Path Integral and Functional Methods in Quantum Field Theory

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Lecture – 20
Asymptotic Condition Kallen – Lehmann representation - II

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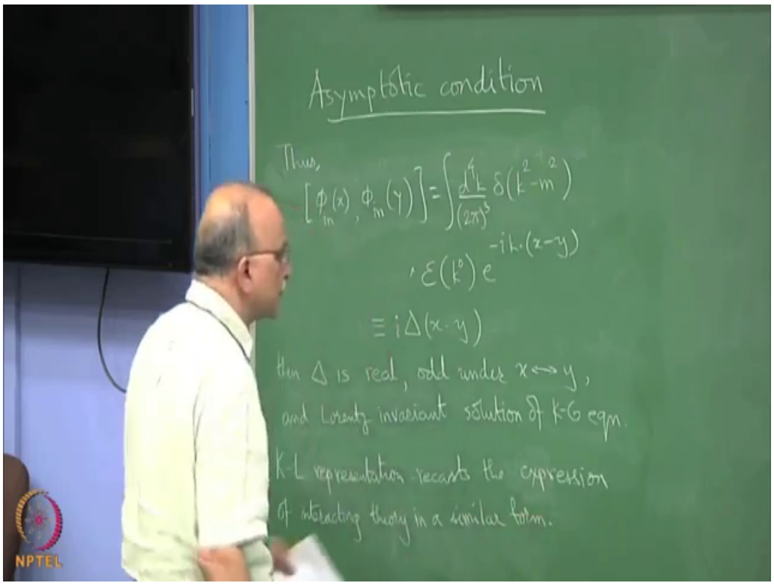

$$[\phi_i(x),\phi_i(y)]=\int \frac{d^4k}{(2\pi)^3}\delta(k^2-m^2)\epsilon(k^0)\exp[-ik.(x-y)]$$

which is firstly a real function because, well that is clear in this form. Because it contains this and minus its complex conjugate. So, if you take out an i then it will become real. So, this is purely imaginary, because you are taking expression minus its complex conjugate. So, it is purely imaginary or under exchange of x and y, that it should be because the commutator has to be this so, you can check that this integration gives all and it is Lorentz invariant and it also solves the Klein Gordon equation.

So, thus if you define this to be equal $i\triangle$ **(x-y,)**because it will just hit, if you setup the K-G operator in x coordinate or y coordinate it will go and hit the corresponding one in then it will just give 0 because it is the free field. Now, what you want to show is that the general one boils down to some kind of a form that looks very similar to this aside from an overall density, that multiplies it, which is the finer version of just doing putting a constant we will see that it has a more detail and gives a k dependent density, which multiplies it. Now to do this, we begin with, $<0|[\varphi(x), \varphi(y)]|0>$; after all guaranteed everything only under a matrix element not as an operator equality. So, what we do now is that we rewrite $\varphi(x) = \exp[ip.x]\varphi(0)\exp[-ip.x]$. So, this is by translation invariance. This translation invariance is actually corner stone of all of the quantum field theory that is used in S matrix theory. So, P as you recall is the P^{μ} are the generators of translations according to Norther's theorem right. So, the point is the origin of field theory should not matter.

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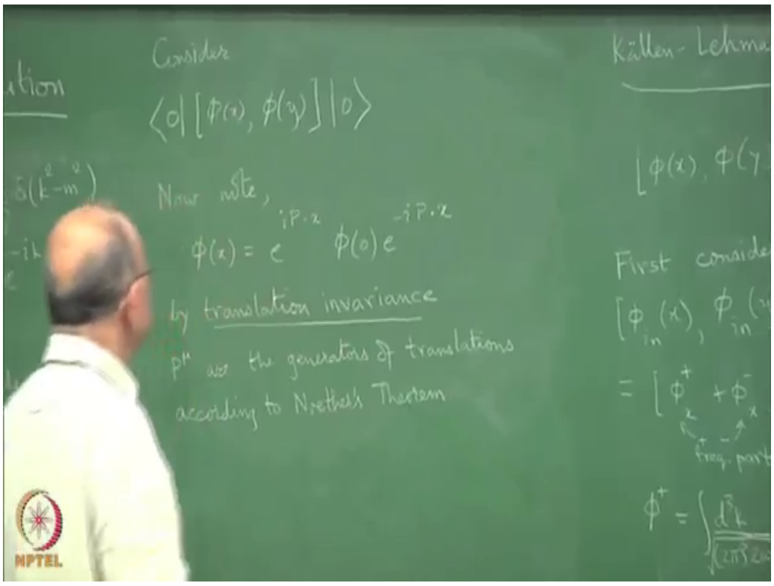

