Path Integral and Functional Methods in Quantum Field Theory

Prof. Urjit A. Yajnik
Department of Physics
Indian Institute of Technology, Bombay

Lecture – 21
Gauge Invariance – Minimal Coupling

Today, we begin with the topic of Yang Mills fields, because these are become the corner stone of all of modern physics.

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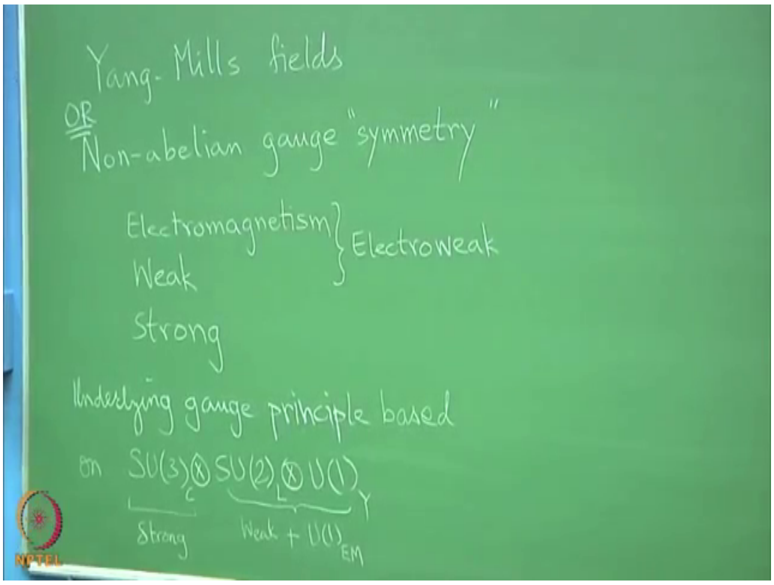

And in other words this topic is called non-abelian gauge symmetry gauge. What is interesting about this concept is that all the particle physics we know is very elegantly described in terms of this principle. What it does is that it makes all couplings universal like the value of the electron charge, everything else is a multiple of that. And therefore, all the electromagnetic interactions are determined by one charge, similarly the weak force and the strong force. All the three are described by the same quote symmetry principle and so, I put quote marks on this I will explain what that means. And what we find is that these two are sort of inter twined.

So, together they form what we now call electroweak theory. So, there is an under lined gauge principle based on the group $SU(3)_C \times SU(2)_L \times U(1)_Y$ where C is color, this is strong force and these two together give the weak force plus $U(1)_{EM}$. So, this is an advertisement or trailer if you like of what this topic is all about ok. So, thus all the known forces at terrestrial level are completely described by this particular framework.

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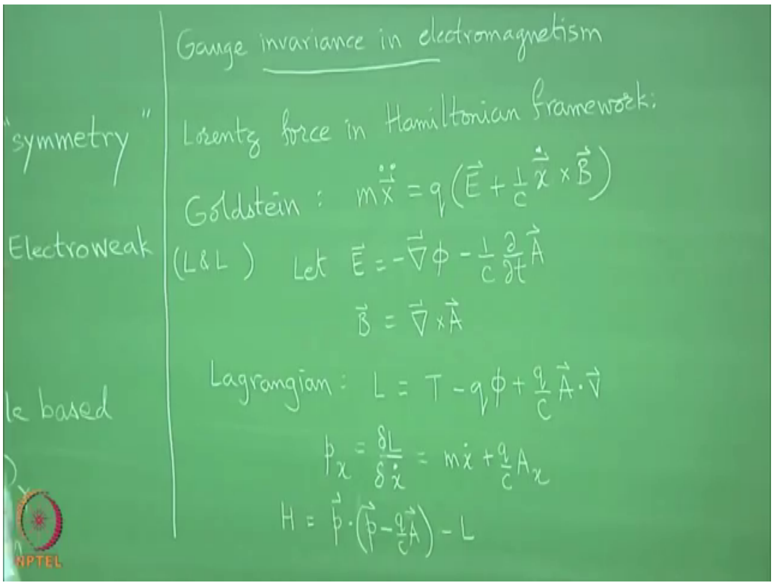

So, let us begin this by seeing what is the usual idea of gauge invariance or the electromagnetism. So, before we get to become very sophisticated, let us try to track back how one reduces this and how possibly the first people who made this observation about this kind of symmetry, saw this and that starts with the Lorentz force in Lagrangian framework or Hamiltonian framework.

