Path Integral and Functional Methods in Quantum Field Theory
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Lecture – 17
Asymptotic Theory – I

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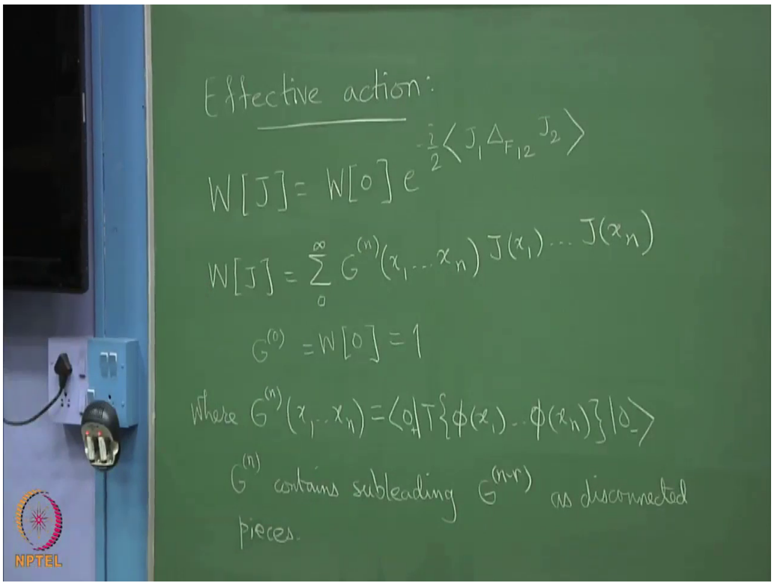

So, what we had was

$$W[J]=W[o]e^{-\frac{i}{2}\langle J_1 \Delta_{F_{12}} J_2\rangle}$$

and then there were two things; one is that this W generates all the possible Green's functions. So,

$$W[J]=\int \sum_0^{\infty} \frac{i^n}{n!} G^{(n)}(x_1, x_{2,} ....., x_n) J(x_1), J(x_2), ..., J(x_n) d^4 x_1 ...... d^4 x_n \quad .$$

And the zeroth order is 1, $G^{(0)} = 1$ which is some kind of initial condition. I think that is consistent because if I set n = 0 then none of these are there and I have this, but it has no argument at that point and J is also equal to 0. So, it is what we have in front. Let us just the convention where the $G^{(n)}$ are the time ordered n point functions and technical it is 0_+ and $0\_$ signs. Now here argued that this is slightly too much information is not packaged very well because this $G^{(n)}$ contains disconnected pieces as well. So, ideally we want to define connected Green's functions.

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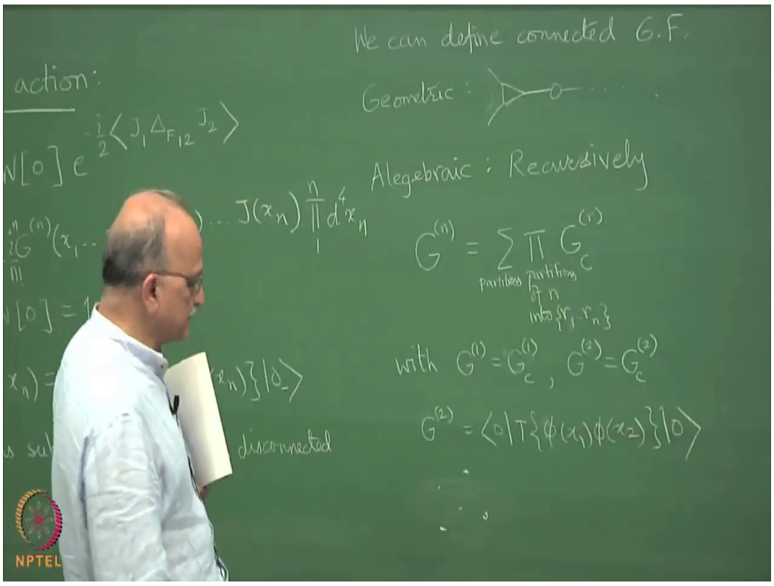

