Path Integral and Functional Methods in Quantum Field Theory
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Lecture – 23
Gauge Invariance- Abelian Case

So, before we go to the non-abelian case I just wanted to highlight some of the salient points of the abelian.

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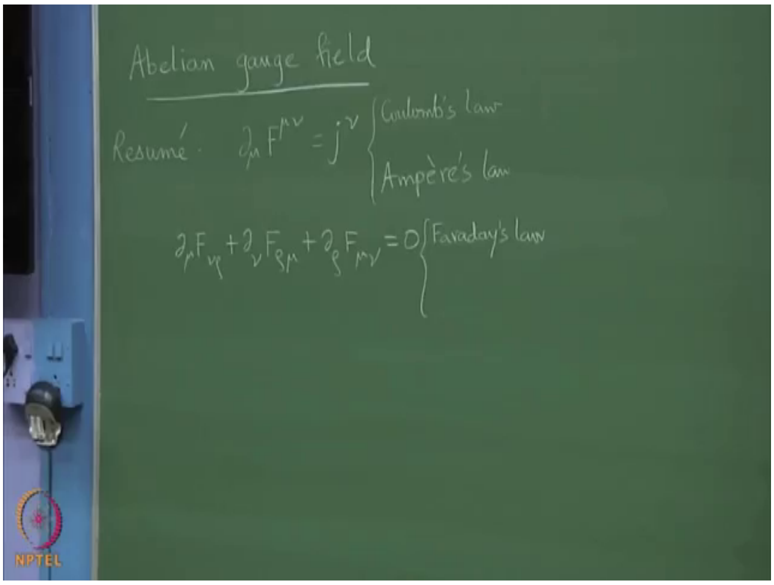

So, resume is that we have $\partial_\mu \mathbf{F}^{\mu\nu} = \mathbf{J}^\nu$ which happens to be Coulomb's law and Ampere's law together. So, in the laws of electromagnetism only the Coulomb law and the Ampere law has a current as a source which generates curl of a magnetic field.

The other two equations simply become identities. These two are the Faraday's law which has no source and it remember it only says that rate of change of magnetic flux in a region causes electric field to be set up and that electric field is non conservative; the distinction between the EMF generated in Faraday's law and Coulomb law electric field is that Coulomb law electric field is generated by a source. So, what is a conservative field?

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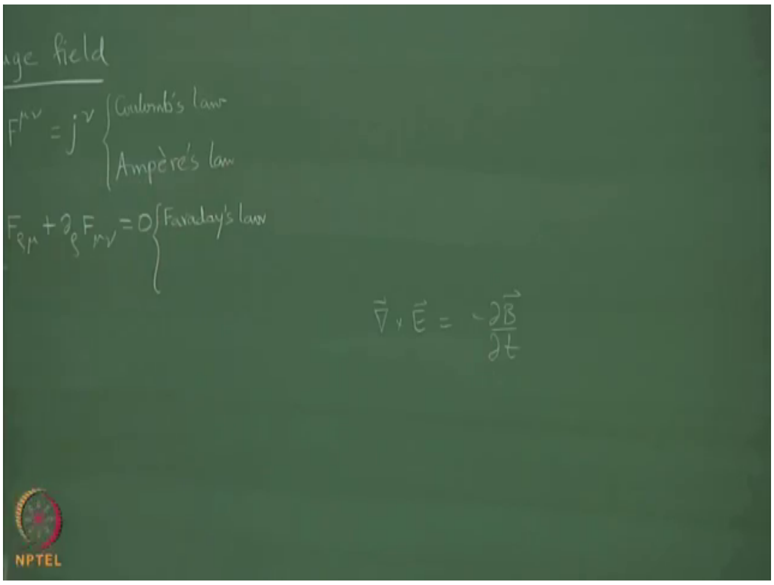

$\nabla \times \vec{B} = 0$. So, the Coulomb law fails obey this, but here of course, it is equal to -∂B/∂t. So, the E field that is generated by this is not a conservative vector field.

