift the phase of electron wave function by $e^{ig\lambda(x)}$. Suppose here I firstly, promote the overall phase to be a function of x, so these θs to be function of x and then I introduce the correct a mu field which absorbs the derivatives of that theta which enters this.

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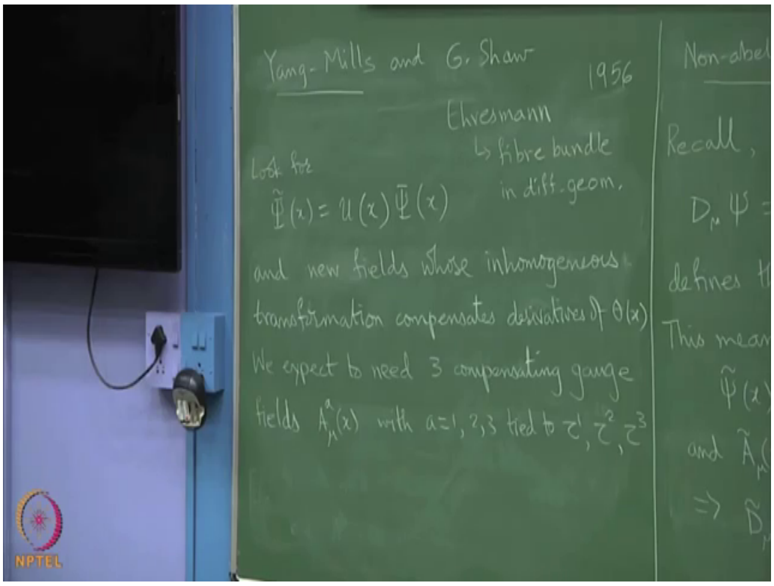

So, this is Yang and Mills and somebody called G Shaw ok. I think he became a Professor also somewhere, this G. Shaw did what yang and mills become famous for in his PhD thesis. He submitted that thesis got his PhD and forgot all about it and the whole world was celebrating yang and mills for having invented the thing.

So, but somehow that name is never mentioned, but it is actually Yang-Mills and Shaw, because G. Shaw also observed the same things and proposed the same thing. But is also interesting is that in mathematics Ehresmann also proposed the same thing at almost the same time. And he called it fiber bundles. All of this happened in 1956.

So, all the same idea emerged from several sources about the same time, Ehresmann's motivation was probably just to generalize the covariant derivative, which was too much tied to space time, tangent spaces and so on so, in differential geometry. So, here what we say is look for transforming $\tilde{\psi}(x)=u(x)\psi(x)$ and new fields whose inhomogenous transformation compensates derivatives of θ(x) ok. So, here we observe and I have told about SU(2) so, many times so, many of you may get bored of this.

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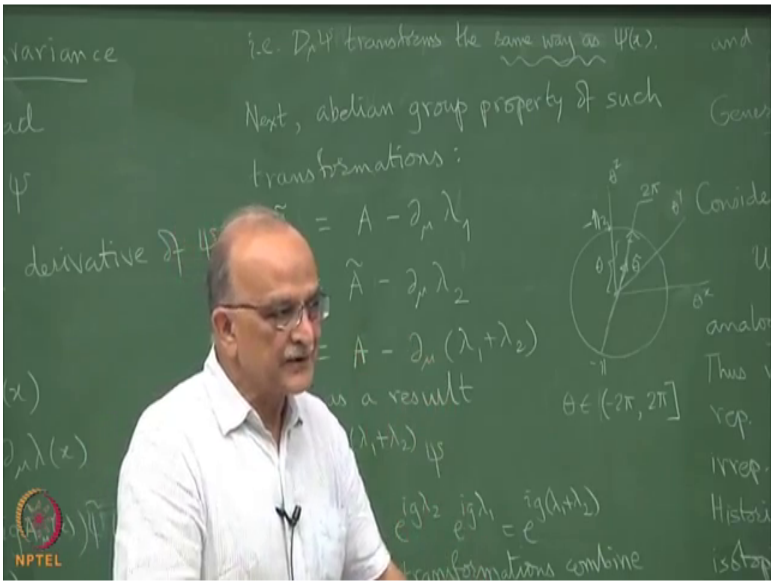

But, let me tell you once again that SU(2) can be thought of as a solid sphere of dimension 3, you try to think of sphere in arbitrarily large number of dimensions, what is the main property of a sphere, it is homogeneous. Wherever so, you first think of the spherical shell, which is an S^2 it is properties that wherever you are on it the space looks the same. So, it is a space of constant curvature or it is a space which is completely homogeneous, it looks the same and it is compact it is not indefinite.

