Path Integral and Functional Methods in Quantum Field Theory
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Lecture – 22
Gauge Invariance – Geometric Picture

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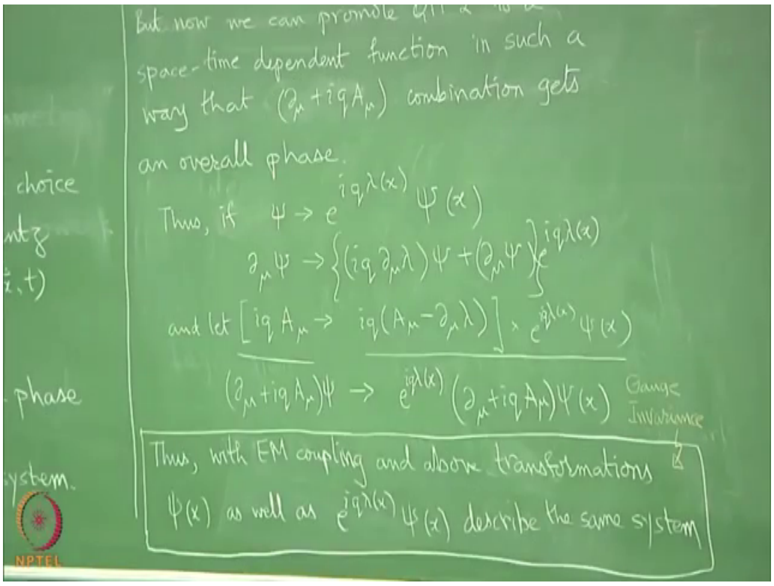

Now, one of the main consequences of this gauge invariance is that all the charges you have in the system have to have only integer multiples of q because when λ shifts by 2nπ/q you have to get back the same answer.

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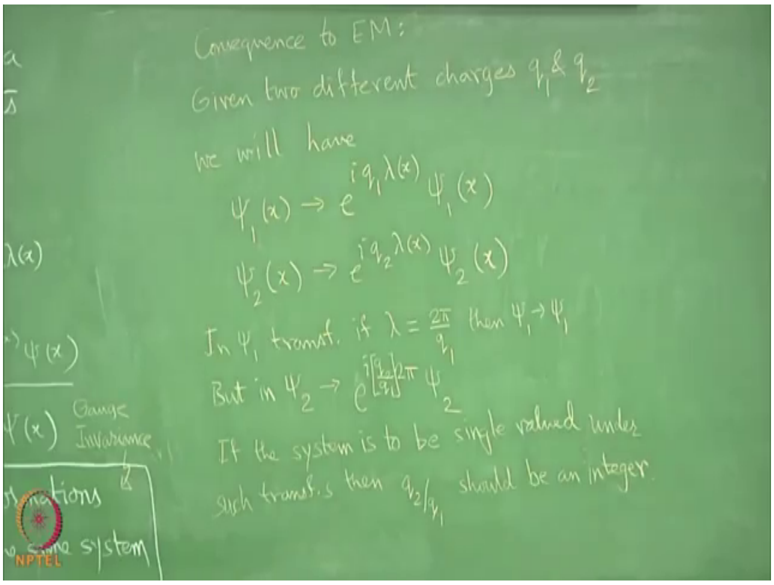

Given two charges, this combination derivative is called covariant derivative. But the point is that if in ψ_1 transformation if $\lambda = 2\pi/q_1$, then there is no transformation $\psi_1 \to \psi_1$. But, in ψ_2, we will have $\exp[i\,(q_2/q_1)2\pi]$.

So, if the system is single valued under such transformations, then we need that (q_2/q_1) must be an integer ok. I guess I have to supplement it by some way in which you can compare the ψ with it is original and so on, but the up short is that if the system is actually unchanged in one sector, then in the other sector should not be stuck with some phase.

So, if q_1 is the smaller number then (q_2/q_1) should be integer or vice versa whichever is smaller the other one has to be an integer multiple of that. So, an interesting actual thing that happened was that Landau and Lifshitz were working out a possible low energy theory for superconductivity.

