Path Integral and Functional Methods in Quantum Field Theory

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Lecture – 15
Effective Potential – III

So, let us begin with a little bit of recapitulation of what we have been doing. What we have done is defined the external force method.

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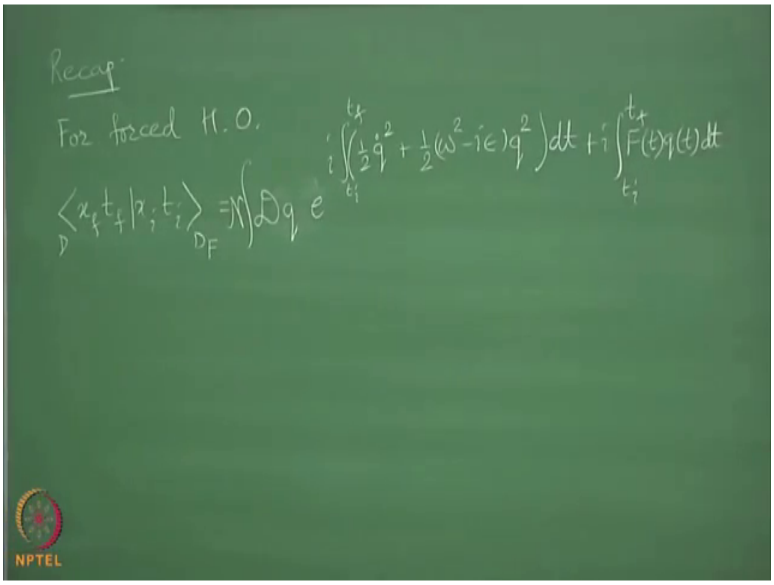

So, for the case of harmonic oscillator we defined in the Dirac picture to be equal to integral well actually we did only. So, we did configuration space. This would be the usual Lagrangian and to this we add

$$\langle x_f t_f | x_i t_i \rangle_D = N \int D\phi \, e^{i\int_{t_i}^{t_f}\left(\frac{1}{2}\dot{q}^2+\frac{1}{2}(\omega^2-i\epsilon)q^2\right)dt+\int_{t_i}^{t_f}F(t)q(t)dt}$$

So, this introduction of F is essentially a trick in the presence of F it looks like this because q's are going to get integrated out. So, there will be no functional dependence left. So, J is the bookkeeping function that will keep track of what is happening and when we do that then we do one more modification which is that change to vacuum to vacuum.

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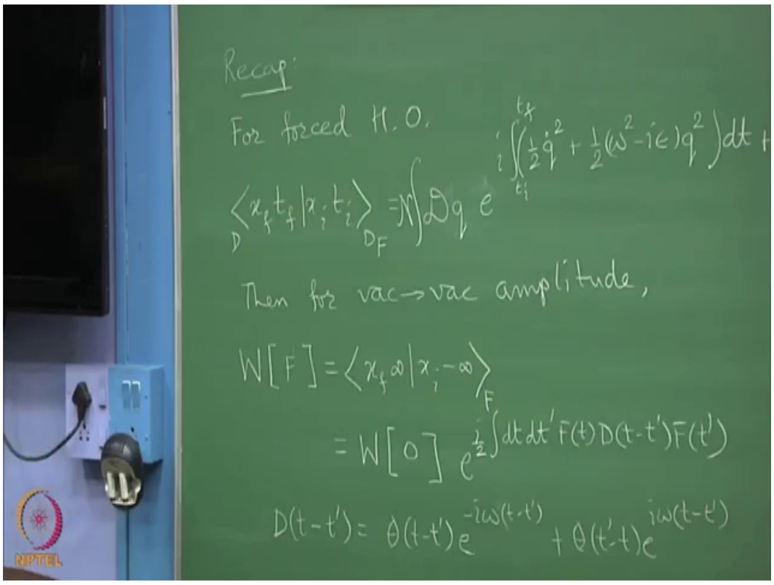

