Path Integral and Functional Methods in Quantum Field Theory

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Lecture - 19
Asymptotic Condition Kallen- Lehmann representation – I

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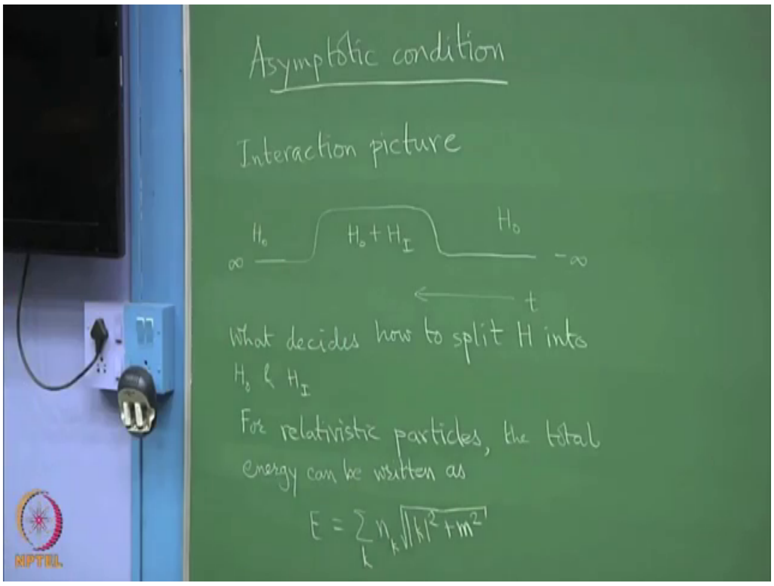

So, what we were discussing was Asymptotic Condition and what we want to say is that we use interaction picture and interaction Hamiltonian terms are only in between. So, it is H_0 here and H_0+H_I over here, and this is our time. But you might wonder what decides what is H_0 and the answer is that for quantum field theory for relativistic particles.

So,if each particle has some kind of meaning on its own, then we expect that the total energy

$$E = \sum_k n_k \sqrt{|k^2| + m^2}$$

where n_k is the number of particles with momentum k and most importantly that m is the physical mass.

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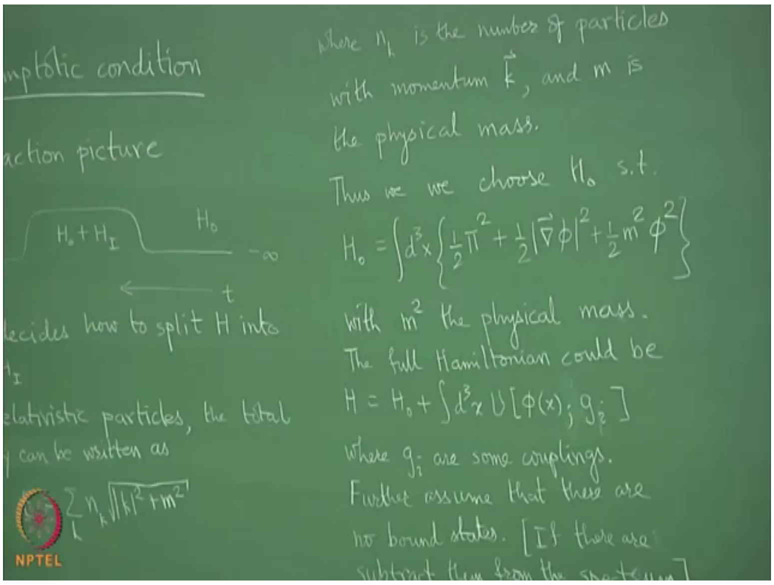

In particular whatever you write has to be valid in the rest frame of at least some set of particles. So, you can boast to that frame and then those particles are at rest. So, it should just be equal to $n_{(k= 0)}$ m.

But if you change to another frame of reference some another k may be set to 0. So, from relativistic invariance and provided there is a particle like picture expect that the total energy of the system would just look like this at any instant of time and therefore, what we expect is that asymptotically as well the free Hamiltonian should be that which contains the physical masses of the particle. So, we choose H_0 s.t.

$$H_0 = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}|\vec{\nabla}\phi^2| + \frac{1}{2}m^2\phi^2\right)$$

with m^2 the physical mass.

