Path Integral and Functional Methods in Quantum Field Theory
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Lecture – 14
Effective Potential – II

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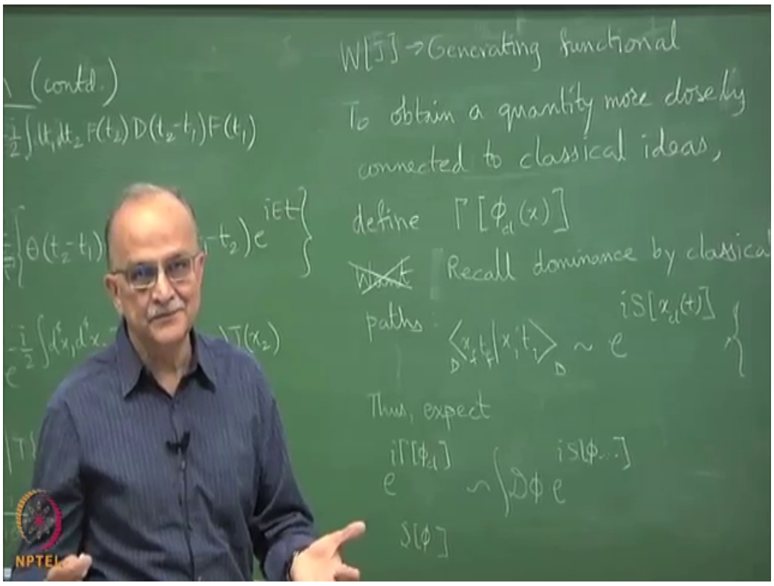

You will complain that we have lost an intuitive feeling about what is going on and what our next goal is to actually derive a different quantity which connects with classical ideas. We want to obtain something which we call $\Gamma[\phi_{cl}(x)]$.

Now, what we want to say is something like this, we want that. So, what do we mean by such an action? So, recall our dominance by classical paths.

So, recall dominance by classical paths where we found that the transition amplitude is e raise to i times, the action on the classical path right $\langle x_f t_f | x_i t_i \rangle_D \approx e^{iS[x_{cl}(t)]} \times \ldots$. In the classical limit this is what we found, if somehow the action was said that it was dominated by mass of a tennis ball or whatever then, its action would just come out and the answer would be just proportional to this.

So, we want here to define expect that when I do all this integration I should come out with $e^{i\Gamma[\phi_{cl}]} \approx \int D\phi \, e^{iS[\ldots]}$. So, there is integration here, but it gives this answer this is also how some of the thermodynamic potentials are defined Gibbs free energy, you have to average over dp, you do the phase space integral of the partition function, but then the left hand side gives the free energy in the exponent $e^{-\beta F[V]} = \int dx\, dp\, e^{-\beta H}$ ok. Well what remains is a its dependence on volume something this is all that remains.

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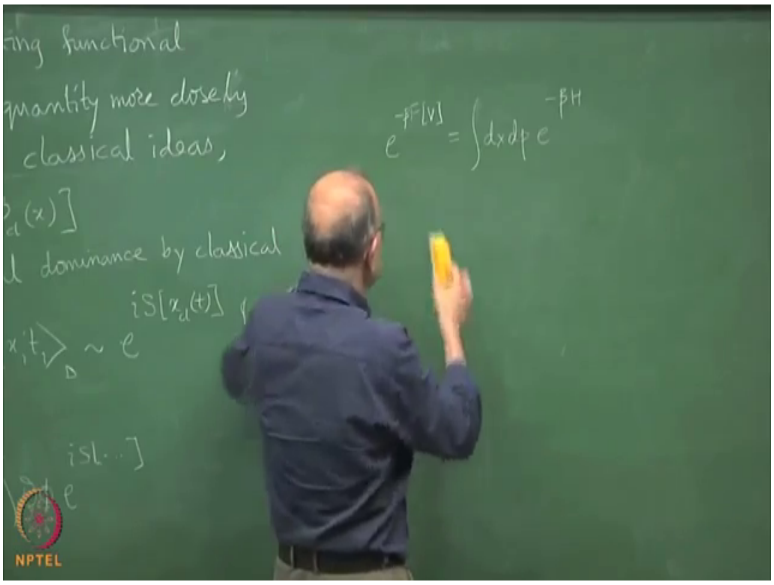

So, in a similar way, so the point is you integrated out all the detailed degrees of freedom and then you come out with a potential which when you minimize you see I am sure this is completely wrong, but I should have brushed up before coming, but you can. So, what do we have for what are the relations $P = -\frac{dF}{dV}$, something like this right and I think that is correct.