So, what happens for SU(2) is you can parameterize this object in this particular way you set up 3 directions. So, totally there are 3 degrees of freedom because there is one over here and there is a unit vector in 3 dimensions, there is one for each of the τ there is a component, but it is unit vector. So, it has only 2 degrees of freedom and the third one is this.

So, suppose we set up that θ^x, θ^y and θ^z, I set up a unit vector $\hat{\theta}$ in this space and then I rotate by some amount. So, this θ means a rotation of some amount. So, we say whatever amount by which you are going to rotate you put it on this ok, so, this is θ.

So, $\hat{\theta}$ exhausted 2 degrees of freedom to choose the direction and then how much you rotate is set up along that. So, any element like this can be taught up as some vector like this right, in 3 abstract dimensions whose axis are labeled some θ^x, θ^y and θ^z, but we know that SU(2) because this has expansion $\cos\frac{\theta}{2} I + i sin\frac{\theta}{2}\hat{\theta}.\vec{\tau}$.

Therefore, we see that if we reach $\theta = 2\pi$ then we get $\cos\pi.1$. So, we get a -1 whereas, $\sin\pi = 0$, so, this term goes to 0. So, if I reach $\theta = 2\pi$ I get minus 1 outside right minus identity at the surface in any direction.

So, if I go from -2π to +2π along any direction, I always start with -1 go through +1 when everything is 0, u becomes equal to +1 and then return to -1. So, this is the kind of space it is and with different $\hat{\theta}$ I should exhaust going from -2π to +2π, but you do not have to struggle too much about this open thing, because outside it is exactly equal to -1 when you reach that it is actually just one point. Although it is drawn as a solid ball the outermost surface of this ball is abstractly speaking only one point ok.

So, we are actually looking at a genuinely a 4 dimensional object in it’s 3 dimensional projection. So, since a few people do look puzzled I will spend a little more time to explain why what this is. We do a 1 dimension lower analogy. So, suppose I was an ant which can only move around in 2 dimensions, but wanted to visualize a compact 2 dimensional sphere ok, but I do not have a third dimension in which to erect that sphere.

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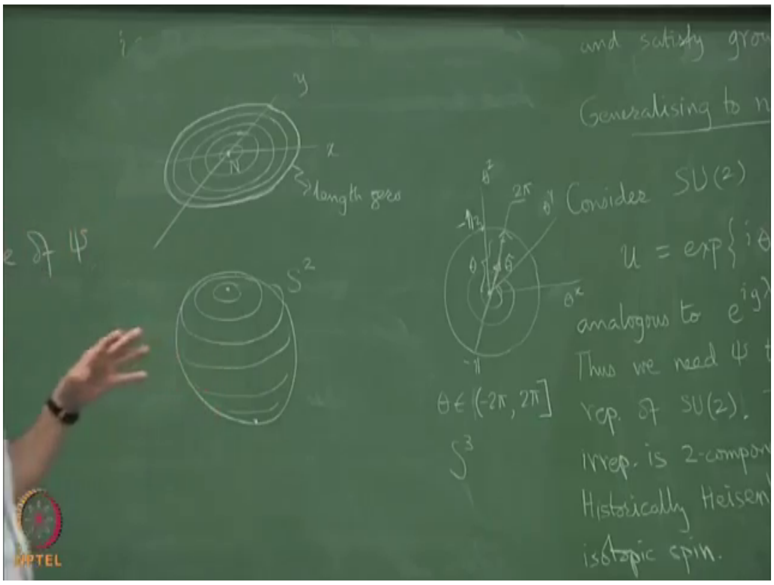

Then, what I would have is the plane and suppose in the plane I set up a disc. So, I will just call it x and y, because we do not have the meaning of group theory there, but this is a compact 2 dimensional sphere as we visualized having only x and y. Normally, would think that a shell would require you to at least living 3 dimensions to visualize it, but it is only a 2 dimensional space, it is intrinsic dimensionality is only 2.

So, we can make it even out of a flat surface like this, what you do is you sit at the center. So, let me draw the real thing that we know. So, suppose this is what we call S^2 a spherical shell, it will have a North Pole and the South Pole. So, let us map the North Pole to this point and then I start drawing circles which grow in size. So, correspondingly I draw the only thing I forego is that the length of this circle does not match the length here. Here the length keeps growing the circumference of circle keeps growing, here as I come closer to South Pole the circles begin to grow smaller until the shrink to a point ok.