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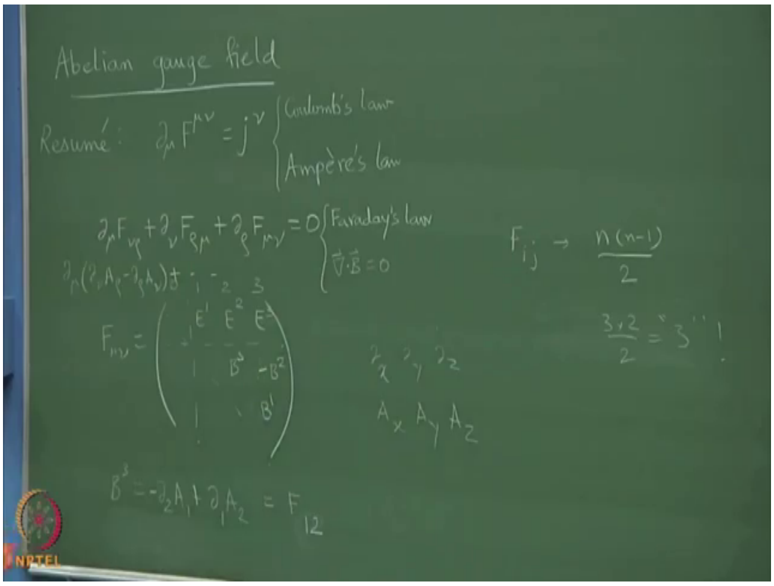

So, I hope everybody knows that there is a mapping from $F_{\mu\nu}$ language to E, B

$$F_{\mu\nu} = \begin{bmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{bmatrix} .$$

So, these laws reproduce this. The main thing I want to emphasize is to remember that the $F_{\mu\nu}$ is intrinsically anti symmetric. In fact, this E and B split is completely artificial it was convenient for low energy physics because you are only looking for three vectors and the E field is certainly a genuine three vector, B field is not a genuine three vector, but it is a good coincidence of three dimensions that the curl has exactly as many components as the usual vector field right.

An anti symmetric F_{ij} has as many components as n(n-1)/2 in n dimensions. So, in three dimensions it will have exactly 3. So, this is a miracle of three dimensions. So, the anti symmetric tensor will have exactly same number of components as an ordinary vector and that is why you have all these things that tell you if you rotate like this then this points this way like a hair coming out of somebody's head no it is not like that, it is really a anti symmetric tensor. And so here it all comes together nicely and they fit together as a anti symmetric derivative of four vector field and then you can say that this equation is an identity.

If $F_{\mu\nu}$ is anti symmetric then it is easy to check that regardless of what $F_{\mu\nu}$ is. So long as it is it changes sign under exchange of the two objects or so long as it is expressed like this $\partial_\mu \mathbf{A}_\nu$, you can just check this out all the terms will add to 0. So, it is an identity. So, the great Faraday's law is essentially an identity because obvious and the other law is of course, which has no name which is the $\nabla.B = 0$. So, those are automatic identities that follow from the $F_{\mu\nu}$ being anti symmetric. Now, then the question is how do you write the action for such a system and there are some remarks which we do not want to go too much into the details of.

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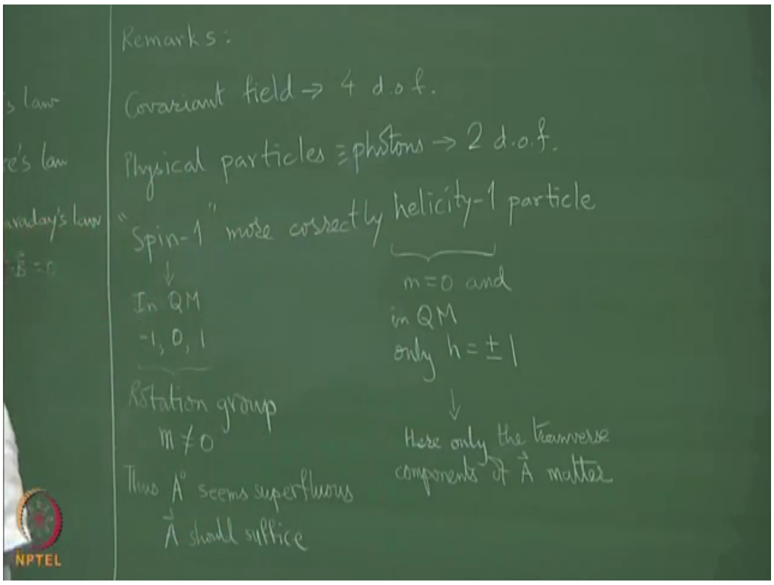