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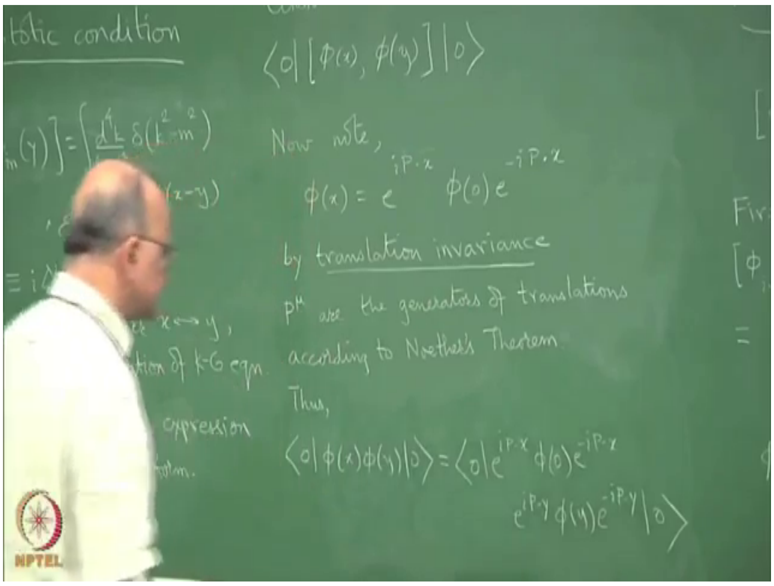

So, that is first statement. Therefore, if we take

$$\langle 0|[\phi(x),\phi(y)]|0\rangle=\langle 0|e^{ip.x}\phi(0)e^{-ip.x}e^{ip.y}\phi(0)e^{-ip.y}|0\rangle \quad .$$

And, what I do now is I insert complete set of states which are momentum eigenstates.

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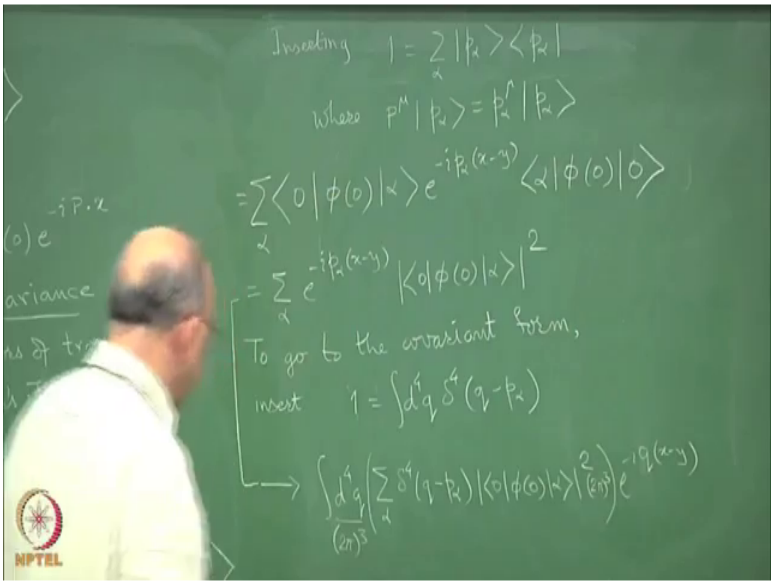