So, therefore, Hamiltonian could be $H = H_0 + \int d^3x\, U[\varphi(x); g_i]$, let us not put derivatives for the time, keep it simpler, take this kind of a Hamiltonian density. So, it could be like this and you would contain some couplings g_i and we also assume that there are no bound states, well if they are then we will kind of leave them out ok.

So, if they have the reasoning does not collapse, so subtract them for the time being. So, so long as the existence of bounds states is not going to change the physical mass of the particles this assumption is ok. So, the point is that we assume that the full Hamiltonian has a spectrum like this.

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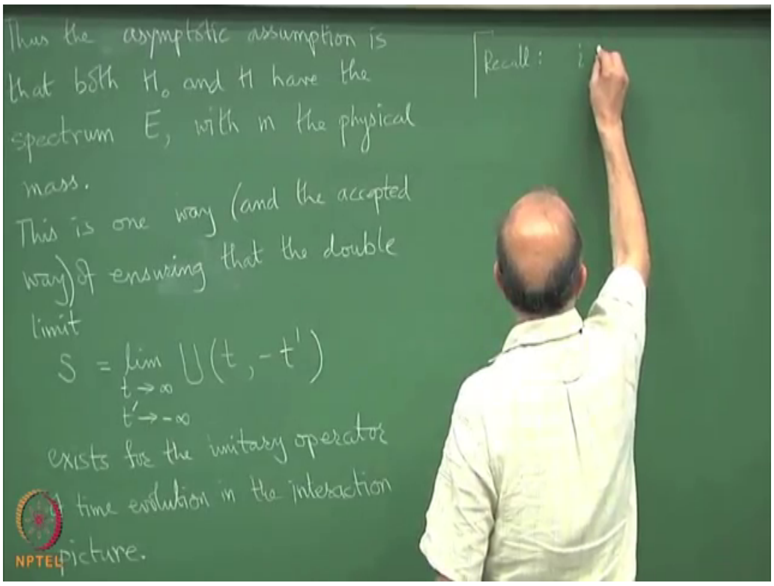

So, this is required or this is one way of ensuring and the accepted way of ensuring that the double limit S exist for the unitary operator of time evolution. So, the interaction picture is defined by this making the split H_0 and H_I and you know how it works.

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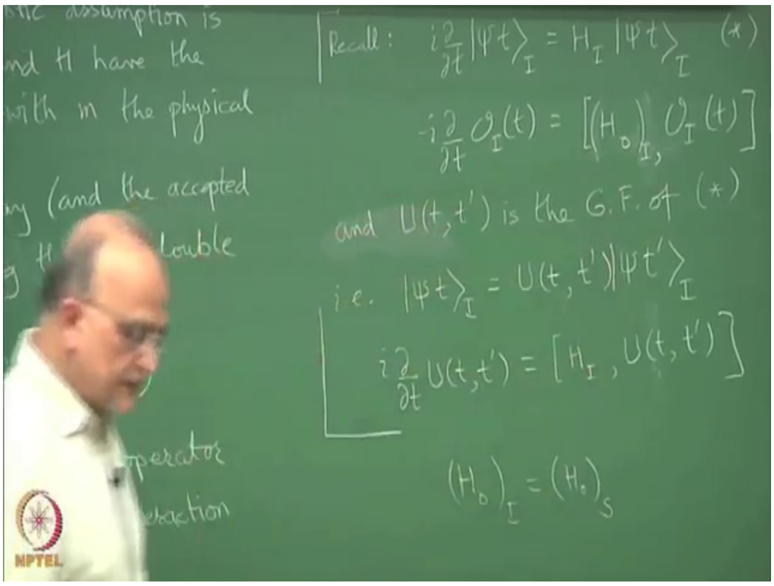

So, quick recall that in the interaction picture $i(\partial/\partial t)|\psi\ t>_I = H_I|\psi\ t>_I$ and

$$-i\frac{\partial}{\partial t}O_I(t)=[(H_0)_I, O_I(t)] \quad .$$

So, H interaction in the interaction picture is same as H_0 in Schrodinger picture. So, the point is this is how you define the interaction picture and then because of this the time evolution and U is the Green's function of this equation.

So, you want to write $|\psi\ t>_I = U(t, t')|\psi\ t'>_I$. It is something that just propagates it from that time to this time and therefore, satisfies the differential equation

$$-i\frac{\partial}{\partial t}U(t,t')=[(H_0)_I, U(t,t')]$$

and this has opposite sign of the general rule for the operators.