So, in superconductivity one deals with a charged effective field which is a scalar. So, this is the so called condensate but it is charged.

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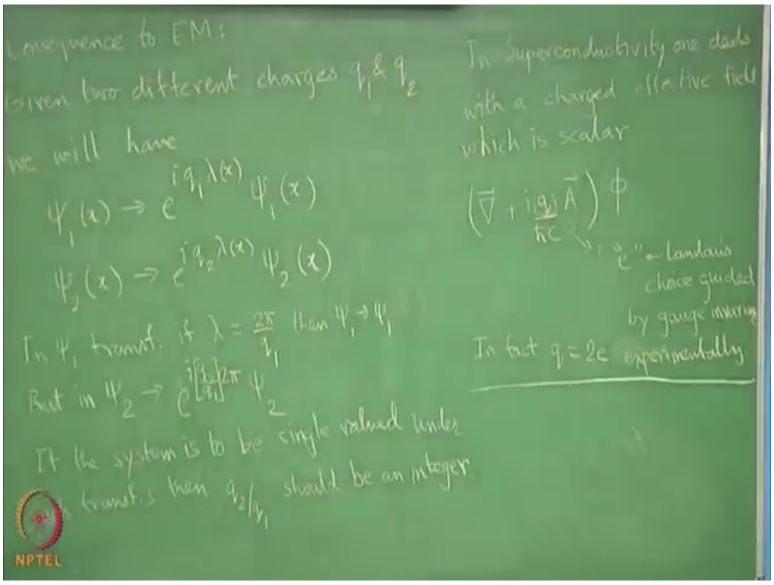

So, one had to write for it something like that $(\nabla + \frac{iq}{\hbar c}\vec{A})\phi$. So, Lifshitz recalls in his tribute to Landau that what should this q be and Landau said just put electron charge because if it is not compatible with electron charge then gauge invariance will be a problem. So, he just said let it be equal to e. It is twice e because the condensate is actually $\psi\overline{\psi}$, electron pair condensate.

So, Landau's choice q=2e for superconductivity ok. Now, that in hindsight and retrospect sounds a very simple outcome, but it is a very deep one because if you are a theorist trying to propose something to experimentalist to look for. And if you got confused at that point you would say, well it can come out any charge then experimentalist will say well so, what should charge should we look for and so on.

Well, it look exactly like ordinary electromagnetism just turned out with twice the charge instead of just q it iself; it could out come out 4, but it would be integer multiple of the charge for compatibility; at least otherwise under closed loops you would not recover single valued systems. There is a beautiful application where actually the system can come to $2n\pi$ times it is original value when there are magnetic flux tubes threading the system ok. So, it is the gauge function λ is actually not single valued if you have obstructions where you cannot shrink loops, but we will discuss that separately.

So, we can complete the discussion of U(1) gauge invariance at this point. Now, we go on to the more advance topic of non-abelian gauge invariance. So, now this gauge invariance as I was mentioning here was proposed by Weyl that had a relation to something else that was going on in general theory of relativity. So, we can comment a little bit on this whole idea of gauge invariance of symmetry and in the literature there is actually a difference of opinion how should one refer to this.