You can do it two ways geometrically which we are drawn last time. It is sort of obvious what connected means in this simple pictorial terms and last time we are drawn three diagram small because we wanted to get it what is one particle irreducible which is a little more advanced idea not concerned which it right now. So, geometrically it is clear what it means, but algebraically what you do is define it recursively that is you say that n’s level normal Green's function is

$$G^{(n)} = \sum_{partitions} \prod_{partitions\ of\ n\ into\ r_1,\ldots.r_n} G_c^{(r)} \quad .$$

And we claim that the 2-point function is the first connected Green’s function with $G^{(1)} = G^{(1)}_c$, $G^{(2)} = G^{(2)}_c$. So, $G^{(2)}$ as we know is this the 2 point function, the Feynman propagator and is a connected Green's function. So, diagrammatically we draw it simply as a line and that is certainly connected. Then you can easily see that you can recursively build up because at 3-point function would have to be a product of 2-point functions times a point or it would have to be a graph like this and then so on.

So when you go to higher you would have a set of points for n-point function, you would have some number of points and then you have to see what are the ways of connecting them and n point function is said there it will be product of different ways of partitioning the points and drawing connected graphs among them and what does that mean it means that one has to be made up of lower order connected graphs and so on. So, once you are define the 2-point function you are fine ok. So, you can recursively define like this and now this is what I am not proved so far.

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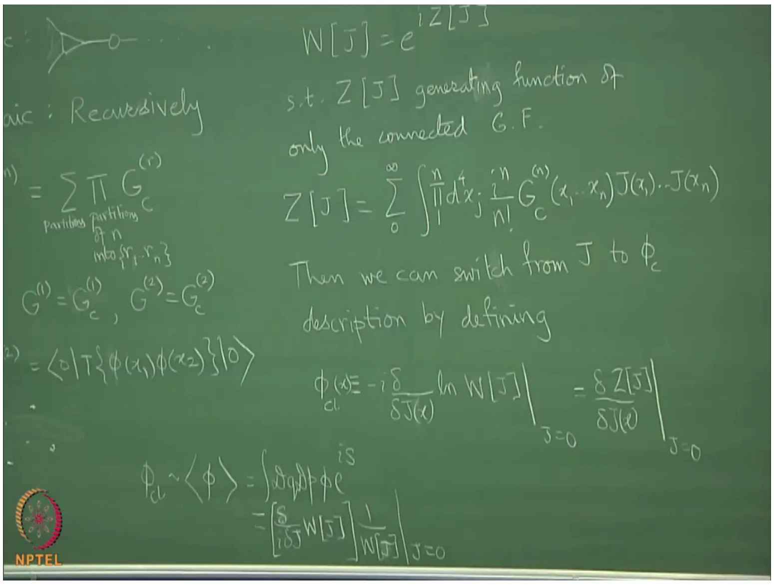

And I actually just meant to summarize what we have done so far is that then we can prove that and here the definitions differ a little bit here. So, here we use W[J] = $e^{iZ[J]}$ such that Z[J] actually generates only the connected Green's functions rather it is generating function of the connected Green’s function,

$$Z[J]=\sum_{0}^{\infty}\int\prod_{1}^{n}d^4x_j\frac{i^n}{n!}G_{(c)}^{(n)}(x_1,x_{2,}.....,x_n)J(x_1),J(x_2),.....,J(x_n) \quad .$$

So, after this the trick we used was to do this Legendre transform. So, we were then looking for a way of having a functional which is functional of a space time field not a auxiliary current. So, we wanted it in terms of more physical entities rather than some auxiliary entities. So, now, the story of what is effective action starts after the this background. Then we can switch from J to φ_c description by defining φ_c to be variation of the Z.

So, intuitively we want to think of φ_c simply as equal to the classical this c is not that c and. So, that to do with connected, but this is just classical. So, φ_{cl} should basically be

$$\phi_{cl} \sim \langle \phi \rangle = \int Dq\, Dp\, \phi\, e^{iS} = \left(\frac{\delta}{i\delta J} W[J]\right)\left(\frac{1}{W[J]}\right)_{J=0} \quad .$$

So,

$$\phi_{cl}(x) = \left(\frac{\delta}{i\,\delta J(x)} \ln W[J]\right)_{J=0} = \left[\frac{\delta Z[J]}{\delta J(x)}\right]_{J=0} \quad .$$

So, we define the desired φ_{cl} and this is a trick that works because φ is essentially like an extensive variable. So, you can do a Legendre transform. So, of course, it comes out correct, but why you would think of Legendre transform I do not know. So, this trick is essentially due to I think man call Jona-Lasinio who was I think showing there is student or collaborator did some clever things is not very well known outside, but once you read the literature you will find them.