So, U is the unitary operator that implements time evolution in the interaction picture and the S matrix is defined as the limit when you take t and t' to go to $+\infty$ and $-\infty$. So, the point is that the region where you can set $H_I= 0$ becomes smaller and smaller as you go further and further away, and this limit has to exist and the limit exist provided you assume that the spectrum remains the same and that the H_0 assumes the same spectrum as H as you go to $+\infty$ and $-\infty$.

So, this is the sort of nuts and bolts thinking behind making these provisions. So, this is end of the recall. Now with all this last time we saw that the field in the interaction region will have more content than the field in the asymptotic region. So, the thing is the other important theorem of field theory is that we argued that the $\varphi_{in}(x)$ at $t = -\infty$ (and $\varphi_{out}(x)$ at $t = +\infty$) create only one particle states. So, we say $<1|\varphi_{in}(x)|0> = 1$. So, the creation operator from this will create one particle out of this, its overlap with this will be 1 and then there is the delta function which will give you 1. On the other hand for φ at finite times in the presence of full Hamiltonian to have more content than this and therefore, we expect that the free one particle state is less than 1. So, we set and this is where we introduced the wave function renormalisation. So, you have probably done renormalisation or are going to do. So, this already sets the story for what has to happen; in the free Hamiltonian we have a mass parameter and we have these couplings. So, when we write this Hamiltonian formally we have so called unrenormalized masses and couplings.

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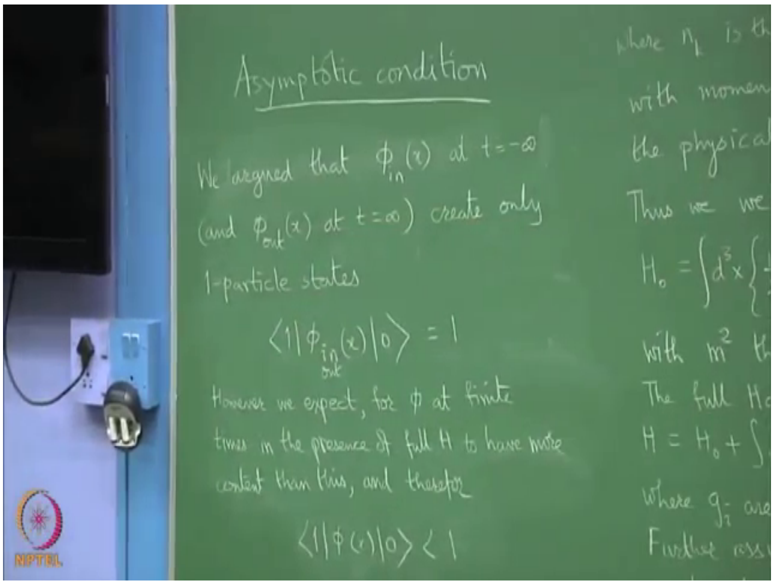

But eventually after you take account of interactions within the S matrix formalism, you have to ignore some of the diagrams and renormalize the couplings. The mildest kind of renormalisation that is required is to change the normalisation of the field itself and that is already seen at this level you do not have to do any diagrammatic calculations to see that you actually need to renormalize the wave functions.

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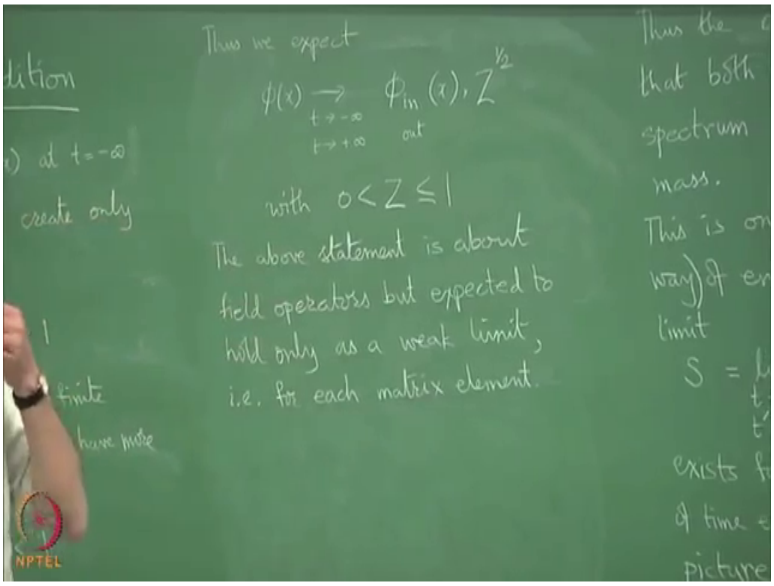