So, I will just keep two things together and then for vacuum to vacuum amplitudes which we call

$$W[F] = \langle x_f \infty | x_i - \infty \rangle_F = W[0] e^{-\frac{i}{2}\int dt dt' F(t) D(t-t') F(t')}$$

where D becomes the Feynman propagator is equal to this language what is it

$$D(t-t') = \theta(t-t') e^{-i\omega(t-t')} + \theta(t'-t) e^{i\omega(t-t')}$$

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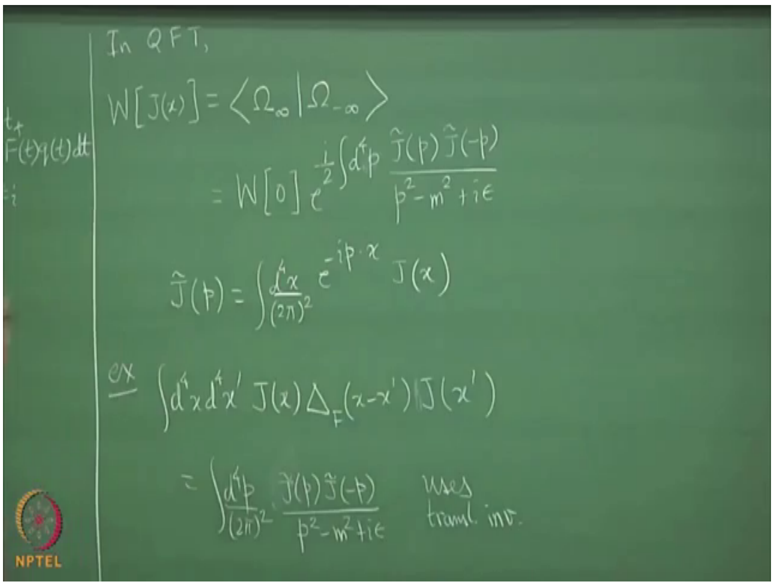

Now, this whole thing just carries over to field theory where instead of t we have x, y, z and t. So, in QFT we will have W[J] where now J is function of x which is the so called vacuum to vacuum amplitude and that has the form

$$W[J(x)]=\langle\Omega_{\infty}|\Omega_{-\infty}\rangle=W[0]e^{-\frac{i}{2}\int d^4p\frac{\tilde{J}(p)\tilde{J}(-p)}{p^2+m^2+i\epsilon}}$$

with $\tilde{J}(p)=\int\frac{d^4x}{(2\pi)^2}e^{-ip.x}J(x)$. So, this is very similar to this. Check that

$$\int d^4x\,d^4x'\,J(x)\Delta_F(x-x')J(x')=\int\frac{d^4p}{(2\pi)^2}\frac{\tilde{J}(p)\tilde{J}(-p)}{p^2+m^2+i\epsilon}$$

So, that is same form as this just that it is converted to this. The important property we use is that the Feynman propagator is function only of the difference of the arguments. So, there is translation in variance.

So, in 2 x; x is not really required one p is enough in the momentum space ok. So, there is a delta function. So, uses translation invariance. So, this is the sort of starting point of the next part and then we had said that any Green’s function can be defined.

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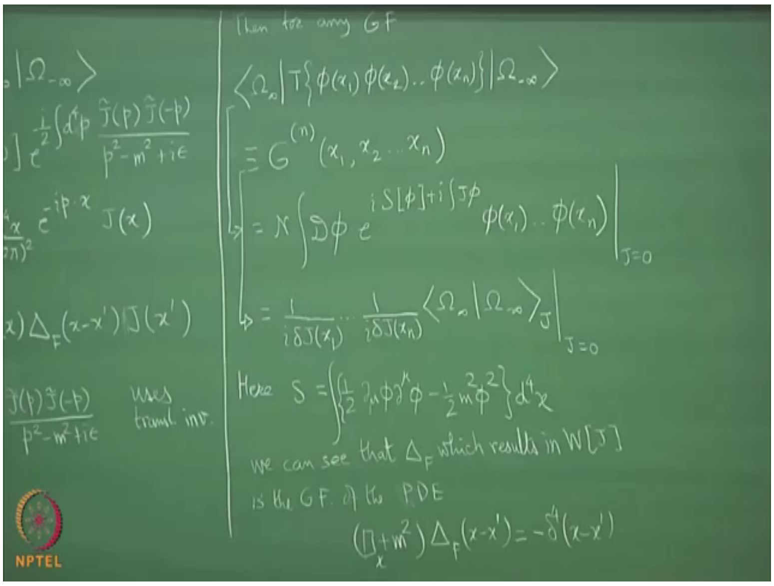

So, which is actually. So, Ramond does not somehow write this T product definition, but I think that is what it has to be. So,

$$G^{(n)}(x_1,...,x_n)=[\frac{\delta}{i\delta J(x_1)}....\frac{\delta}{i\delta J(x_n)}\langle\Omega_{\infty}|\Omega_{-\infty}\rangle]_{J=0}$$

So, this is actually same as this.