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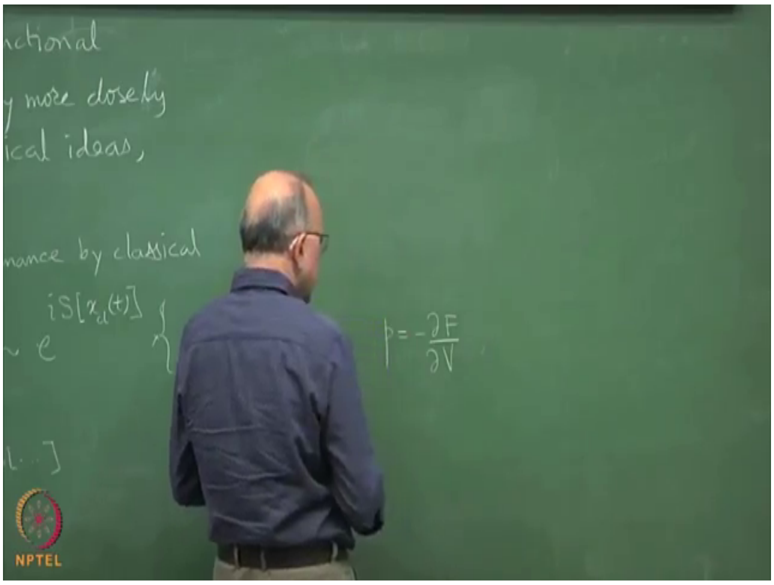

So, you want a potential what I mean is a classical potential of some kind which when you vary with respect to its argument you get your desired quantities out of it. So, after you have integrated out all the details the final answer depends only on the extensive variables, the volume is an extensive variable in thermodynamics, pressure is a local variable, intensive variable.

So, you can vary that potential with respectively extensive variable and get a intensive variable as the answer. Similarly here we expect that after all the complicated stuff here is over I come out with a clean and neat functional which mimics the classical behavior ok. Except for the problem that this is also a functional of ϕ, so what is this ϕ and what is that ϕ because this ϕ is all being integrated over ok.

So, I have to somehow find a remainder extensive variable which remains after all the integration here is carried out. This is done by a clever method called Legendre transform and beautifully enough there is a relationship between the extensive variable we expect classically and the variables we get from this W[J].

So, the first observation is that the Green's function is a twice variation of yeah the generating function. So, I am just trying to announce to you with the result, we somehow want an object Γ which is going to be the quantum version or the quantum corrected version of S, the good old action of classical mechanics, but now it is function of some slightly different argument.

So, if this was a naive field, this will be dressed up quantum field in some sense, but it will have classical values and the Γ will be the effective action for that system this is the idea. So, then you can see the whole system as evolving as if it's a classical system in terms of some argument called ϕ_{cl}. So, the question is how do you link the two?

So, as I said earlier it is like the potentials one difference in thermodynamics. It is not the naive classical field that you would have put inside the ordinary action, it is dressed up by quantum mechanics, it will be a classical variable; but not what you would have expected naively classically, it encapsulated it the quantum effects and the function Γ encapsulates all the quantum effects.

So, that now you can evolve this quantum system as if it is classical, you can study the quantum system as if it is classical by just varying, studying Euler-Lagrange equation.

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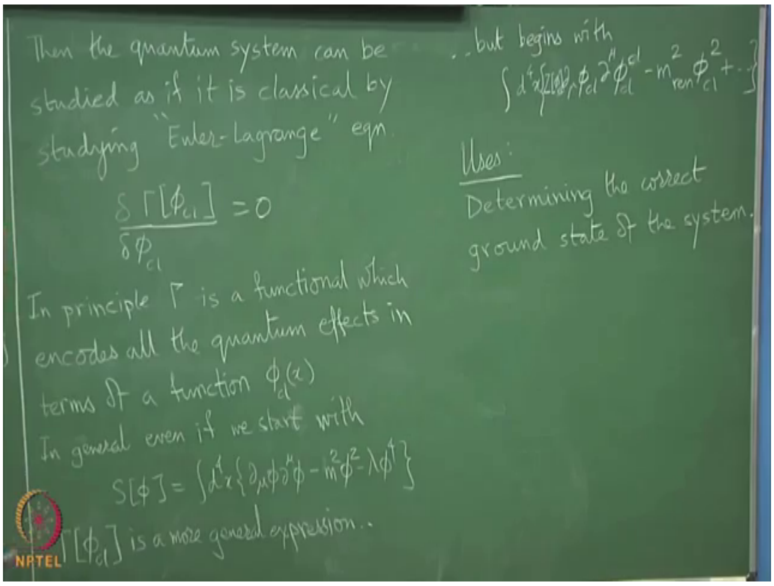