So, if I was an ant confined to this and I just make a new rule ok. Although it looks like I had this I ascribe it a smaller length and smaller length I change my unit as I go further and further out you know it is a $\sin\theta$ factor it. So, I make it is smaller and smaller until when I reach the outermost circle of the disc I declare it to be size 0 ok.

So, I am confined to a plane, but I make my distance rules or length measurement rules in such a way that when I reach this circumference of the disc I declare it to be size 0, then I have effectively visualized and S^2 sitting in a plane ok. Something that normally would require 3 dimensional to visualize I have done using a 2 dimensional space. So, this thing where you have a sequence of 2 spheres right each one is a 2 sphere, each value of θ allowing $\hat{\theta}$ to vary over everything is a sequence of 2 spheres, which eventually gives you only 1 point.

So, this is like the North Pole of the S^3 and finally, this is the South Pole of the S^3 embedded in 3 dimensions ok. So, SU(2) is basically a is as a point set it is isomorphic to S^3 and in this particular way you can visualize it like this. So, there are 3 degrees of freedom that describes this group 3 parameters. If, we are going to do this kind of transformations we will need at least 3 compensating gauge fields right.

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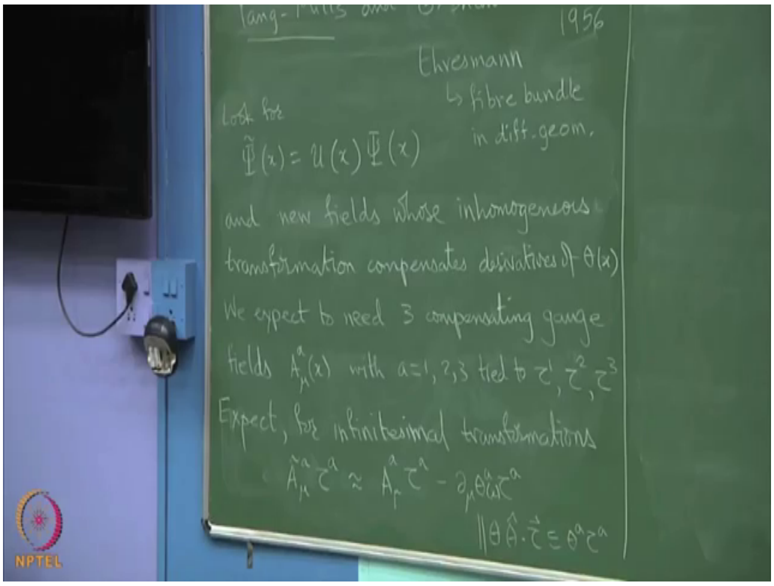

So, note that we could have always written $\theta \hat{\theta}.\vec{\tau} \equiv \theta^a \tau^a$, where each unit vector component will get multiplied by θ to produce θ^a and it can be written like this. So, that is what we are writing.

So, just as in the abelian case we had A^μ going to old A^μ minus derivative of some λ, here we will make that θ as a function of x and then this is how we expect that this will happen, I now jump a little bit.

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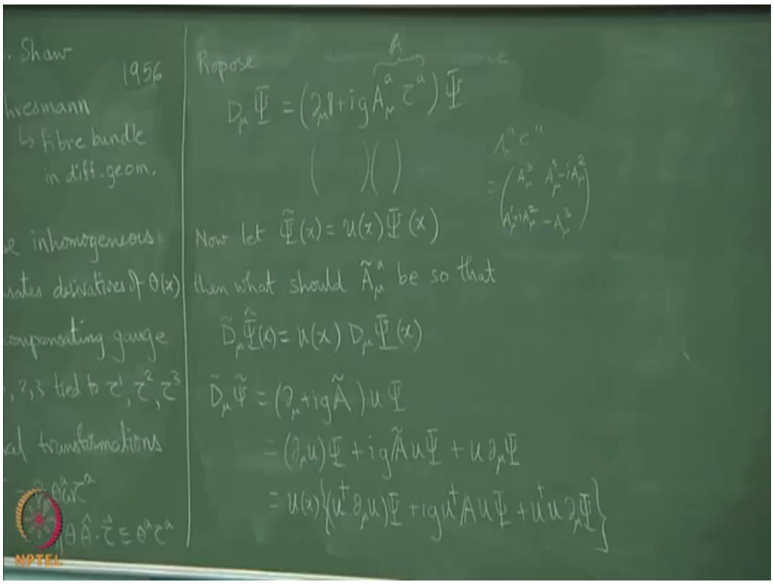