So, this becomes equal to the translation operator acting on the vacuum will just all be 0's. So, that part becomes 1. So, we get equal to

$$\sum_{\alpha}\langle 0|\phi(0)|0\rangle e^{-i p_\alpha(x-y)}\langle\alpha|\phi(0)|0\rangle=\sum_{\alpha} e^{-i p_\alpha(x-y)}\left|\langle 0|\phi(0)|\alpha\rangle^{2}\right| \quad .$$

So, as you know if we had only the free field, then only the 1 particle state would have contributed. Of course, this is p_α, so it is in terms of total momentum. So, it is not in terms of number eigen states, but this summation would be much smaller if it was free field.

But, now it contains everything. And of course, we have to put the terms with x → -x exchanging x and y so that we get the other term with a minus sign. Now, we want to be cleverer with this and what we do is we want to write this in the covariant form, because we want to tied up with that thing. So, to go to the covariant form insert $1 = \int d^4q\ \delta^4(q-p_\alpha)$; so, then this will become equal to

$$\int \frac{d^4 q}{(2\pi)^3} \sum_{\alpha} (\delta^4(q-p_{\alpha})|\langle 0|\phi(0)|\alpha\rangle^2|(2\pi)^3) e^{-iq(x-y)} \quad .$$

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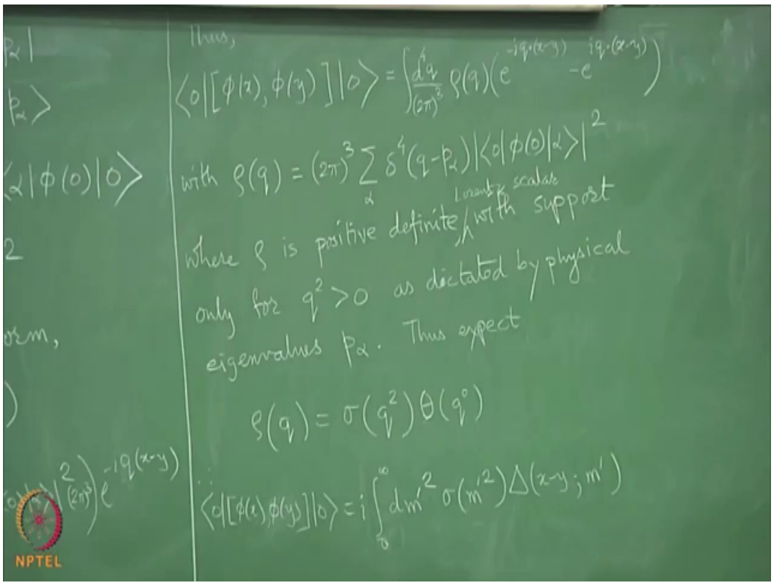

Therefore, it looks like we can rewrite our original expression

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = \int \frac{d^4 q}{(2\pi)^3} \rho(q)(e^{-iq(x-y)} - e^{iq(x-y)}) \quad .$$

So, we of course gathered everything as that was not looking nice into this thing, which is a density, which is obviously positive definite ok. Not only that it is nontrivial only for q in the positive light cone. So, because this delta function sets any of the values of the p_{α} which are all physical momenta.

So, now comes the final rearrangement or relabeling is to say that $\rho(q) = \sigma(q^2)\Theta(q^0)$. So, function only of q^2 and is non-zero only for $q^0 > 0$ is the Heaviside theta function. The further thing that it is function only of q^2 is because the left hand side is Lorentz invariant right. It has Lorentz scalars which are Lorentz invariant, is commutator of two Lorentz scalars. So, this is the Lorentz scalar.