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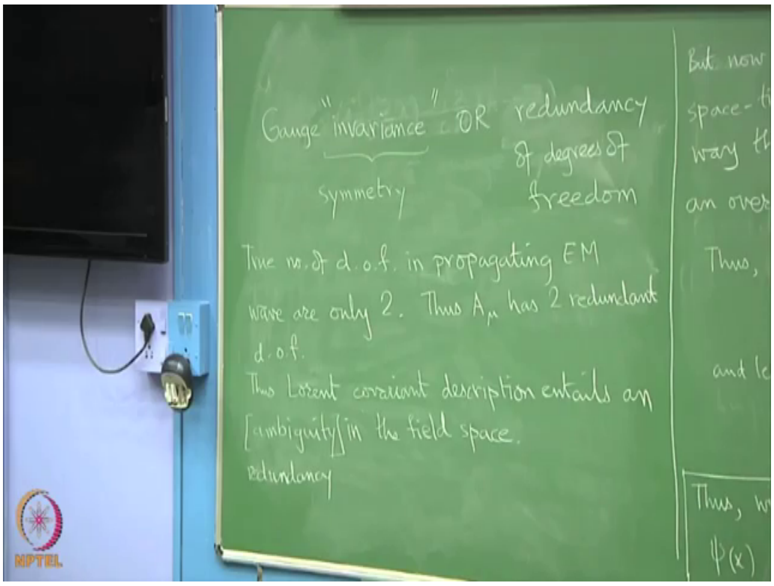

So, if you think as if it is this profound invariance that you can introduce magically some λ which traverses through all the equations, but never appears in the final physics answers you may I think that this is some kind of an invariance and often called a symmetry, but the other opinion is that you just have extra degrees of freedom that you putting so that the formalism looks covariant.

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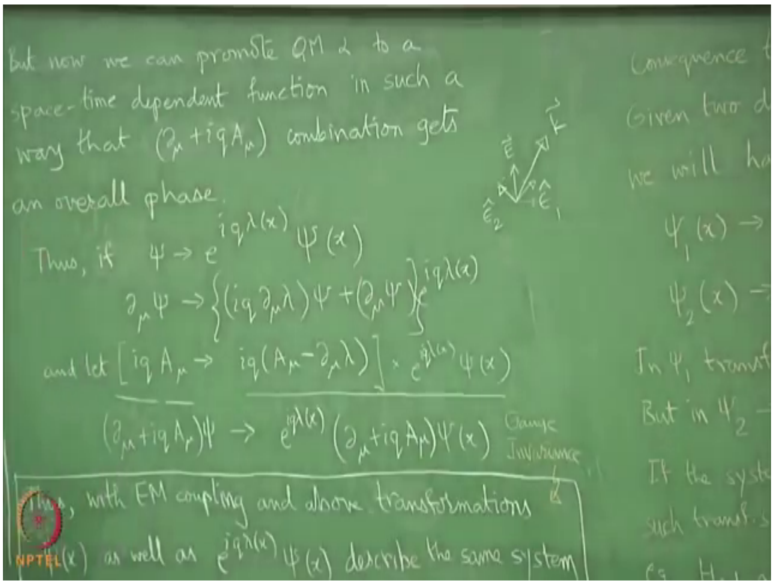

For example, electromagnetism the propagating electromagnetic wave is essentially a plane wave and if you have some preferred directions ε^1 and ε^2, these are unit vectors then a typical electron electric field will have specific projections along this and the direction of propagation is normal to this plane but ε^1 and ε^2 are perpendicular to the direction of propagation. So, electromagnetic field is transverse in a propagating wave, it can do strange things within this plane it can be rotating or doing anything, but it remains perpendicular.

Therefore, there are only 2 degrees of freedom and the B field is determined by E field completely from the Faraday's law. The true number of degrees of freedom in propagating electromagnetic wave are only 2. So, what are 4 A^μ's doing? Now the A^μ has a great advantage because it is a covariant expression, Lorentz covariant field.

So, all the description, all the physics you want to write you have a test to check whether the equation are physically relevant by checking that they remain Lorentz covariant. So, it is good to have a Lorentz covariant notation, but if you do this, then in the internal space, in the space of possible field values you have some redundancy to the extent of an entire space time dependent scalar field. So, thus Lorentz covariant description entails an ambiguity in the field space or rather redundancy not ambiguity. It is ambiguous to the extent that you can add or subtract, but you can therefore say that there is a redundancy of the description.

But, most of the time, in most textbooks and most discussion this thing is emphasized for it being some kind of a symmetry ok. So, if you think that this is some kind of a paradox of contradiction well answer is that both points of you are correct and ultimately the equations remain equations. This is what is good about physics or science. You can put what word you like, but that is the covariant derivative you will have to use. So, the answers will all remain the same.