So, if you see Goldstein's book which is only one we used to use when we were young, nowadays the book has a co author and it is somewhat re written. But if you see Goldstein's book this occurs fairly early and that book contains the proof that and it would probably be also there in Landau Lifshitz classical mechanics

$$m\ddot{\vec{x}} = q\left(\vec{E} + \frac{1}{c}\vec{x} \times \vec{B}\right).$$

So, what we do is we propose at this point that E is derivable from a scalar potential

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial A}{\partial t}.$$

we know that, this can always be done Faradays law and Ampere's law together will allow you to do this and $\nabla \times \vec{B} = 0$, will allow you to write B in terms of a vector potential. In that case using this we can rewrite the Lorentz force in Lagrangian form as

$$L = T - q\phi - \frac{q}{c}\vec{A}.\vec{V} \quad .$$

And therefore if you find the canonical momentum

$$p_x = \frac{\delta L}{\delta \dot{x}} = m\dot{x} + \frac{q}{c}A_x \quad .$$

So, the canonical momentum of electromagnetism already has the potentials in it not just the velocities. So, you can then construct the Hamiltonian out of this

$$H = \vec{p}.(\vec{p} - \frac{q}{c}\vec{A}) - L \quad ,$$

but the point is the canonical momentum now looks like this.

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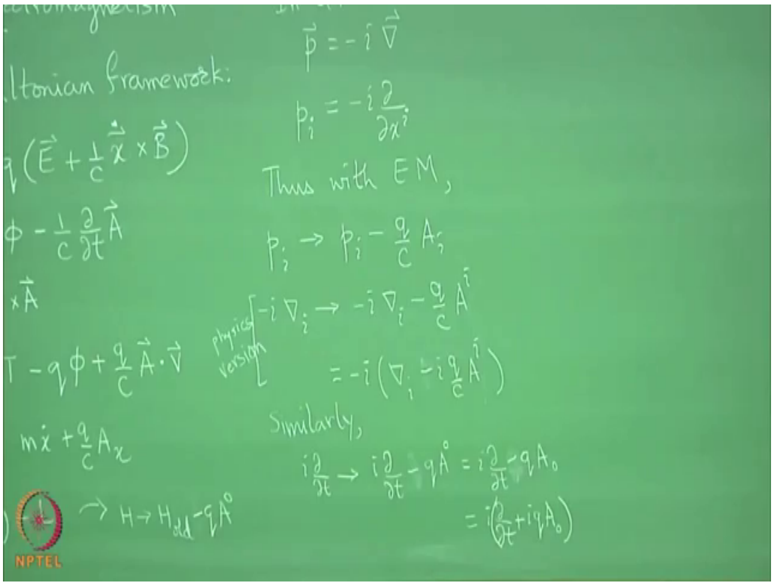

So, in quantum mechanics we know that $\vec{p} = -i\nabla$ and eventually we have to get to a covariant form and there is some mismatch between the way we write here and the covariant notation. So, for example, the gradient operator. So, if we have to write this as a in index notation then the gradient operator is automatically a covariant vector, contra variants are written with topper index and co with down index, mainly to remember whether you are free what whether you are referring to the original frame of reference or it is dual wall frame of reference.

In electromagnetism what we need is that $p_i \longrightarrow p_i - (q/c)A^i$ and as for as I remember there is a mixed notation here, it turns out that one of these indices is up instead of down ok. So, if you take x to the contravariant vector then this A_x is covariant and so there is an opposite sign, but this is the prescription that correctly works.