So, whatever happens on the right hand side it has to come out a Lorentz scalar. And, if you look at this whole thing this is a Lorentz invariant measure, this is a Lorentz invariant expression. So, this be Lorentz invariant. So, it is function only of the Minkowski q^2, but now we look at our expression here, this had almost the same things in it except for the $\Theta(q^0)$ becomes our this function $\varepsilon(\mathbf{k}^0)$. We want to pretend that this is an integral over this dq^2 only and for $q^2 > 0$. So, let me just write it as,

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = i\int dm'^2\,\sigma(m'^2)\,\Delta(x-y\,;m') \quad .$$

So, this is a step that says it amounts to changing the definition of your mass. So, you have to read it backwards that with this form which looks like that, but with q^2 is fixed, but it is the mass values that keep changing. And, with a weightage factor that is this σ which is extracted from this density and that is what this expression looks like.

This probably needs a little more elaboration, but this is what we will claim. So,this is already derived free field function, but with different values of the invariant in it. So, if we go back we can think of this as the q^0 with a delta inserted instead of that delta we inserted in the free field we have this density.

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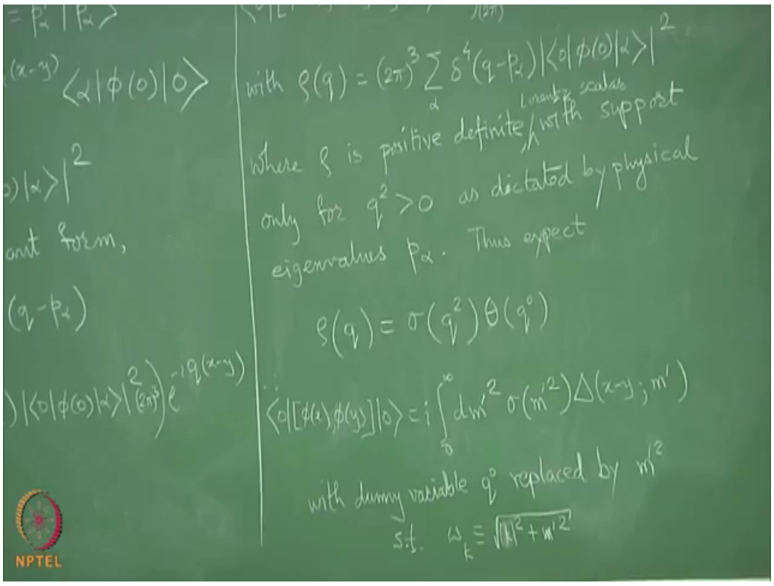

But, we write a different dummy variable. It is forced to be positive because it is support only when q^2 is positive and the theta function is there. So, we call it m^2 right. And, whatever remains is an expression that is corresponding to delta function. So, it becomes as if it is the free field delta function instead of ω_k's, you will be using the q^0 replaced by m^2. So, k^0's inside these will be determined by the condition that, they will refer to $(m')^2$ such that $\omega_k = [k^2 + (m')^2]^{1/2}$.

So, the q^0 integration is now same as the k integration for the individual ones except that it is quote on shell on mass shell provided you put the mass value corresponding to this integration variable. So, it is a just relabeling of the integration variable and imposing the theta to take it from 0 upward. Now, the power of this formula is that it says that the most general one, the interacting commutator is just like summing over the free particle once.

So, that particle picture kind of persists even in your interaction region. Accept that you have to sum over a large number of them that corresponds to all the possible momentum eigenstates. This is what we tried to argue when we said this $Z^{1/2}$ thing.