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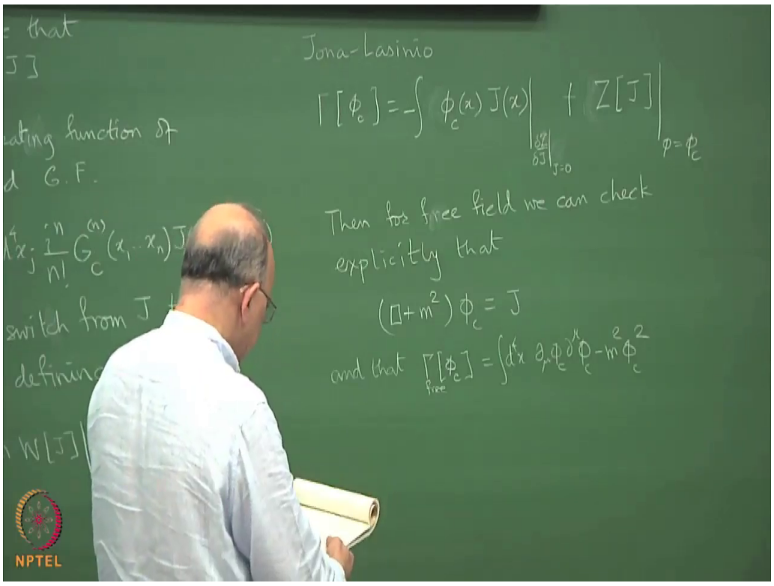

So, one of them is this and then define

$$\Gamma[\phi_c] = -\left(\int \phi_c(x) J(x)\right)_{\left[\frac{\delta Z}{\delta J}\right]_{J=0}} + Z[J]_{\phi=\phi_c} \quad .$$

Then we can check for the free field case very easily that the $\Gamma[0]$ turns out to be nothing, but the free field that $(\Box + m^2)\,\varphi_c = J$ and

$$\Gamma_{free}[\phi_c]=\int d^4x(\partial^{\mu}\phi_c\partial_{\mu}\phi_c-m^2\phi_c^2) \quad .$$

So, that is rather reassuring and it generates the interpretation that the quantum gamma the quantum action is going to be the classical one plus some quantum collections.

Now, you may say this looks rather to simple how is this going to work when I have more complicated situations and the answer is that in the presence of a potential we play further tricks.

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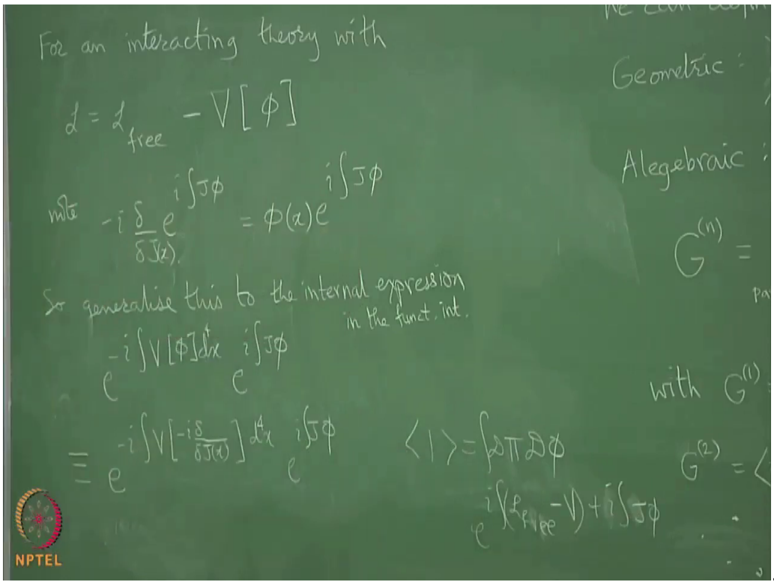