This you know, this is the definition if you like we know that if in the path integral if you insert certain operators you will automatically get the expectation value of the T product between this end points, but that is what we are calling Green's function now. But, clearly this can then be got by using this trick by using this expression because all we have to do is vary with respect to i times J.

So, after doing this one evaluates it at J equal to 0. So, this is J, but evaluated at J equal to 0. This amplitude will not contain any ϕ's. This amplitude is this right it does not contain any ϕ because ϕ got integrated out, but there is a J dependence. So, if I vary this with respect to J it will bring down a $\phi(x_1)$ it will bring down a ϕ. So, I can populate this numerator by the required operators by acting back d/dJ on this right, very standard trick for generating functions. And so, this is what we finally, say is our Green's function $G^{(n)}$ is this.

So, this is how far we got and then we want to. So, we check then that this Δ_F emerges more or less automatically. If you start with the free theory or the free just has we started here with the harmonic oscillator which serves as a emblem for what is the word it is emblematic of the somewhere where we wrote S right. So, it is a free propagator. So, that looks very much like $S=\int[\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi-\frac{1}{2}m^2\phi^2]d^4x$.

So, this is what the free action is and from that we generate this Δ_F which we can checks satisfies which works as Green's function of the classical differential equation. Here Green's function in the traditional sense, no quantum mechanics just in the partial differential equation. Green's function of the PDE and by the conventions

$(\Box_x+m^2)\Delta_F(x,x') = -\delta^4(x-x')$. So, this can also be checked. So, you can just see that is what that is exactly what you get, so, far so good.

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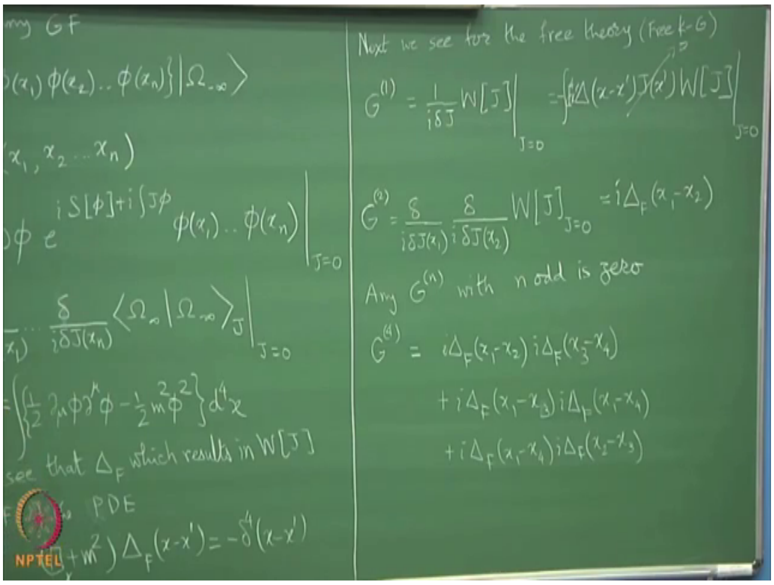

Now, we can do a very simple calculation in this particular case. So, for the free theory there are no interaction just the Klein-Gordon is that if I calculate $G^{(2)}$. So, $G^{(1)}$ will be 0 right because if I vary with respect to G what happened J. So,

$$G^{(1)}=[\frac{\delta}{i\delta J}W[J]]_{J=0}=-[\int d^4x'\Delta(x-x')J(x')W[J]]_{J=0}=0$$

So, we are left with because this is in the exponent. This whole thing is in the exponent. If you vary this W[J] with respect to J.