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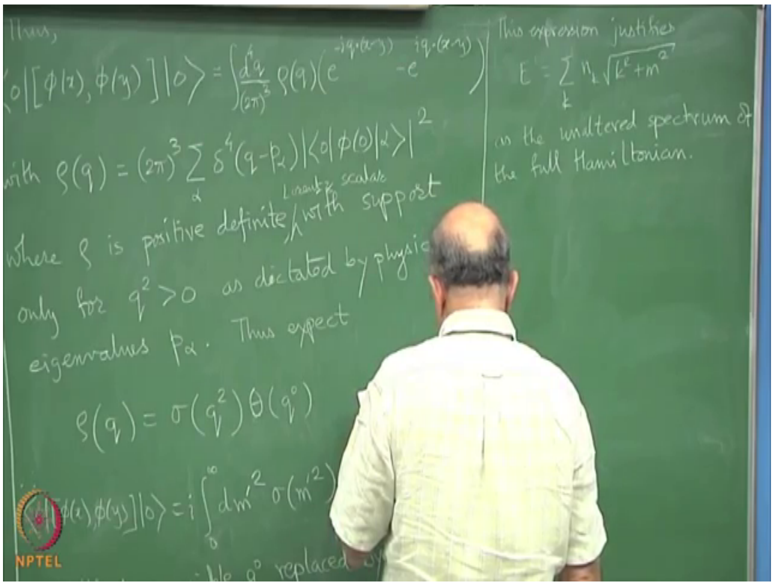

That not only one particle states, but all kinds of other states, that are so long as charge is conserved. So, all the pair productions and everything else is generated by these interacting fields, but we insist that that the energy expression looks like it is some over. So, this is compatible with

$$E = \sum_k n_k \sqrt{k^2 + m^2}$$

as the unaltered spectrum of the full Hamiltonian. Not only that we now claim that we can isolate the part corresponding to one particle states or the free particle state out of this.

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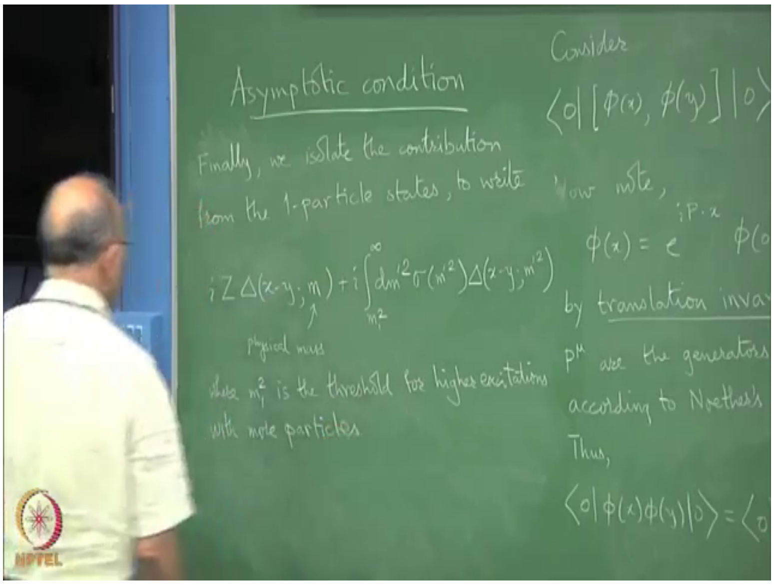

And, note that this is nothing but if you replace α by 1 particle state then this is nothing but the wave function of the single particle. So, this is equal to then,

$$iZ\Delta(x-y;m)+i\int_{m_1^2}^{\infty} dm'^2\sigma(m'^2)\Delta(x-y;m')$$

Where, m_1^2 is called threshold for higher excitations with more particles. This is the final statement. If, we go back to our assumption that we will take only polynomial interaction terms remember we wrote the kinetic part and plus the $U(\varphi)$ or $V(\varphi)$ with only φ and not any derivatives.