So far in interacting theory L = L_{free} -V[φ] so long as the V is a functional only of φ and not the derivatives, we can write that

$$-i\frac{\delta}{\delta J(x)}\exp[i\int J\phi d^4x]=\phi(x)\exp[i\int J\phi d^4x] \quad .$$

So we generalize this to the integral expression in the functional integration

$$\exp[-i\int V[\phi]d^4x]\exp[i\int J\phi d^4x] \quad .$$

So, if I had a φ here what I did was to hit here by $e^{-id/dJ}$. If I now have a whole thing like this all I say is that

$$\exp[-i\int V[-i\frac{\delta}{\delta J(x)}]d^4x]\exp[i\int J\phi d^4x] \quad .$$

So, this is the big formal step. So, what we are trying to do is I have the full path integral which is $\int D\Pi \, D\varphi \, \exp[i\int (L_{free} - V + J\varphi)\, d^4x]$. Now, I know how to take care of L_{free} along with $J\varphi$, but I do not know what to do with this V. So, what I do is I rewrite as this operator acting on what is going to come there.

But now this is this does not involve φ any more it only involves J. So, I pull this out completely of the path integral and I say that therefore

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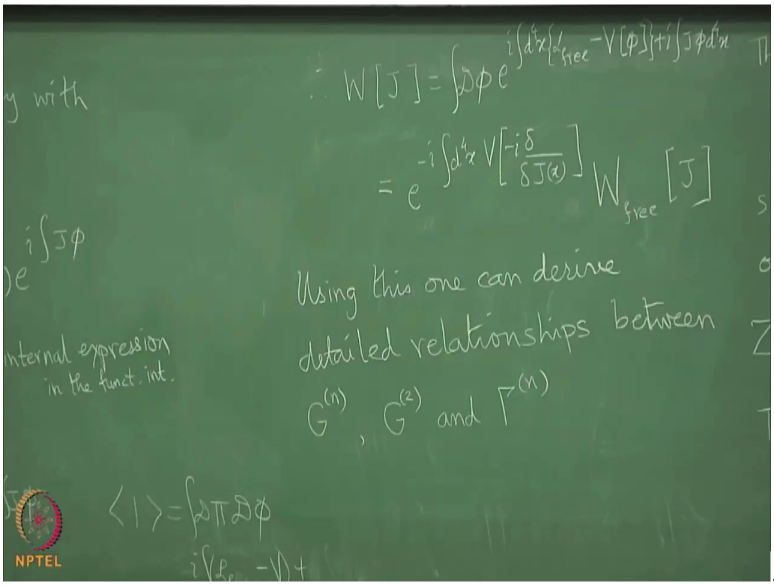

$$W[J] = \int D\phi \exp[i\int d^4x\,(L_{free} - V[\phi])] \exp[i\int J\phi\, d^4x]$$

$$So, W[J] = \exp[-i\int V[-i\frac{\delta}{\delta J(x)}]\, d^4x]\, W_{free}[J] \quad .$$

So, this is just a clever trick for writing this out to satisfy oneself that one can formally write something. In practice it will work because the V will be usually just φ^4 or something like this and it will be a it is a monomial of some kind.

So, you can always use it to extract Green's functions up to a particular order and they will involve not just 2 point functions, you remember in free field theory the only thing we got out was 2 point functions or their products. So, here you will begin to get more things out and I did intent to do it, but somehow I want to do it next time ok.

So, you can use this to derive detailed relationship between $G^{(n)}$ and $G^{(2)}$ and $\Gamma^{(n)}$. Now this may look a little bit of what does this mean. The answer is that actually the n point function, $G^{(n)}$.