So, propose that $D_\mu \psi = (\partial_\mu + igA_\mu^a \tau^a)\psi$, just as we had in that case except that instead of single A^μ I have a triplet and the if you want there is an identity matrix multiplying this. So, this is a 2×2 matrix acting on a 2 vector. What is $A_\mu^a \tau^a$? It is equal to

$$\begin{bmatrix} A_\mu^3 & A_\mu^1 - iA_\mu^2 \\ A_\mu^1 + iA_\mu^2 & -A_\mu^3 \end{bmatrix},$$

it is a 2×2 matrix and it will act on this Heisenberg's 2 component nucleon wave function. So, now, we ask if we let $\tilde{\psi}(x) = u(x)\psi(x)$ then what $\tilde{A}_\mu$ do we need so that $\tilde{D}_\mu \tilde{\psi}(x) = u(x) D_\mu \psi(x)$ same story as before. So, the point is this is a space time dependent object a derivative action it will differentiate this as well, but somehow by magic we want to end up with this only multiplying this without any derivatives.

So, the derivatives have to be compensated by the transformation of A_μ. So, right everyone is clear about this. So, all we need to do is to carry this out and see what happens. So,

$$\tilde{D}_\mu \tilde{\psi} = (\partial_\mu + ig\tilde{A})u\psi = \partial_\mu u\psi + ig\tilde{A}u\psi + u\partial_\mu \psi$$

where $A = A_\mu^a \tau^a$. So, what we should do, but with one this to look like u times something right. So, I just pull out a u because I want the answer to look like u time something.

So, I put everything else inside

$$u(x)[u^{+}\partial_{\mu} u\psi + igu^{+}\tilde{A}u\psi + (u^{+}u)\partial_{\mu}\psi] \quad ;$$

however, simple trick it looks today that we can just teach like this was the greatest invention of the last century.

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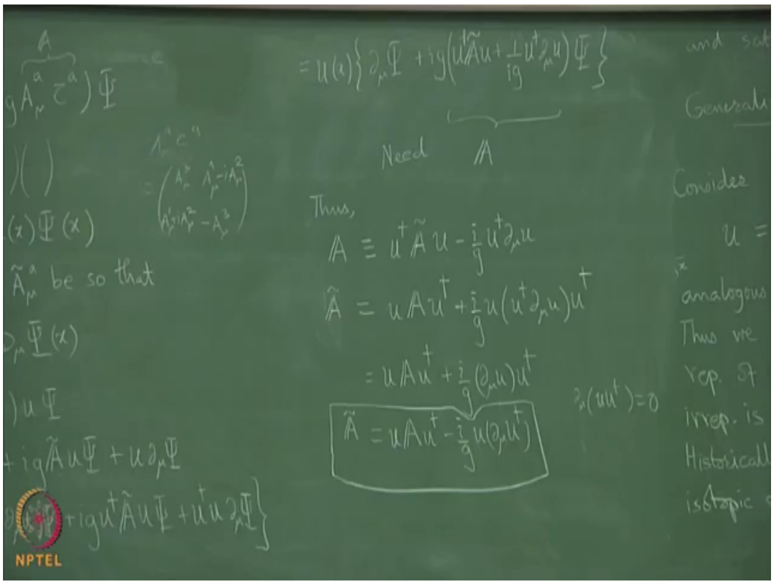

So, this is equal to $u(x)[\partial_{\mu}\psi + ig(u^{+}\tilde{A}u + \frac{1}{ig}u^{+}\partial_{\mu}u)\psi]$.

So, we want this to read the old derivative, the old derivative was just A there. So, it follows that this is nothing, but A. So, the transformation law can now be written

$$A \equiv u^{+}\tilde{A}u - \frac{i}{g}u^{+}\partial_{\mu}u \quad .$$

Equivalently $\tilde{A} \equiv uAu^{+} + \frac{i}{g}(\partial_{\mu}u)u^{+}$, but recall that u is a unitary matrix. So, $uu^{+} = 1$.

So, often people write it in this form $uAu^{+} + \frac{i}{g}u(\partial_{\mu}u^{+})$ because $\partial_{\mu}(uu^{+}) = 0$, so $\partial_{\mu}(u)u^{+} = -u(\partial_{\mu}u^{+})$. So, this is our great transformation law, the greatest applicable theoretical discovery of last century may have been many others, but this is certainly the all the forces of stand of all the nuclear forces obey this law.