So, our next hope is that we calculate $G^{(2)} = i\boldsymbol{\Delta}_F (x_1 - x_2)$. So, now we calculate the $G^{(4)}$ by the same method. So, we can see that $G^{(3)}$ has to be 0, any odd G.

Now, the so, okay the point of doing this exercise is that we will begin to see the particle interpretation of this big formalism we have plugged in a field ϕ into the action, but what we come up with where did we write the Lagrangian. So, we are dealing with the field theory, but after we do this end point function we eventually come out with a particle like interpretation because we already has this $G^{(2)}$ which is the Feynman propagator which as you know from your Quantum-III has basically this kind of behavior, it propagates positive frequency particles forward negative frequency particles backwards.

So, one often draw this $G^{(2)}$ as a line joining the points x_1 and x_2 create a particle here destroyed there or describe antiparticle there and created it here. We will do it in a minute, but just to tell you in advance why we are doing this. So, we are doing this to see that the particle like interpretation emerges automatically from the Green's functions of this theory.

So, $G^{(4)}$ then is going to be you can see actually what happens. If I vary with vary 4 times the point is that when I vary with first two of the arguments I will get a Δ_F down when I vary with the next J these things cannot give any contribution. So, again more Δ's have to be brought down from the exponent of W. So, what will happen is that I will get,

$$G^{(4)} = i\Delta_F(x_1 - x_2)\, i\Delta_F(x_3 - x_4) + i\Delta_F(x_1 - x_3)\, i\Delta_F(x_2 - x_4) + i\Delta_F(x_1 - x_4)\, i\Delta_F(x_2 - x_3)$$

This is the only I mean this is the there are 2 other terms of this kind, but essentially you get pairwise products you cannot get anything new from the what comes down all the what comes down in the numerator or the main x line because you can vary it only twice and that just produces a Δ. The next nontrivial answers come when you vary again twice one more Δ comes down, except that all the orderings are possible.

So, since you are going to differentiate with respect to all the 4 and set J equal to 0 in the end temporarily you can get various combinations down. So, you will also have this. So, all the permutations happen this is happening because we have a free theory there are no interactions ok. So, now, complicated connections between these lines get generated, but what we do is we represent this diagrammatically.

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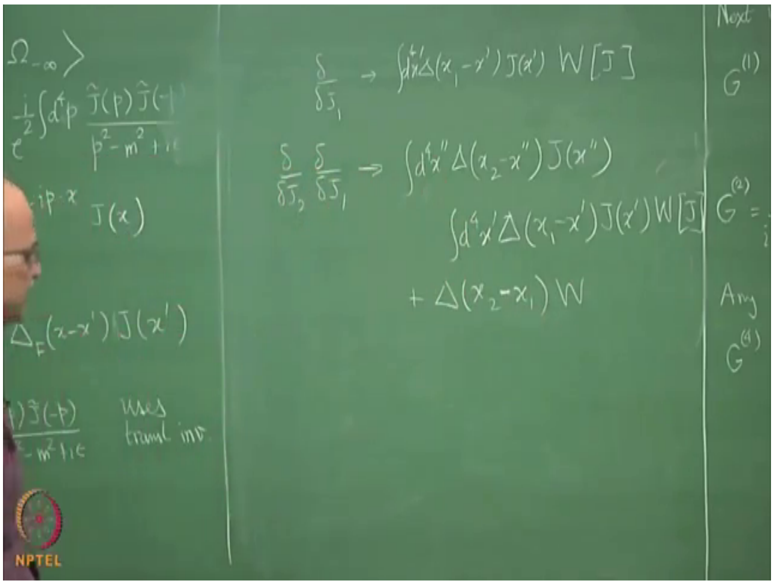

So, when you do

$$\frac{\delta}{\delta J_1} \rightarrow \int d^4 x' \Delta(x_1 - x') J(x') W[J]$$

$$\frac{\delta}{\delta J_2} \frac{\delta}{\delta J_1} \rightarrow \int d^4 x'' \Delta(x_2 - x'') J(x'') \int d^4 x' \Delta(x_1 - x') J(x') W[J] + \Delta(x_2 - x_1) W$$

But, now I am going to act with more J's. Now, those J's can remove this or this or bring down a more J and so on. So, in the end you end up with all the combinations like this. So, this is also an exercise. Just check this in detail, exercise 2.