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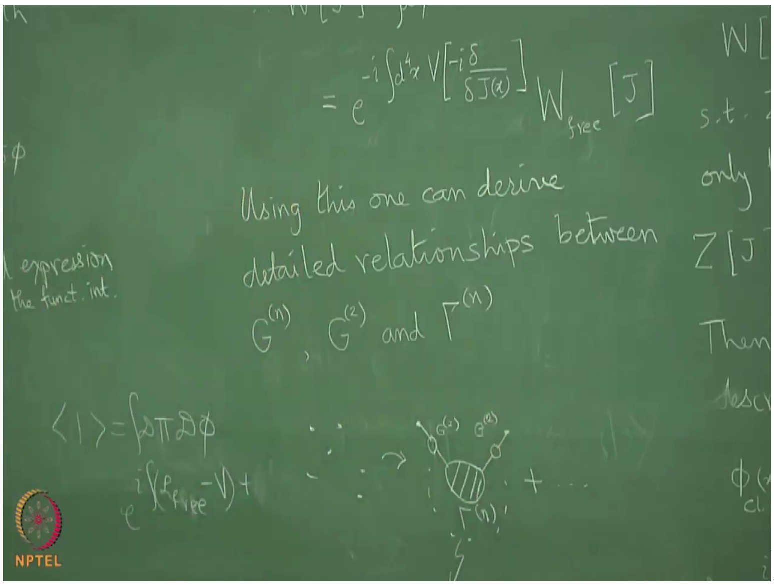

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So, it has some n points right that is what $G^{(n)}$ means, but this thing when it is connected boils down to lines like this with $G^{(2)}$ is inserted, but then and irreducible piece which will be $\Gamma^{(n)}$ and plus lower order terms ok.

So, what one finds is that the n point function as it can contain the lower pieces which are products of lower connected parts at the nth level you will find that the most non trivial term has the structure of a vertex, some block which you cannot further resolve a one particle irreducible diagram. So, this will be a times lines with the $G^{(2)}$ insertions on them. So, this is how the particle and vertex interpretation emerges the fully interacting quantum field theory.

So, long you still use perturbation theory because after all we pretended that you could treat V after you do the free field think. So, you have to think of this as in some sense small otherwise these manipulations do not mean that much, but to the extent that you can try to isolate the effects of V, those effects will can be broken up in to an irreducible n point function times just 2 point functions. So, that is the idea of defining the effective action. So, that these are the physical meanings of the various quantities that we have been introducing.

And $\Gamma^{(n)}$ is what will enter into the expansion of the Γ in terms of it is argument $\varphi^{(n)}$ just like the GJ where the $\Gamma^{(n)}$ are

$$\Gamma[\phi_c]=\sum_n \frac{1}{n!}\int \Gamma^{(n)}(x_1,....,x_n)\,\phi_c(x_1,....,x_n)\prod_1^n d^4x_j \quad .$$

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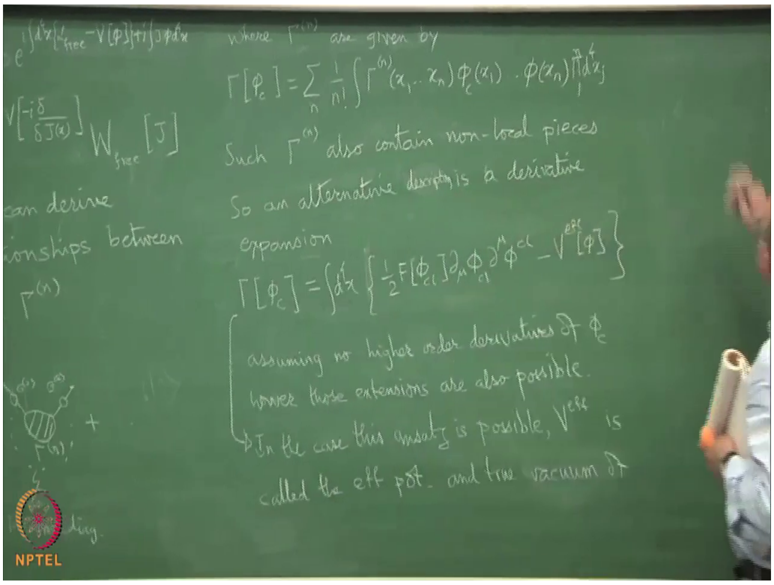

So, if you expand the Γ I like that then those $\Gamma^{(n)}$'s have the significance of being the highest order non reducible part in the n point Green's function. Final comment is that this Γ there are now derivatives appearing here now of φ. So, it is functional only of φ, but we do know that in fact the free part will contained in $\partial_\mu\varphi\partial^\mu\varphi$.