So, finally this happened, but another finally, is that now if I take time derivative of this, which should become $[\Pi(x),\varphi(y)]$. This becomes this provided Lagrangian is of the form $(\partial^\mu\Phi\partial_\mu\Phi - m^2\Phi^2 - U(\Phi))$ which ensures

$$\dot{\phi}^2=\pi \quad .$$

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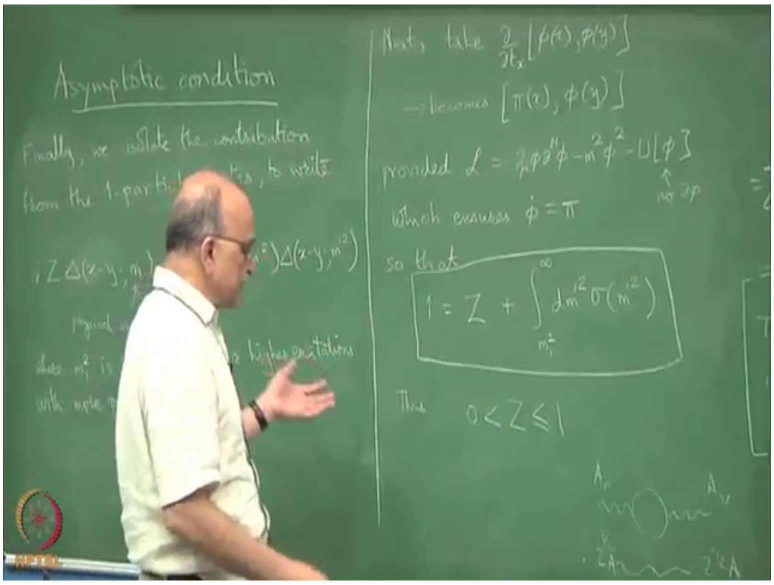

However, badly interacting the theory is so long as the interactions are only polynomials in Φ, the canonical momentum will simply come only from this term and will be equal to Φ·. So, if I now take dot of this I will get [Π,Φ], but which is -1. So, looking at the full expression here yeah this one. Hitting with this I will get a -1 here, but on the right hand side I will get a -1 and on this side I will get same things, because these are all delta of, so there also all commutators like that right.

So that we get

$$1 = Z + i\int_{m_1^2}^{\infty} d\,m'^2\,\sigma(m'^2) \quad .$$

All those things become 1, because they are commutator canonical commutators. So, they become 1 or -1 depending on which order you differentiated, but this is what it becomes. So, this is a very nice result, which shows explicitly that 1 is equal to Z plus some positive stuff ok.

So, Z has to be between 0 and 1. Now, this was a very important thing back in the days of renormalization, because everybody was so scared that everything was diverging and there were infinities. So, Kallen Lehmann were the ones to prove this representation first to reassure people that the wave function renormalization is going to be less than 1. But, if you do it in quantum electrodynamics then it comes from the photon wave function renormalization comes from this graph. Because, it is $A_\mu A_\nu$. And this piece when integrated as to be reabsorbed as just ordinary propagator with a redefinition, it could be a $Z^{1/2}$ at each end.

This two particle content has to be absorbed in the free propagator well. So, you do not get any dot there you only get renormalization, but this thing is log divergent, it is not finite at all. So, there is a psychological contradiction between the 1 loop perturbation theory answer for what Z is and this rigorous proof which did not involve any assumptions about anything except that the interaction is purely polynomial ok, that is the only main assumption involved. So, Kallen latter proves a theorem that says even in quantum electrodynamics out of the 3 renormalization constants, this renormalization and renormalization of electric charge and of the electron mass. Out of those 3 at least one has to be finite cannot be infinite. And, that goes back to being able to prove this for the single particle. So, this is only one field. Now, you had several fields and some complicated things were happening, but he proved a non perturbative theorem that at least 1 out of the 3 renormalization constants remains finite even in quantum electrodynamics.

And, then one fine day his plane crashed. So, otherwise Kallen was quite a, I mean challenge to Schwinger and others because he was proving various theorems without having to take records to real perturbation theory, but well that is what the outcome of all this consideration.

So, field theory gets very suttle because you are really dealing with uncountable infinities at uncountably infinite number of space time points. But, you can still carry out manipulations under integral signs and protected by something's all the other. And, you can prove some general things that remain valid and they give you quite a bit of insight into how field theory works and how it relates to the S matrix.