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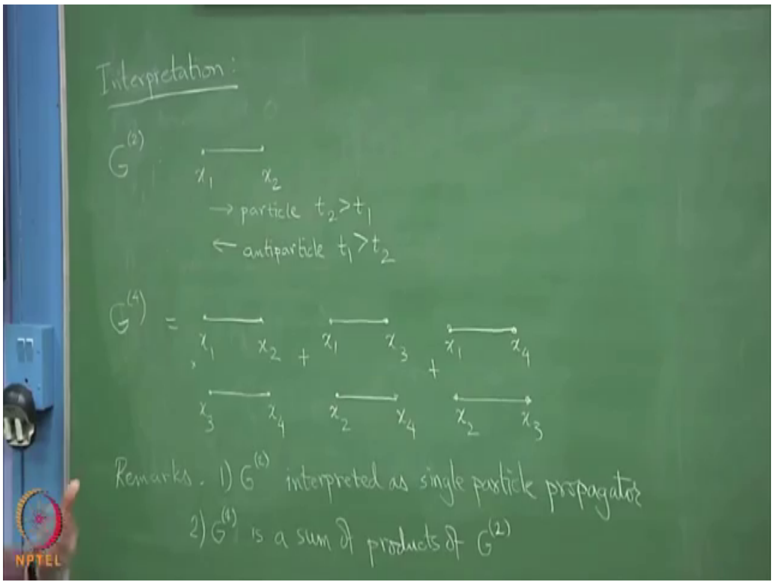

So, the $G^{(2)}$ you interpret simply as creation of a particle at x_1, I am going to x_2. So, I will write it like this particle with $t_2 > t_1$ or like this antiparticle if $t_2 < t_1$. So, that is the detailed interpretation of this I mean this is what the Feynman propagator actually gives and we think of it is a particle being created here and transported there or antiparticle created at t_1 and send to t_2 ok. Then the $G^{(4)}$ has interpretation that it is equal to x_1, x_2 times; so, there is a cross between the two x_3, x_4.

So, these is a amplitudes are just multiplied to each other where the two products of the 2 Δs plus sign. So, this plus sign in the sense of linear superposition. So, in quantum mechanics this four-point amplitude is a sum of 3 possible ways it can happen with equal weightage. So, either x_1 goes to x_2 and x_3 goes to x_4 or x_1 goes to x_3 and x_2 goes to x_4 or x_1 goes to x_4 and x_2 goes to x_3.

Now, there is one advanced lesson here which the book does not spend time explaining at this point, but he just drops the answer and I have also chosen to postpone it because we are not done interacting theory any interacting theory at. But, the point is that the fact that we come out with this kind of a combinatoric answer sum of various products of some more basic Green's functions. So, what has happened is this of course, we already saw. But, the more interesting remark is that $G^{(4)}$ is simply linear combination of products of $G^{(2)}$. In fact, in a free theory it is somewhat boring. Even if you do $G^{(56)}$, you will only get products of 2-point functions because there is nothing else that can happen.

So, due to this being a free theory all higher $G^{(2m)}$ will be sums of products of m $G^{(2)'}$s. It can be shown that the combinatorics of it works out correctly. So, in the more general case new irreducible vertices. However, we still get this products of the lower Green's functions anyway in the higher one.

In this redundancy of seeing the lower ordered ones appear in higher order ones can be removed by a very clever trick and that trick is that you look at that you take a log of this. The way these are the combinatoric factor comes out exactly correct so that so, let me write 3 a and 3 b.