So, there is an alternative expansion which is a derivative expansion. So, such $\Gamma^{(n)}$ also contain nonlocal pieces. So, you see it is just $\Gamma(n)\,(x_1,....,x_n)$ times a product of fields, but which are at different points. So, if it was a local field theory. So, what I mean by this functional if it was a local functional then there should be a $\int(d^4x_1,....,d^4x_n)$. So, that everything collapses to same point, but the general one is not like that; obviously, because there are derivatives.

So, an alternative definition is a derivative expansion or alternative description rather not definition where we anticipate our traditional free field and simply say it is equal to

$$\Gamma[\phi_c]=\int d^4x\left(\frac{1}{2}F[\phi_{Cl}]\partial_\mu\phi_{Cl}\partial^\mu\phi_{Cl}-V^{eff}[\phi_{Cl}]\right)$$

assuming no higher order derivatives of φ_c; however, that extension is also possible.

Here by proposing that the only derivative terms I have are $\partial_\mu\varphi\partial^\mu\varphi$ up to some functional which is function only of φ, but not of derivatives and whatever remains I call V^{eff}. This is a local expression. It is powers of φ at the same point x, there is only one x integration now ok. So, this is a local theory because everything in it is local to one point x. In the case this ansatz is possible V^{eff} is called the effective potential and it's minima give you the true vacuum expectation value of φ

$$\left[\frac{\delta V}{\delta \phi_c}\right]_{\phi_i}=0,\left[\frac{\delta^2 V}{\delta \phi_c^2}\right]_{\phi_i}>0 \quad .$$

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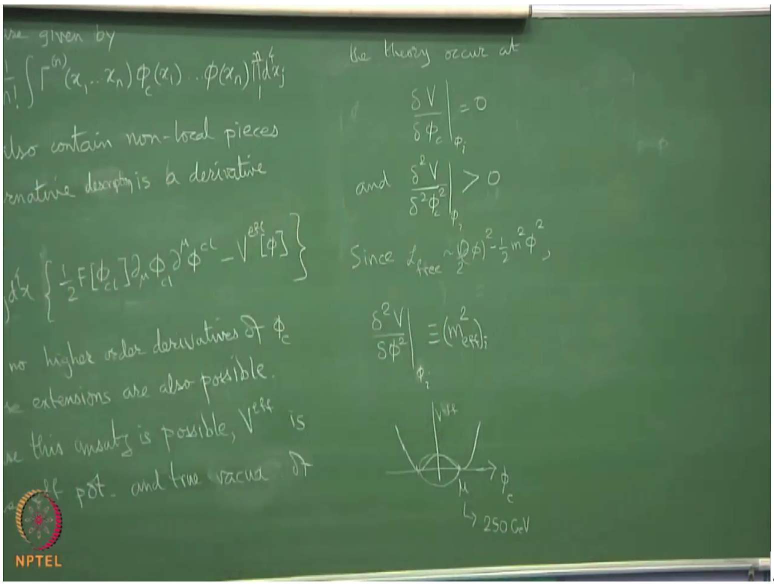

So, there could be several such φ_i and the $d^2V/d\varphi^2$ essentially as the interpretation of the mass.

So, since $L_{free} = \frac{1}{2} (\partial\varphi)^2 - \frac{1}{2} m^2\varphi^2$,

$$\left[\frac{\delta^2 V}{\delta \phi^2}\right]_{\phi_i} \equiv \left(m_{eff}^2\right)_i \quad .$$

If you have complex field then you will not have a ½ factor what you have to do take care of that ok. So, this is what we mean by our symmetry breaking. This is where we cannot explain to general public what is a Higg's mechanism all about.

So, now, you really know what the Higg's vacuum means. So far you are just varying V which was not the V effective, but luckily because the theory is renormalizable polynomial remains the same, but the coefficient get quantum correction. So, they are not just λ/m^2, the minima, but including quantum corrections so, but this is the real meaning of what symmetry breaking is. If the effective potential develops is then it is there.