But, one thing is that when you come to the particle interpretation you know that the photons have only two polarizations. So, there are only 2 degrees of freedom, but there are 4 degrees of freedom here. So, in the the covariant field description we have 4 dof, physical particles have only 2 degrees of freedom.

So, there is a redundancy and we can see the redundancy easily because actually we know that it is a spin 1 particle, more correctly helicity 1 particle. Now we know that if it was spin 1 then from in quantum mechanics there will only be we only expect -1, 0 and 1 projections and helicity 1 actually means that it is mass less helicity is in the massless limit +1 or -1, the 0 part drops out because that you get the spinless component of the multiplet is available only if you go to its rest frame.

So, if you have a massive particle which has spin 1 if you go its rest frame then this after all is the helicity. So, you can get the spin 0 component if you can go to its rest frame, but if you can never go to its rest frame then it will be always spinning and you can never overtake it. So, you cannot convert a plus helicity particle to negative helicity particle, but here you can get it to come to 0's and the helicity will then become 0.

Now, therefore, we observe that out of the four components this is more or less what rotation group needs; because it is a massive particle you can be in its rest frame and then rotations really are enough to characterize what are the degrees of freedom in it. So, actually therefore, fourth component A^0 seems superfluous the three A^i should be enough, but when we go to this then here even only the transverse components matter.

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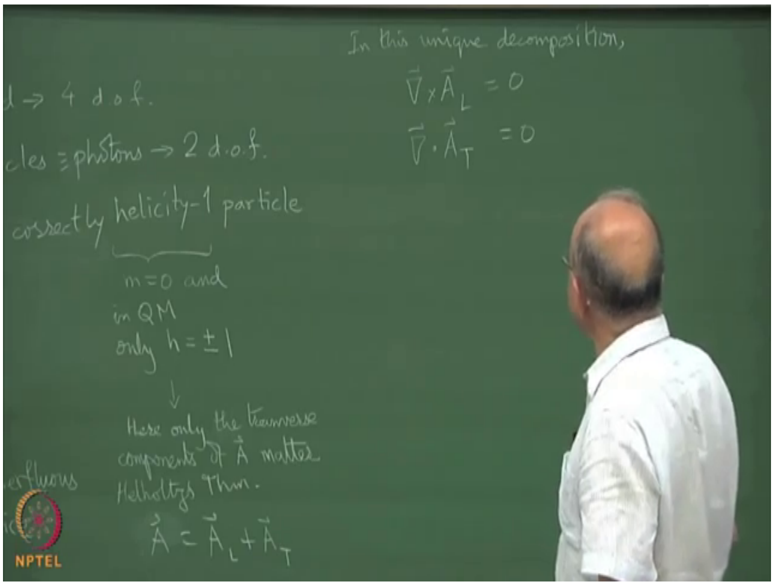

Now there is Helmholtz theorem from vector calculus that says that any vector A can be written as $A = A_L + A_T$. The transverse component has divergence 0 and the longitudinal components curl has to be 0.

So, there is a further restriction because the longitudinal part is set to 0 to get the final 2 degrees of freedom. So, that is how the connection between the covariant field description and the photon or particle description works. Now because of all this the Lagrangian has a specific form.