So, Euler-Lagrange equation says that on the classical trajectory variation of the action is 0, so that is what will happen if you do this $\frac{\delta\Gamma[\phi_{cl}]}{\delta\phi_{cl}}=0$. So, this is a tall claim nobody actually achieves this, but the point is you can formally define such a object and you can get reasonably good estimates of it in perturbation theory for all the leading terms.

So, finally, we will find that in principle Γ is a functional of a field which has encoded all the quantum effects, actually it is only a variable, so I do not know where we should not say that ϕ encodes anything is just an argument of the functional. So, is a functional which encodes all the quantum effect in terms of a function ϕ we keep calling it classical simply because we are going to treat it as classical, such that the quantum system can be studied as above.

So, we will find that Γ will be. So, to give an example start with $S[\phi]=\int d^4x[\partial_\mu\phi\partial^\mu\phi-m^2\phi^2-\lambda\phi^4]$. The S the Γ will come out it will be, but begins with $\int d^4x[Z[\phi_{cl}]\partial_\mu\phi_{cl}\partial^\mu\phi^{cl}-m_{ren}^2\phi_{cl}^2+..]$, it will look similar, but then it will have more complicated terms in it because the effects of the phi to the 4 coupling with m squared phi squared ok. There could be a Z in front which can also be function of phi classical, there would be a wave function renormalization of the fields and this is how it will look.

So, it will look similar, but it will have captured all the quantum effects. Now, you may ask why are we struggling to do all this and the answer is in two parts, in the simpler situations it allows us to determine the ground state of the system the collective ground state of the system.

Now, we tell children that if you put a negative mass squared, then you have a you know symmetry breaking potential and all this. So, all this is high school algebra, that thing can be rigorously defined obtained if you do this if you calculate. So, if you put negative mass squared, then if you calculate the effective action correctly you will actually find that the minimum of the potential is at the correct nonzero value of ϕ and it will not be the trivial $\sqrt{\frac{m^2}{\lambda}}$ or whatever you find from here.

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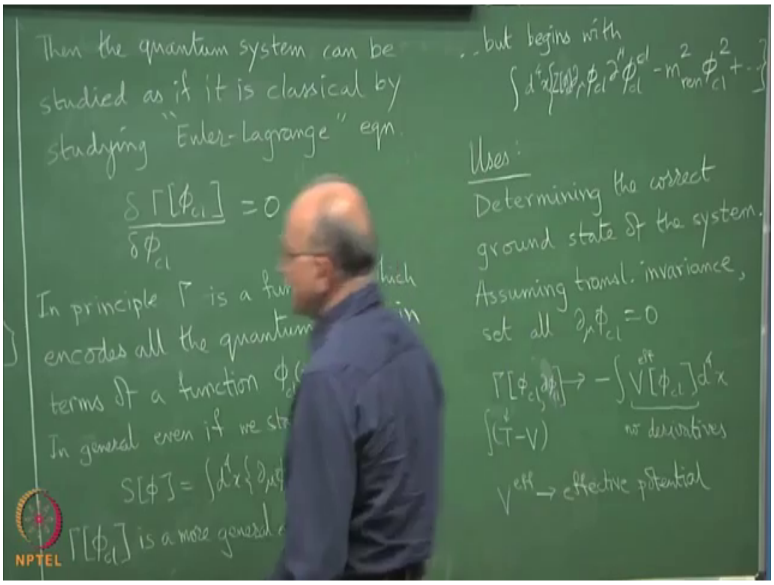

So, determining the correct ground state of the system this is usually done by, so assuming translation invariance. So, we said all derivative terms to 0 and then we get $\Gamma[\phi_{cl}] \rightarrow \int V^{eff}[\phi_{cl}] d^4x$ because Γ has the form of T - V after alright integral good old Lagrangian definition.