So, whenever you see minus $i\nabla$, you replace it by $[-i\hbar\nabla_i - (q/c\hbar)A^i]$, but that is same as $-i[\nabla_i - i(q/c)A^i]$. So, this is sort of the physics notation and additionally what happens is that the Hamiltonian gets from the -L it gets a $+q\phi$.

So, this also means $H \longrightarrow H_{old} - qA^0$ and therefore,

$$i\frac{d}{dt} \rightarrow i\frac{\partial}{\partial t} - qA^0 = i\frac{\partial}{\partial t} - qA_0 = i\left[\frac{\partial}{\partial t} + iqA^0\right] \quad .$$

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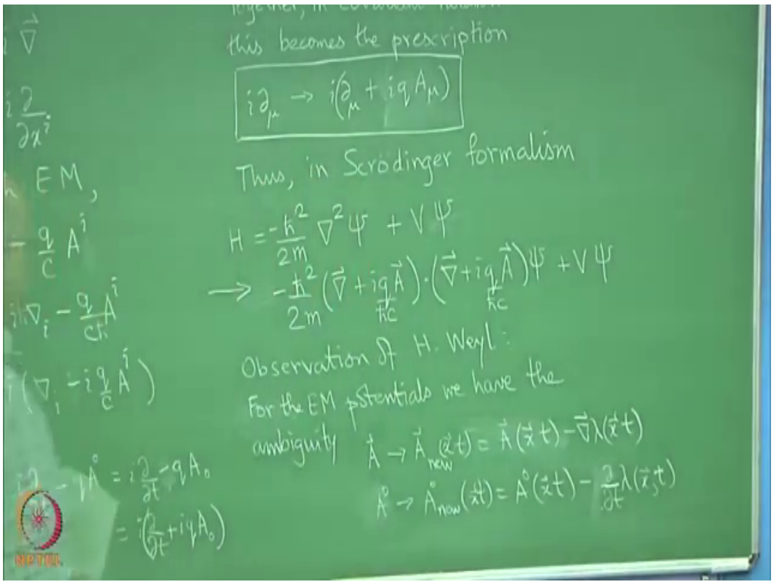

So, what we will adopt is that together $i\partial_\mu \rightarrow i[\partial_\mu + iqA_\mu]$ and this is what we will be using. The only problem is that you have to check that it reproduces Lorentz force correctly, if you started with the idea of describing physics correctly and it does match.

So, if this is so then so what? Well the interesting point is that this is the way to couple electromagnetism to charges, the coupling between charges and the electromagnetic fields is through this prescription only. Whenever there is a canonical momentum you replace it by canonical momentum plus this contribution from the gauge fields. So, in quantum mechanics, where we have a Schrodinger equation where the Hamiltonian begins with

$$H=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi \quad .$$

So, this goes over to

$$-\frac{\hbar^2}{2m}\left(\vec{\nabla}+\frac{1q}{\hbar c}\vec{A}\right).\left(\vec{\nabla}+\frac{1q}{\hbar c}\vec{A}\right)\psi+V\psi \quad .$$

This was observation of Herman Weyl that we have an invariance in electromagnetism.

So, for the EM potentials we have the ambiguity that the vector potentials can be replaced by

$$\vec{A}\rightarrow\vec{A}_{new}(\vec{x},t)=\vec{A}(\vec{x},t)-\nabla\lambda(\vec{x},t) \quad .$$

Again I am using a mixed notation because that gradient is actually from the covariant notation whereas, I am writing upper index A’s, that does not matter is just to observe this they were all sign will not matter. So, the point is if you change A by the gradient of a space time function λ, then the curl of the new A is same as curl of the old A.