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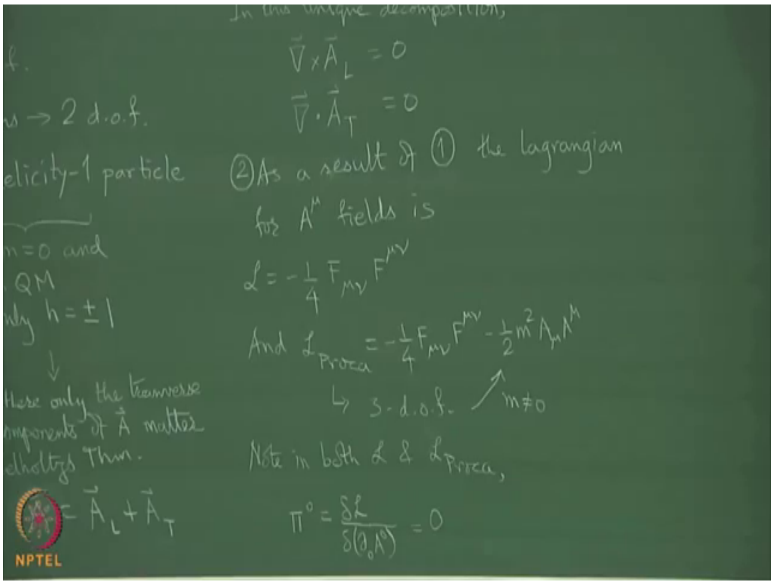

So, in the Proca Lagrangian one puts back m^2 and then there are 3 degrees of freedom because mass is not equal to 0. So, that is this case and we can see that A^0 is superfluous. Because for both the Lagrangians, the canonical momentum

$$\pi^0 = \frac{\delta L}{\delta(\partial_0 A^0)} \quad ,$$

but there is no time derivative of A^0 because this is always anti symmetric whereas, this is $\partial_0 A^0$ there is no such term going to occur in $F_{\mu\nu}$ right. So, the Lagrangian density does not contain $\partial_0 A^0$ at all and so, this is equal to 0.

So, it basically says the canonically conjugate quantity to A^0 is 0 there is really no dynamics in A^0 field and that is what we mean by saying A^0 is superfluous, another way of saying it is. So, this is an exercise, check that the most general lagrangian density for 4-vector field A^μ up to quadratic order in first derivatives nothing terribly a complicated right. Since as a Lagrangian it is you know it starts with kinetic energy, so, it has square of the time derivative. So, in covariant notation it would involve all the derivatives up to quadratic order.

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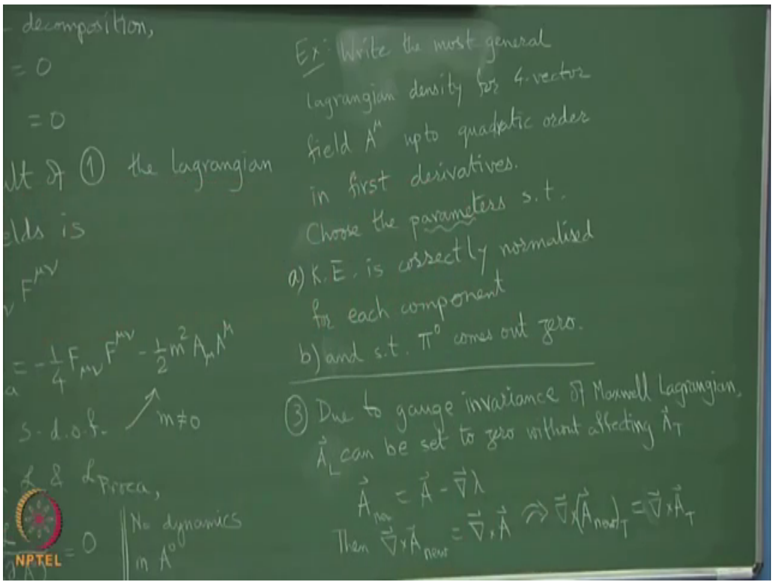

So, covariant Lagrangian that is of this form. So, here the parameters will be arbitrary ok. So, choose the parameters such that kinetic energy is correctly normalized for each component and such that π^0 comes out 0 ok. So, it will look a little mysterious, but once you start writing you will know what this is all about.