So, thus we expect that at t = +∞ and -∞, $\varphi(x) \rightarrow \varphi_{in/out}(x) Z^{1/2}$. And that Z we expect same for in and out because after all the evolution is unitary. So, the normalisation from this end to that end does not change. So, Z is the same in both in and out regions, but this is what we expect because Z is the number between 0 and 1. It becomes 1 if it is free field theory and further the caveat is that the above is not an operator, although this is about quantised fields.

So, when we withdraw from making the full operator assumption to only for matrix elements, what it means is that if you take higher powers of this statement then they may not hold because then you have to do matrix multiplication of and you get a more complicated answer. So, do not expect that $\varphi^2(x) \rightarrow \varphi_{in}{}^2(x)Z$, but φ itself we expect this to happen.

Because the φ^2 operator then has intermediate contributions from various other states. So, it may not hold. So, now I want to show a slightly intricate derivation I hope that I get it all right.

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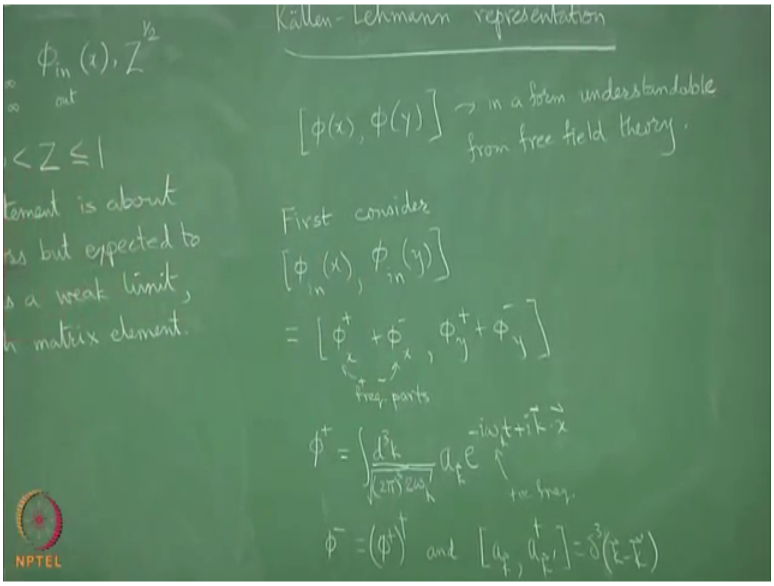

So, this is called Kallen-Lehmann representation. So, what is the representation is about? So, the representation is about the $[\varphi(x), \varphi(y)]$ in a form understandable from free field theory, this is what the representation does. It rewrites the general interacting field commutater in terms of expressions that refer to free field quantities. First begin with $[\varphi_{in}(x), \varphi_{in}(y)]$, φ_{in} could be out also, but let us just do φ_{in}. consider this commutator. So, this is not time order product, this is genuine $[\varphi(x), \varphi(y)]$ which you know has to be

0 at equal time, the canonical commutation relations say that this has to vanish at t=0 , but it is for general arguments x and y.

So, we have to first compute this. So, the easiest way to do it is to say I will split it up into φ_{in}^+, φ_{in}^-. So, we will drop the in for the time being. So, $[\varphi_x^+ + \varphi_x^-, \varphi_y^+ + \varphi_y^-]$ where these are the positive and negative frequency parts. So, they contain a and the negative frequency part contains a^+. Just since we have not done field theory fully together, let me just say specifically in my normalisation

$$\phi^+ = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} a_k \exp[-i\omega_k t + i\vec{k}.\vec{x}]$$

and φ^- is the dagger of this.

So, plus just means positive frequency and the -i occurs because of the Schrodinger choice of +id/dt as the energy. So, this is what we mean by the positive frequency part and our convention is that

$$[a_k, a_{k'}^+] = \delta^3(\vec{k} - \vec{k}') \quad .$$

So, in these commutations this with this will give 0 because they both contain annihilation operator.