But, by thinking that it is some kind of a symmetry a lot of satisfaction was drawn. The most important practical implication that it results in is this uniqueness of the charge that you have to introduce and this is what we will see when we got to non-abelian case. So, I will just say the real gain is universality of interaction. So, what this means we will see later. But, idea is what we discussed there it is only one charge value and it is integer value. So, that reduces the number of independent couplings to be read from nature right.

If you have a metal, then the it is elastic constant changes as you go from one metal to the other. Won't it not be very nice if all the elastic constants for integer multiples of each other, will be a boring world, but and maybe in the nano world something like that happens. But, that kind of universality reduces the need to do fresh experiments or writing different description for different substances. They all have a common description.

So, gauge invariance is also sometimes called local symmetry; we also have the usual QM ambiguity is called global symmetry. In fact, most physicists now like to think that the QM ambiguity is something superfluous, but in fact the global symmetries lead to conserved charges. So, global means not space time dependent; local mean space time dependent. These result in conserved charges by Noether's theorem, whereas local symmetries enforce that the Lagrangian can only be the covariant derivative.

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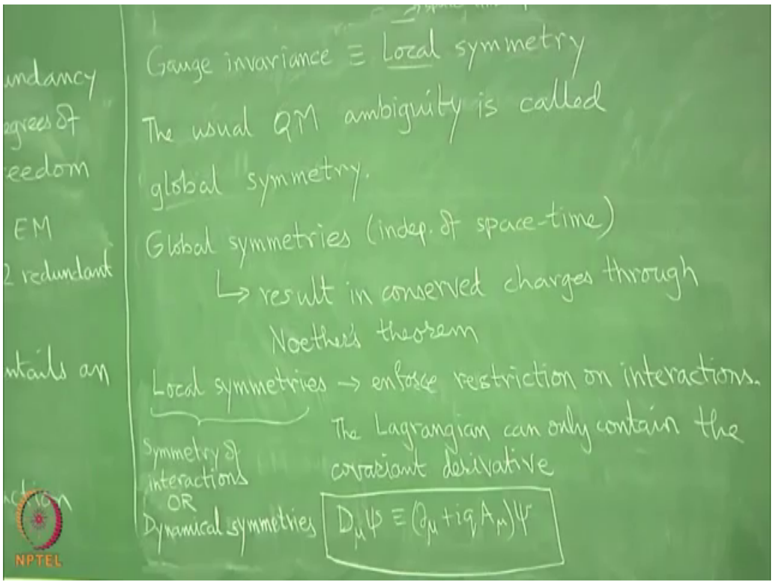

So, you are not allowed to put only the ordinary derivative $\partial_\mu \psi$, you have to put it in this combination only. So, it restricts the kind of interactions you can have and as we also saw this coupling or charge that is a numerical factor, that number is also fixed and at best you can add integer multiples of that. So, that is the restriction put by local symmetries.

And, so sometimes Weinberg calls these symmetry of interactions or dynamical symmetries. And, these are real symmetries because they give you conserved charges. This thing review no new conserved charges, but they enforce the kind of interactions you have. So, that is the distinction between the two kinds of symmetries.

Now, there is also a geometric picture associated with gauge transformations which we can used to extend this idea to the non-abelian case.