So, my claim is that this $F_{\mu\nu}F^{\mu\nu}$ if you will expand it out in terms of derivatives of A is quadratic in derivatives of A, but it is not the most general one and you will be led to this particular one by choosing the parameters in it correctly ok. Now coming back to this part, so, just we saw how A^0 is removed to get radiation we need to also remove the third component which is this longitudinal component which is the nuisance and we can see the gauge transformation. So, now remark number 3.

So, we note that the Proca Lagrangian does not have gauge invariance, if you shift A^μ by $\partial_\mu\lambda$ then the $F_{\mu\nu}$ is unchanged. So, $F_{\mu\nu}F^{\mu\nu}$ remains, but if you shift A^μ by $\partial_\mu\lambda$ then this term gets messed up and $A^\mu\partial_\mu\lambda$ kind of terms appear in the Lagrangian whereas, λ is not something physical. So, you do not want it to appear or in other words this term will not respect gauge invariance.

But the Maxwell Lagrangian we can always and shall longitudinal part is 0, A_L can be set to 0 without affecting a tranverse. So, I am sorry these things are look I mean dropping them as remarks, but after you have understood what the problem is and if you start thinking about it then these are the answers.

The real difficult part is to think of the questions. So, once if you get sufficiently familiar with Maxwell's theory then you begin to ask the questions for which I am giving you the answers ok. So, you will have to go back and forth in thing why we are making these statements. So, this is easy to see because $A_{new} = A-\nabla\lambda$, it does not affect transverse part by making sure that $\nabla^2\lambda$ is 0 and it does not enter the longitudinal part at all,

$$\nabla\times\vec{A}_{new}=\nabla\times\vec{A}\Rightarrow\nabla\times(\vec{A}_{new})_T=\nabla\times\vec{A}_T \quad .$$

because the longitudinal part does not enter the curl at all, but it is an identity to right if you take curl of this on both sides then you get this, but that is automatically 0. So, that equation only affects the transverse parts therefore the gauge transformation only effects the longitudinal parts.

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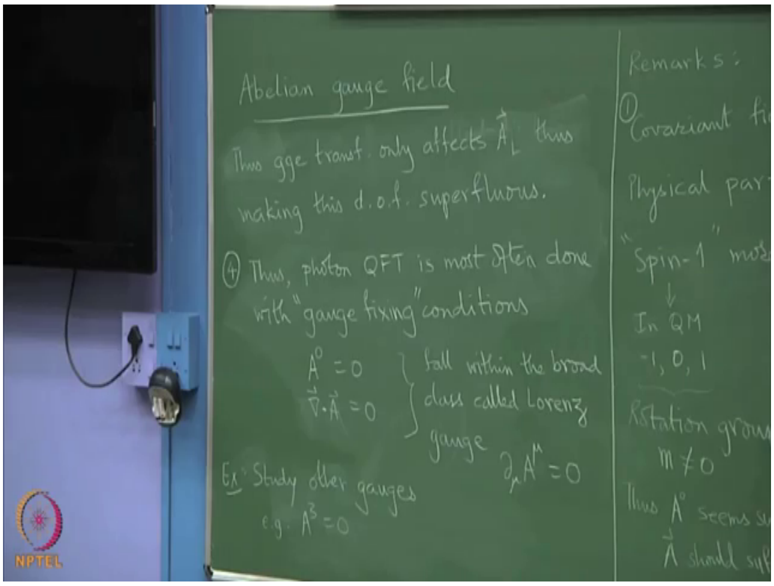

And therefore, making this degree of freedom superfluous and one way that people impose this. So, now, let us just summarize everything in a sense in textbooks you will see what I am about to say next and if you try to understand why they say that these are the reasons for it. So, number 4 thus photon QFT is most often done with gauge conditions, they are called gauge fixing $A^0 = 0$ and $\nabla.A = 0$. And these both are within the broad class called Lorenz gauge and this Lorentz is not Mister Hendrik Antoon Lorentz, but some cousin could not write a t in it which is $\partial_\mu A^\mu = 0$.