So, this relation does not change and you can also set

$$A^0\rightarrow A^0{}_{new}(\vec{x},t)=A^0(\vec{x},t)-\frac{\partial\lambda}{\partial t}(\vec{x},t) \quad .$$

This works because if you look at the E field we can actually check about the 1/C etcetera

$$\vec{E}=-\vec{\nabla}\left(A^0-\frac{\partial\lambda}{\partial t}\right)-\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{A}-\vec{\nabla}\lambda\right) \quad .$$

So, this works out correctly if we put a 1/C here. So, because of that notational problem with up and down indices, I think I need here a plus sign. So, we put a plus sign here in any case as we are saying it will not matter. So, this will become equal to

$$-\vec{\nabla} A^0 - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \quad .$$

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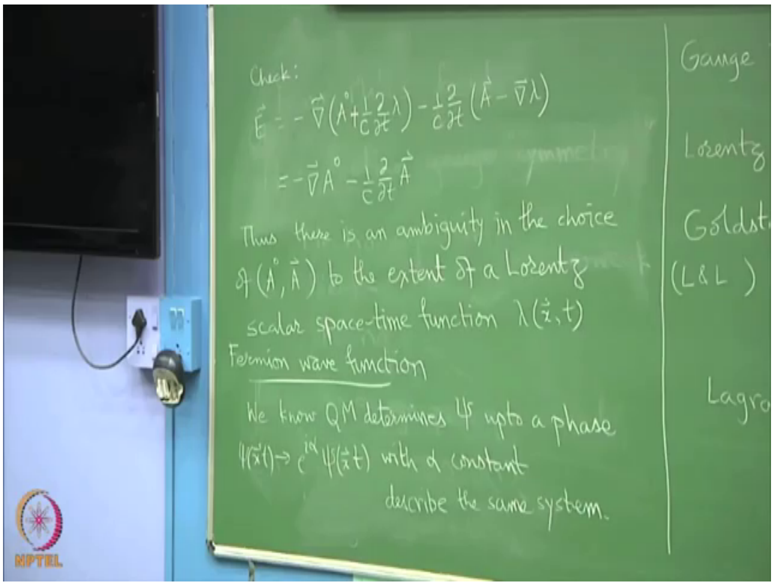

Now, I have used the word ambiguity which is correct from the goodold differential equations point of view because given physical E and B, you can always find ϕ and A as solutions of these differential equations because they are first order differential equations, all these are coupled partial differential equations.

The point then is that solution will not be unique in fact, it will have a huge ambiguity, this if you have tried to solve first order PDE is by substituting into each other you know that unlike putting initial condition in ordinary differential equations, here the initial conditions are functions of the variable which is not involved.

When your partial differential equations all you know is

$$\frac{\partial}{\partial x}F(x, y) = x \Rightarrow F = \frac{1}{2}x^2 + G(y) \quad .$$

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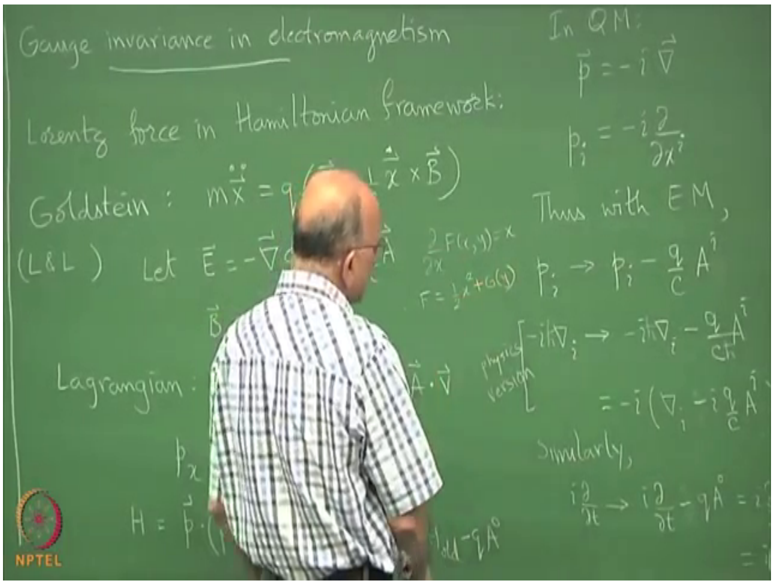

So, when you are solving them simultaneously the other conditions then help you to determine G, but overall you may be left with a whole functional ambiguity in solutions of such partial differential equations and that is exactly what happens here, you try to find this, they are not uniquely determined by these two equations and so, there is a functional ambiguity and that anyway you observe enters through derivatives, it is not directly that there is and there is only one scalar function.