The T part will get set to 0 all the derivative terms there will I mean this Γ can be a very unwieldy beast, it can have all kinds of high level derivatives. All of those get set to 0 and you will be left with this $V[\phi_{cl}]$ which has only polynomials, so only powers of, so this has no derivatives.

We call it effective potential and the vacuum expectation value that we expect from that translation invariant vacuum.

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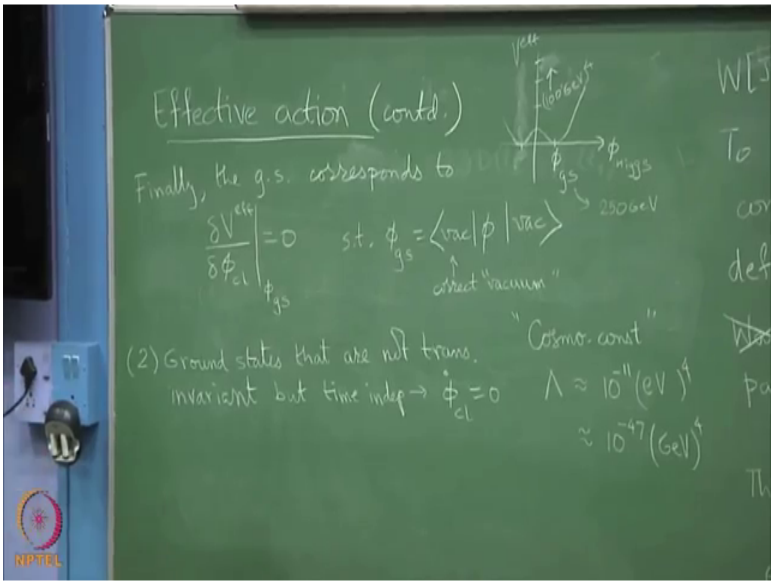

So from $[\frac{\delta V^{eff}}{\delta \phi_{cl}}]_{\phi_{gs}} = 0$, whatever that gives is the ground state value of ϕ, but interestingly you can also find other non trivial solutions. So, this is use number 1, but you can also find ground states that are not translation invariant, but time independent otherwise you will not have a ground state. So, $\dot{\phi} = 0$, so all the quantum effects are encoded in the coefficients of this effective potential.

Student: I get a value for ϕ_{cl}.

Yes.

Student: That is.

That will be the expectation value of the field.

Student: Ok.

That will be the vacuum expectation value of the field.

Right; such that $\phi_{gs} = \langle vac|\phi|vac\rangle$ where this will be the correct vacuum also. Of course, vacuum is not the correct word ground state is the correct word, but people use the word vacuum. Vacuum means there should be nothing, but what you begin to find is that there are lot of thing is lurking in the vacuum, but anyway that is it will not have any particle like excitations of the ϕ field that is what we mean by vacuum right. So, ϕ will have a value, but there will not be particles whizzing around.

So, in that sense it is not free of everything, but it certainly does not have excitations moving about and this is our current enigma because the electroweak theory is supposed to have a nonzero, Higgs field pervading us like the good hold ether and the Higgs field the observed in LHC was the first excitation above this ether.

So, we are right now living in a Higgs ether with a nonzero value for ϕ and nobody has been able to get rid of this ether idea and the ether idea looks; obviously, wrong because let me some picture like this sorry I should draw it through the center.

So, the thing is you are somewhere here, but if it is so then this is the V^{eff} of electroweak theory though, so this is ϕ_{Higgs} and this is ground state value. The ground state value determines the masses of everything it enters into W and Z boson masses the everybody's mass. Now, the point is that all of this is at TeV scale you calculate it at least in you like 100s of this is this value is some 200 GeV.

So, these values can be thought of in like 100s of GeV, 100 Gev, 200 GeV to the fourth power because its density. You can set it you can say that exactly when gs is 250 GeV is where the electroweak vacuum is and the that vacuum the V^{eff} should be 0 should be the ground state, no energy.

If you do this, then standard model works, but there is a slight ticklish problem which is that all this scale is in GeV scale, but if you have tiny mistake in it to 15th decimal point, it will have residual vacuum energy left that vacuum energy can be seen by gravity which would appear as cosmological constant.

So, cosmological constant today have the so called Einstein's famous $\Lambda \approx 10^{-11}(eV)^4 \approx 10^{-47}(GeV)^4$. So, today we are actually observe vacuum density, vacuum energy density of the order of $10^{-47}(GeV)^4$, this thing requires you to set it to 0 in 100s of GeV units, so you do not care if the this much is residual.