What is nice about Lorentz gauge is that it is covariant $\partial_\mu A^\mu = 0$. So, it is setting some scalar piece out of the 4 degrees of freedom to 0. So, this falls within that right. So, this is the class more general class of gauge this is only one scalar condition. Here we are putting two conditions which also break Lorentz invariance, $A^0 = 0$ and $\nabla.A = 0$ will then satisfy this automatically. So, there are other gauges the other gauges. So, exercise study other gauges for example, $A^3 = 0$.

So, these gauge conditions are put because of 1, 2 and 3 to take care of these ambiguities and where they arise they can be fixed by usually putting these conditions. So, I have said things somewhat backwards trying to give you the reasoning, but normally books will tell you that these are the gauge conditions we use. So, I think we are ready to go to non abelian gauge theory.

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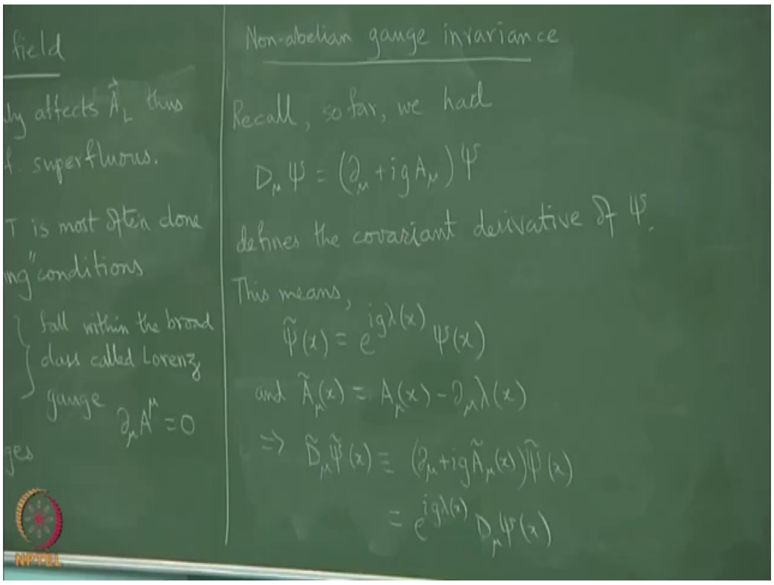

Recall so far we had $D_\mu\psi = (\partial_\mu + igA_\mu)\psi$ as the covariant derivative. This means,

$$\tilde{\psi}(x) = \exp[ig\lambda(X)]\psi(x)$$

$$\tilde{A}_\mu(x) = A_\mu(x) - \partial_\mu\lambda(x) \quad .$$

This implies that

$$\tilde{D}_\mu\tilde{\psi}(x) = (\partial_\mu + ig\tilde{A}_\mu(x))\tilde{\psi}(x) = \exp[ig\lambda(x)]D_\mu\psi(x) \quad .$$

So this is the meaning of the word covariant, it transforms the same where the psi self does, if ψ is transformed to $e^{ig\lambda}\psi$ then the $\partial_\mu\psi$ transforms by the same way; so, the same way as is called covariant.

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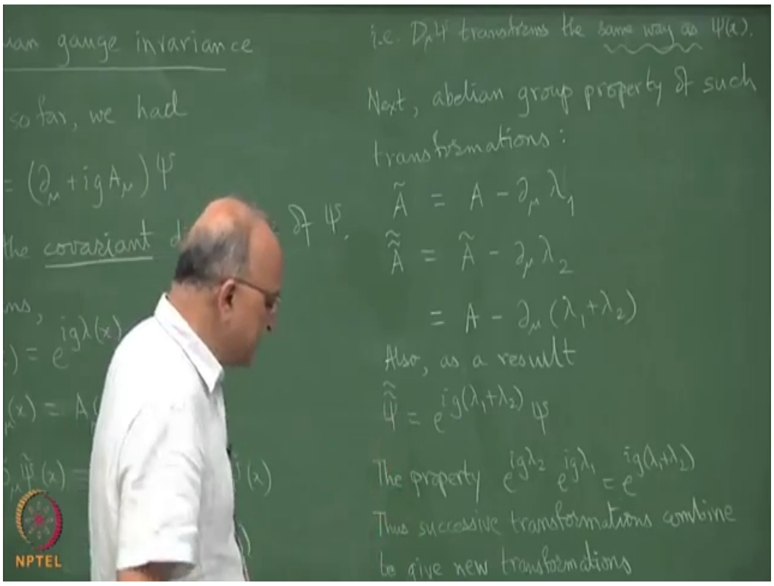