Although there are four components, there is only one overall space time the Lorentz scalar function that enters whose derivatives cause an ambiguity in identifying the gauge potentials. Now one other thing I just would like you to remember is that, a Lorentz scalar is a very strange object. Lorentz scalar is something you are never seen, all the scalars you know of will transform very strangely if you perform Lorentz transformations.

One of our favorite scalar is length, it is invariant under rotation. So, we think it is a scalable length is not a scalar Lorentz scalar. So, most things that you know are actually not going to qualify. In fact, the rest mass is a true scalar, but we already give it away because we say rest mass meaning mass in its own frame of reference, which anywhere reduces to particular choice of frame. So, we do not know any space time fields that are scalars until very recently. All I am trying to say is it somewhat counterintuitive what space time Lorentz scalar would be, but that is what one adds.

Now, that ambiguity is transformed into magically into some kind of a symmetry and that was the observation of Harman Weyl. So, going back to the Schrodinger equation we know that the wave function has a overall phase problem. So, but now Herman Weyl observes that if for this constant α if you instead put this same λ, then it will then you can arrange for this whole object to remain unchanged under this.

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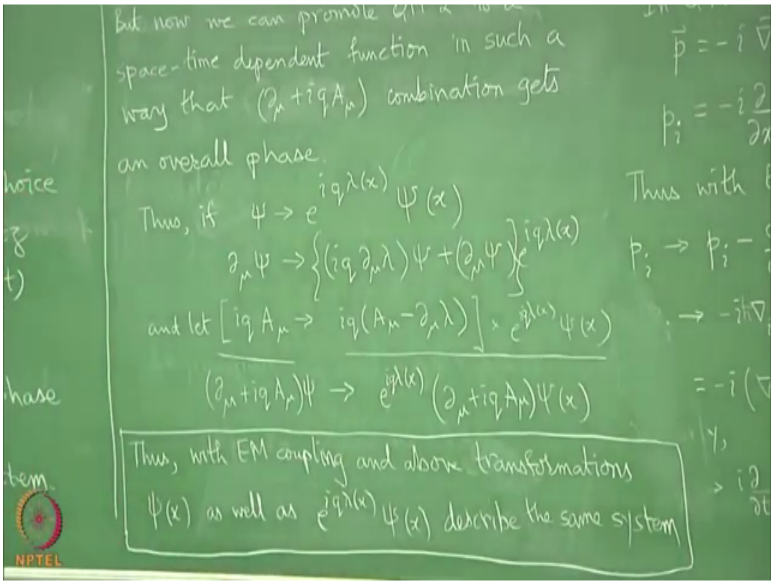

Thus if $\psi \rightarrow \exp[iq\lambda(x)]\,\psi(x) \Rightarrow \partial_\mu \psi \rightarrow [(iq\,\partial_\mu \lambda)\,\psi + (\partial_\mu \psi)\exp[iq\lambda(x)]]$ and let

$$iq\,A_\mu \rightarrow iq(A_\mu - \partial_\mu \lambda)$$

and now then $(\partial_\mu + iqA_\mu)\psi = \exp[iq\lambda(x)](\partial_\mu + iqA_\mu)\psi$.

Now, when you take this and put it here that $e^{iq\lambda}$ will come here, this again will act on this combination, but this combination again will remain unchanged because A will have shifted and you can bring the overall phase all the way here and this ψ also has the same phase. So, you can throw away that same phase the whole Hamiltonian has changed by only an overall phase or you can just replace the ψ by that new ψ with a space time dependent phase it will not change the Hamiltonian. In covariant notation it gets even better because it actually even cancels as we will see for scalar fields. Thus with electromagnetic coupling and above transformations of $\psi(x)$ as well as $e^{iq\lambda}\,\psi(x)$ describe the same system and so this is the statement of gauge invariance.