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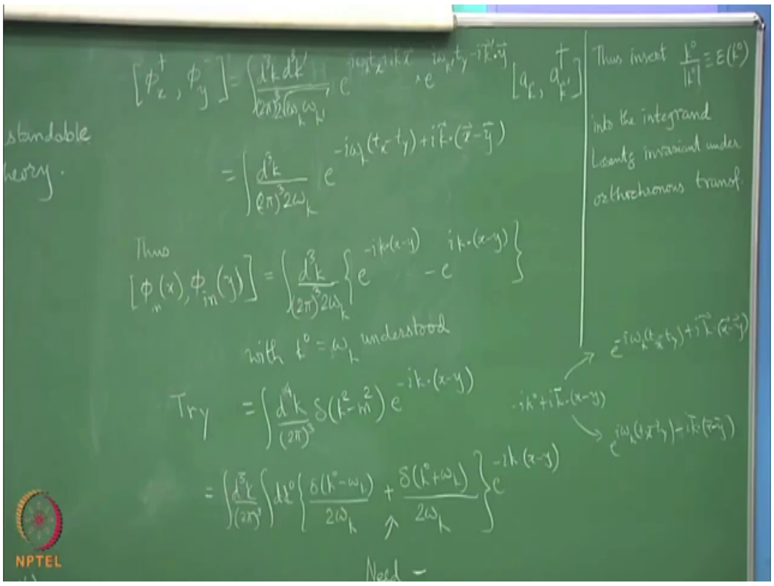

So, we have to worry about $[\varphi_x^+, \varphi_y^-]$ which will be equal to

$$\int \frac{d^3k\, d^3k'}{(2\pi)^3 2\sqrt{\omega_k \omega_{k'}}} \exp[-i\omega_k t_x + i\vec{k}.\vec{x}] \exp[i\omega_{k'} t_y + i\vec{k}'.\vec{y}][a_k, a_{K'}^+]$$

$$\int \frac{d^3k}{(2\pi)^3 2\omega_k} \exp[-i\omega_k (t_x - t_y) + i\vec{k}.(\vec{x} - \vec{y})]$$

So, putting it altogether we will see

$$[\phi_i(x), \phi_i(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \exp[-ik.(x-y) + ik.(x-y)] \quad .$$

So, now, this looks a little awkward you cannot take much sense of it, but you can be clever and rewrite it as a try

$$\int \frac{d^4k}{(2\pi)^3} \delta(k^2 - m^2) \exp[-ik(x-y)] \quad .$$

If you do this then basically this sign will come out wrong. So, let us just check this. So, this is equal to

$$\int \frac{d^3k}{(2\pi)^3} \int dk^0 [\frac{\delta(k^0 - \omega_k)}{2\omega_k} + \frac{\delta(k^0 + \omega_k)}{2\omega_k}] \quad .$$

If you really followed this true, you actually recover each term correctly because all you have to do now is to do the dk^0 integral, set $k^0 = \omega_k$, here which will give you this and in the second term $k^0 = -\omega_k$. So, it will give correctly this with the k^0 understood as $-\omega_k$ and the d^3k signs can always be flipped; you know if it here it will be $-ik^0$+ik.(x-y). When it becomes exp[-i-$\omega_k(t_x-t_y)$ + ik.(x-y)] we are fine, but when it becomes exp[i-$\omega_k(t_x-t_y)$ + ik.(x-y)] you are still stuck with +ik.(x-y), but that can be flip to minus without costing anything because this d^3k can be flipped and the Jacobian of going from k to -k is 1. So, it does not change anything. So, that can be reversed and so, you recover this term as well as this term, but the sign comes out wrong. So, to take care of that and the whole exercise now is if we go towards this then we have a Lorentz invariant expression.

See so far we were just commuting and we were getting something a little unclear, but once you cast it in this and because of the d^3k integral, it is it looks non covariant, but now you have d^4k integral at delta function that involves only a Lorentz invariant product, I mean magnitude of k and this is a Lorentz invariant inner product. So, the whole function becomes manifestly Lorentz covariant except that this sign cannot wrong. So, we need minus sign and to cure that we insert a function $k^0/|k|$ into this.

So, and this you might call $\varepsilon(k^0)$ which is also invariant under allowable Lorentz transformation. So long as you do any normal Lorentz transformation, a axis that is positive time axis will remain positive time axis. So, this is Lorentz invariant under orthochronus transformation, which are the connected set. We have disconnected set if you take the full invariance group of the Minkowski inner product, but the connected one that you use physically is the so called orthochronous one.