Now, the business of abelian. We see that if you carry out successive such transformations then the compound right if I have a λ_1 then I do λ_2. So, if we do $\tilde{A} = A - \partial_\mu \lambda_1$, let us say and suppose I make $\tilde{\tilde{A}} = \tilde{A} - \partial_\mu \lambda$, then it is clear that it is equal to $A - \partial_\mu(\lambda_1 + \lambda_2)$ and similarly and also as a result $\tilde{\tilde{\psi}} = \exp[ig(\lambda_1 + \lambda_2)]\psi$.

Because first transformation $e^{i\lambda 1}$ then you do another one another λ_2 just gets added in the exponent. So, this property that the two gauge transformation just combined to make a one, makes the set of all such transformation an abelian group, because you can of course multiply them in reverse order this answer will remain the same.

So, the successive transformations combined to give new transformations that itself is group property number one right closure if you do two of them you get a new one which is in the same league of things and then you can check associativity, existence of inverse and because it is abelian also that the order does not matter. Identity of course, is $\lambda = 0$.

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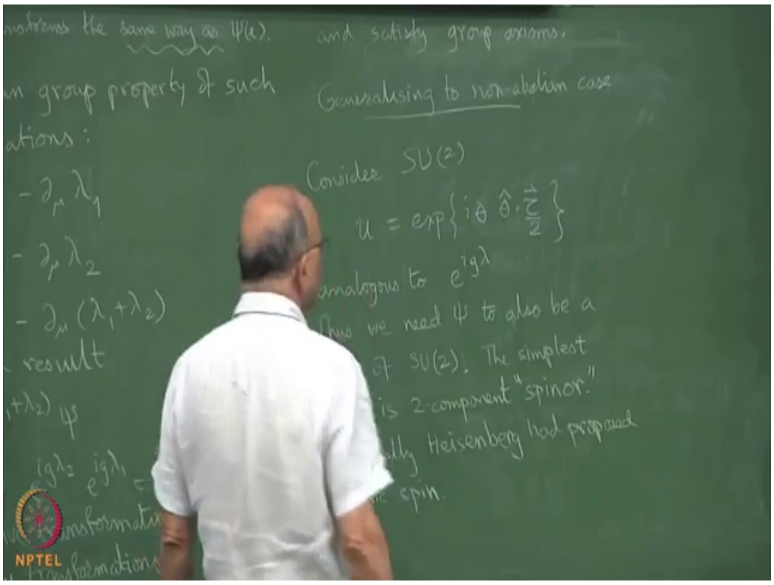

Now, we want to think in terms of generalizing to non abelian case. Here a first very interesting physical property of matter comes into effect because what do we mean by abelian, non abelian etcetera. Well we know there are there is a well known group that can be written in this form this is SU(2) where u can be written as $\exp[i\theta\hat{\theta}\frac{\tau}{2}]$.

Where Θ is the amount of rotation, this is the direction of rotations and the generators are $\tau/2$. So, this is analogous to this $e^{ig\lambda}$, but now we want to we want this to act on something, you know ultimately the covariant derivative was how the $\partial_\mu \psi$ transformed. So, we need a ψ on which this u can act, but this u is 2×2 matrices. So, first I just want to motivate that this ψ you have to deal with has to be also a at least a two component vector it has to be a representation of SU(2).

So, the representations of the abelian group, this is U(1) group you know unitary group, is a magnitude one determinant complex number. So, we need ψ also to be a representation of SU(2) this. So, one does take various possible representations higher order ones as well the simplest is a spinor representation. Now, spinor, do not start thinking of Dirac equation and all that, just means a two component complex valued vector. Historically this already existed.