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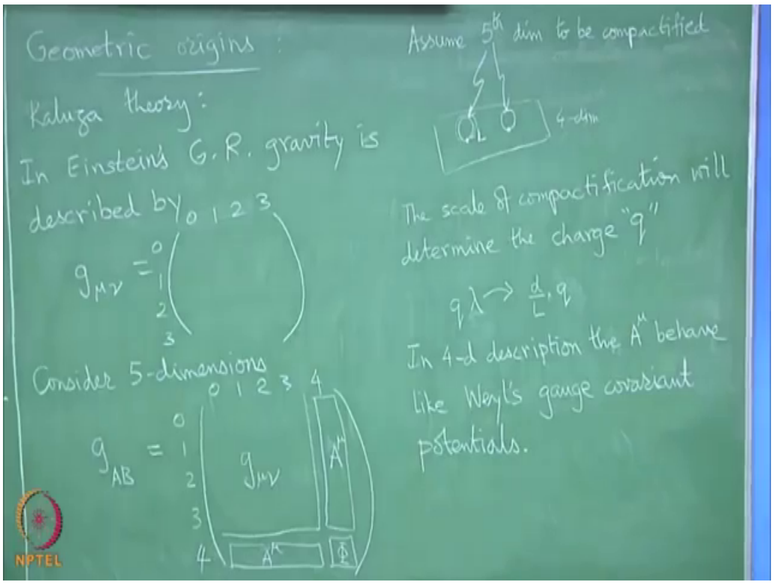

So, now geometric origin is go back what is called Kaluza Klein theories, well actually Kaluza theory. In Einstein's general relativity gravity is described by asymmetric second rank tensor $g_{\mu\nu}$. Now Kaluza proposed that suppose you use 5-dimensions. So, suppose we put g_{AB}.

And suppose that this part of the matrix is the usual $g_{\mu\nu}$. Then you will be left with the components here which because it is symmetric there the same and an extra component here. And, then Kaluza proposed that since we do not see the fifth dimension maybe the dimension is very small, it is compactified. So, although it is supposed to be a 5-dimensional metric, so, assume the fifth dimension to be compactified.

Well, if we do not see it then how will it manifest? Well, the point is it is some kind of a circle. So, this is the 4-dimensional space and then there would be an internal direction at each point right. So, this is the fifth dimension. At every point in space you will have this internal direction which we cannot probe which cause it is too tiny, trying to probe would be equivalent to exciting these.

But, think about it they basically will behave like A_μ field because they are four components. So, you cannot rotate 3 into 4 because this is too small to rotate this into that, but any excitation here to the 4-dimensional observer will look like a 4 vector and that will be an additional thing which would be a new scalar not the one we have been writing before. There would be a scalar.

So, in Kaluza theory the presence of this compactified dimension will appear as ordinary Einstein relativity plus an electromagnetic vector potential and some new other scalar degree of freedom and the scale of this will determine the charge. So, this λ was like a angular variable, but if this circumference is equal to L then the angle would be some distance travels on this divided by that L and then this charge q could as well be thought of as determined by this value L in the units of 1/L ok. So, it counts a units of 1/L.

So, because when d reaches 2π/L is when the system has to come back to itself, but the charge would be determined by the size of the internal dimensions and in 4D description this looks like Weyl's gauge covariant potentials. Now, I jump to the conclusion, but this is the proof that Kaluza gave. So, what Kaluza should was that if you transform it like a 5 dimensional space time metric, but restrict yourself to compactify transformation from this then effectively for this pieces of the metric the look as if you are doing Weyl gauge transformation ok. This is the proof of Kaluza.

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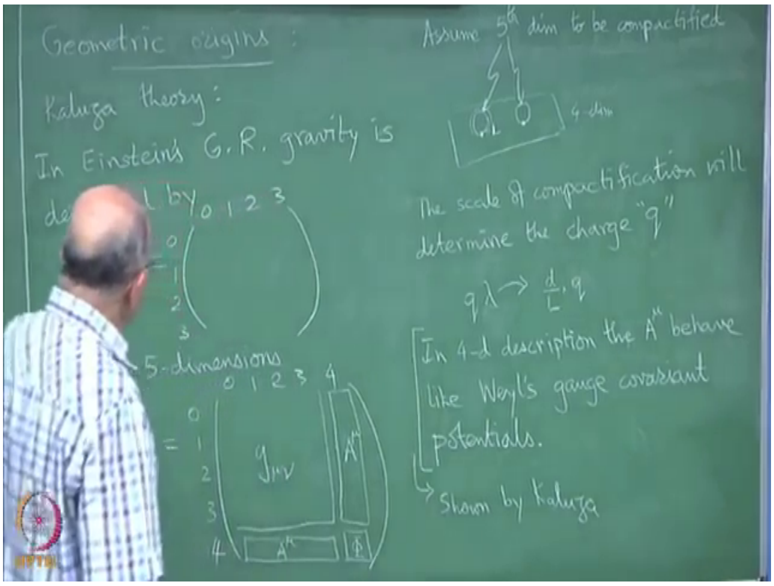

So, you start with full Einstein general covariance. So, thus this is like saying we got gauge invariance out of general covariance.