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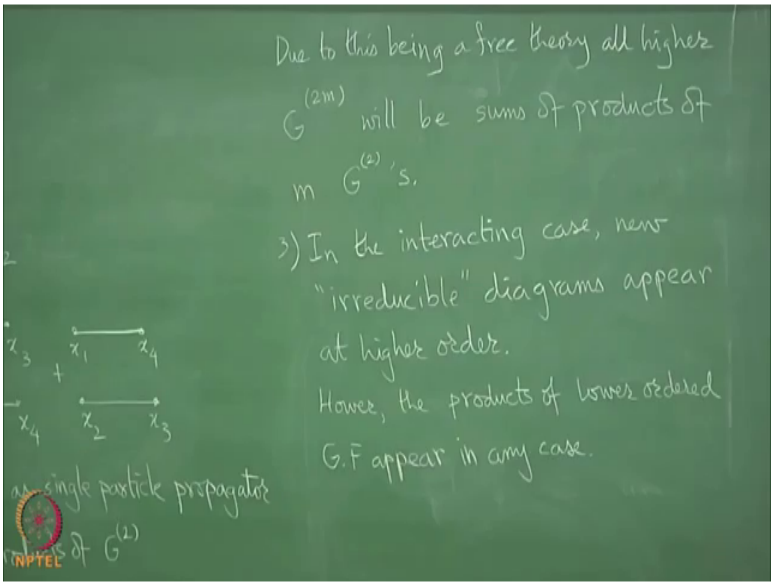

So, due to b, if you do this then because there is an exponent of $W[J] = e^{-iZ[J]}$ whatever is Z there will be various powers of the terms of Z which will get multiplied among each other to produce W in exactly the same way that the lower ordered Green's functions multiplied to produce higher order ones.

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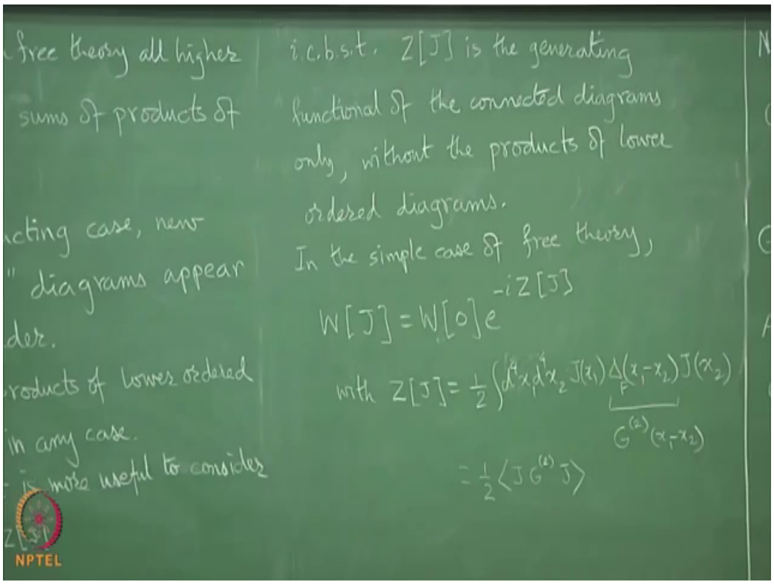

So, the it can be shown that so, this will be done in the future. So, it can be shown that Z[J] is the generating functional of the connected diagrams. So, this is one diagram. So, the this $G^{(4)}$ is sum of 3 diagrams, but this diagram is disconnected because it has some 2 parties not talking to the other two. It is like it has fallen into. So, this is a disconnected diagram this is disconnected. But, one can show that if you take; if you take the log of W which is this Z then you get only the connected once and when you exponentiate this connected diagram generating function then you will generate all the product pieces in exactly such a way as to get the W with products of lower ordered diagrams multiplying it. So, in our case it is somewhat obvious.

So, in the simple case so, free theory we already see this because

$$W[J]=W[0]e^{-iZ[J]}$$

with

$$Z[J]=\frac{1}{2}\int d^4x_1 d^4x_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)=\frac{1}{2}\int d^4x_1 d^4x_2 J(x_1)G^{(2)}(x_1-x_2)J(x_2)$$

and we know that exponentiating just $G^{(2)}$ is going to produce the W.

So, this log of W in the case of the free field theory is simply in simple language it is equal to one $Z[J]=\frac{1}{2}\langle JG^{(2)}J\rangle$ that is all there is and because it is a free theory there is only non trivial diagram you have and it is exponentiation will generate all the possible diagrams of W ok. But, when you have nontrivial physics going on then you will find newer G is at a higher level and that is the aim of my part of the this set of lectures to show that that is what we get, but we will go going through a few more things in between.

So, now we introduce the Green's function last time. One last comment about this is that there is a momentum space version of this business of connected and I should tell you that I am giving you only a preliminary discussion. This discussion goes better than get better and better as you spend more time.