But there is a conceptual problem, why exactly this amount is leftover, if there was an error in this thing being set to 0 exactly there should be maybe 0.7 GeV to the fourth leftover may be millionth of maybe MeV to the fourth leftover, but what is left over is stupendously small on the GeV scale and it is nonzero it has a value.

So, you have to tune this to this, this is like our government trying to check your one rupee transaction through money pay you know. So, it is as if the government's budget would get reset by whether you bought a chocolate today or not well. So, that is extreme fine-tuning that it should have been if it was really determined by this physics, then it should have been in some high scale at that scale it is 0. So, you would say well there is no other physics intervening between electroweak and everything else we know it is electromagnetism at lower scale which is exactly massless and so on.

So, where does this energy come from? It should have been ideally 0, if it was 0 then everybody would be happy because then you would say oh supersymmetry is a principle which requires vacuum energy to be exactly 0. So, you would have said oh it is 0 because of supersymmetry, but it is nonzero and it is stuck at some strange value.

So, now, you have to try to say yeah, but supersymmetry is actually broken very very far away in such a way that it shines a little light here like Birbal's Khichdi or something. So, it is exactly, but then you are put then tuned why it is over there, so that it produces this. So, there are lot of I mean this is how we make our living and making theories for this. So, you are welcome to join lot to be done here.

So, this is an open problem, but this is the machinery that allows you to determine it directly determine it systematically. But there are more since I have got on the topic, let me just say there are more interesting things that you can compute from the effective potential, where use do not set all derivatives 0, but you said that time derivative to 0. So, you are time independent state, but not translation invariant.

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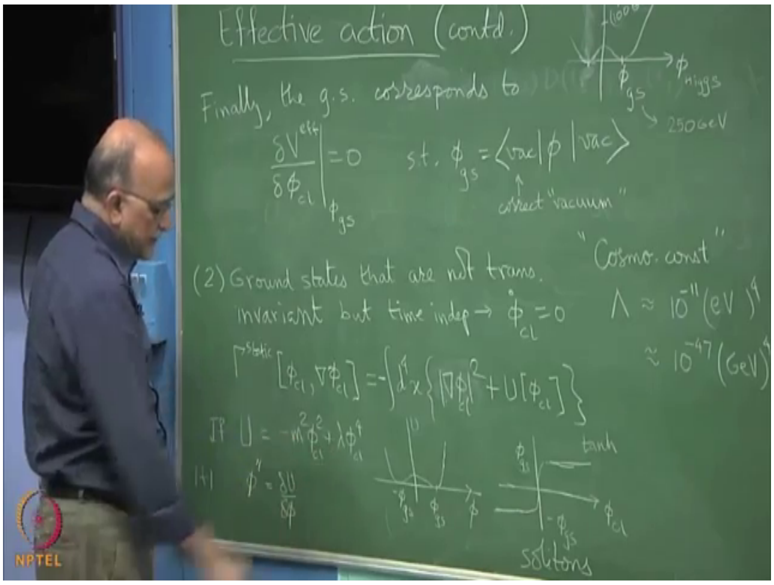

So, in this case you get $\Gamma^{static}[\phi_{cl}]=-\int d^4x[|\nabla\phi_{cl}|^2+U[\phi_{cl}]]$ and you can have something interesting which is. If you now put your favorite Mexican hat potential, then you can get what. So, if $U[\phi_{cl}]=-m^2\phi_{cl}^2+\lambda\phi_{cl}^4$, then you have U that looks like this.

Then you can actually get solutions for ϕ which start at let us call this yeah ϕ_{gs} let us say, so $-\phi_{gs}$ and $+\phi_{gs}$. So, you can actually as solutions which are $-\phi_{gs}$, for most of the time on this side and $+\phi_{gs}$ here and in between the actually interpolate as if it is well interpolate as tan hyperbolic, you can actually solve the differential equation and instead of minus it will be plus well, so yeah plus because it is actually not time derivative.

So, it will be $\phi''=-\frac{\delta U}{\delta\phi}$ you do it in one dimension 1 + 1, so choose only 1 space dimension and if you solve this equation this non linear equation has a nice exact solution has tan hyperbolic. So, you can come out with non trivial ground states, these are called kink solutions or solitonic solutions and condensed matter physics and various and even in optics you can find this solitonic modes propagating, so these are called solitons.