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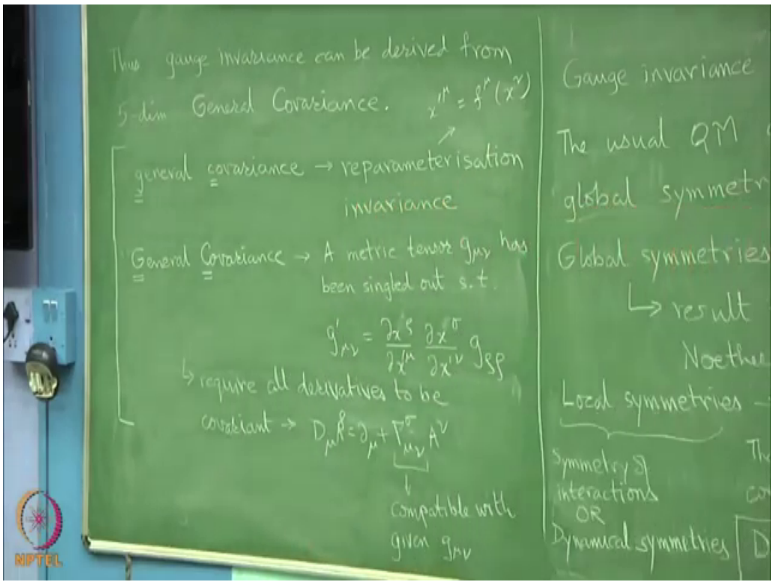

Now, I could also lecture your little bit on what is meant by general covariance, there are 2 parts to general covariance. So, general covariance with lower phase G and C is same as re-parameterisation invariance. This is not a principle of physics. All it says is geometrically you can choose co-ordinate patches as you like that is what parameterization invariance means. It has no physical content what is ever. Lot of people seem to think that sending $x^{\mu} \rightarrow x^{\mu'} = f^{\mu}(x^{\nu})$. This is there because it is a choice of coordinates system. It has nothing physical about it..

However, the general covariance of general relativity is that there is a metric tensor $g_{\mu\nu}$ has been singled out such that

$$g'_{\mu\nu} = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\sigma\rho}$$

that when you make the transformation, this is the transformation the $g_{\mu\nu}$ obeys.

So, you know the transformation rule for g. So, the existence of a selected out metric along with re-parameterisation together that is what constitutes the physical law, that the selected metric has to transform covariantly under the re-parameterisation invariance that together constitutes general covariance and results in further required that all derivatives to be covariant derivatives right where the Γ is derived from that this g.

So, it is of course, this requirement at this is compatible with this is what is called Riemannian geometry, but that this whole package together is what constitutes general covariance. So, this at least is a clarification of how Weinberg's book on general relativity uses these words. You will find that at some point is switches from this to this and the that is the difference between the two.

So, what I was saying was that it is not so much that you have reparameterisation in variance in 5-dimensions that is important, but that actually exist physical fields A_μ which will exactly transform the Weyl way as observed in 4-dimensions under the general covariance of 5-dimensions.

In 5-dimensions you will also introduce it's covariant derivative you have a selected metric tensor and that so in fact, when you propose this curved up you already proposed a specific geometry. So, you proposed a particular metric in those 5-dimensions and that is what and how that transforms is what is the physical principle of gauge invariance. So the gauge invariance can be derived from general covariance in this sense. So, we will stop here today.