So, all such things can be found from quantum theory where you have approximately classical description which you know actually derives from some big daddy quantum theory, but after everything is integrated out the effective degrees of freedom that remain are this ϕ_{cl}, the extensive variable that remains is the volume or just the total number, so it is like this.

So, this kind of expressions can be constructed, that is the meaning of Γ and we want to see how to extract such a Γ out of our W[J], right now we have a description W in terms of J some external current. The answer is that Γ need not always have the naive symmetries that you see in the classic collection, Γ can break symmetry that is spontaneous symmetry breaking.

Yeah the quantum action will not respect the classical symmetries ok. So, it can happen in several different ways and well, so this is actually the classic example the ϕ^4 theory, but with a negative sign for mass squared term the squared term the quadratic term, but positive for quadric this is potential not action. So, minus here, but plus here is like this. So, this theory has let us if I had only a real scalar field, it has $\phi \rightarrow -\phi$ symmetry, but because it is now like this you will have to do quantum theory either here or here you can do it in both places, but if you do it in either of them then you are broken the symmetry.

Now, you do not have the freedom to flip $\phi \rightarrow -\phi$. So, then what will happen is the excitations that you see over it, the quantum excitations will be the small oscillations here. And there then their interpretation is specific to having chosen $+\phi_{cl}$, somebody else can make a choice of putting $-\phi_{cl}$ is quanta will be different, you will have to do some unitary transformation to convert its to your description, but those quanta will not enjoy the $\phi \rightarrow -\phi$ degree of I mean symmetry.

So, the ϕ quantum need not have the classical symmetries of S and it gets a little more complicated as well which we will see later what are called anomalies where it is not even so trivial. Here at least you can even see algebraically that this happens, but there are also ways of quantum mechanics violating classical symmetries which are hidden in the loop expansion, but eventually you find that the corresponding conserved current will not.

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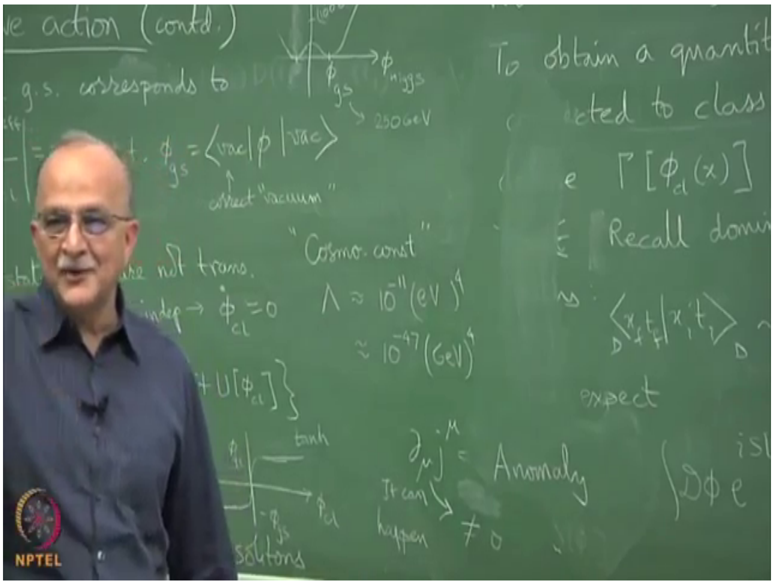

So, for every symmetry there this is of course, a discrete symmetry you can if you make it a complex field, then you will have a real part imaginary part you know Real ϕ Im ϕ then you at least have a continuous symmetry, but there also symmetry will be broken you no longer can rotate. But for this case we know that this is just complex Klein Gordon field, so there is a conserved current right it is. So, it is equal to $j^{\mu}=(\phi^{*}\partial^{\mu}\phi-\phi\partial^{\mu}\phi^{*})$.

So, there is a conserved field, but a conserved current corresponding to that symmetry, but sometimes in quantum mechanics this may not work, $\partial_{\mu}j^{\mu}\neq 0$. So, it can happen that, so this is just for example, you know, but it can happen that $\partial_{\mu}j^{\mu}$ may not come out equal to 0, where you have to interpret this as the full quantum expression for the current.

No, it is not this is true to simple that is not where it happens it happens in the case of chiral fermions this is called anomaly. So, it is a grand preview of what the whole course is trying to do, I do not mind spending a little bit of time because these ideas are all so difficult that it is to repeat them. But now, since we have lost all the time doing this I request you to come prepared next time having read a particular section of Ramond's second edition book the one on effective action.