**Path Integral and Functional Methods in Quantum Field Theory**
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**Lecture – 13**
**Effective Potential – I**

There is this question why we add epsilon term proportional to $\phi^2$ and not to any other operator and the answer is of course, we are trying to add the simplest possible thing that gets our job done. And, it is just convenient to add a quadratic term which can be integrated easily by Gaussian integrals. And so, it will just modify the kinetic terms and therefore, it modifies only the free propagator and does not enter into interactions.

Of course, eventually you are taking $\epsilon \longrightarrow 0$. So, you just want to put it in the most convenient place and such that it damps the integral also. So, actually it is quite a fortuitous thing because originally $\epsilon$ was put by hand, meaning that i$\epsilon$ prescription is one out of many you can have right. I hope you know this.

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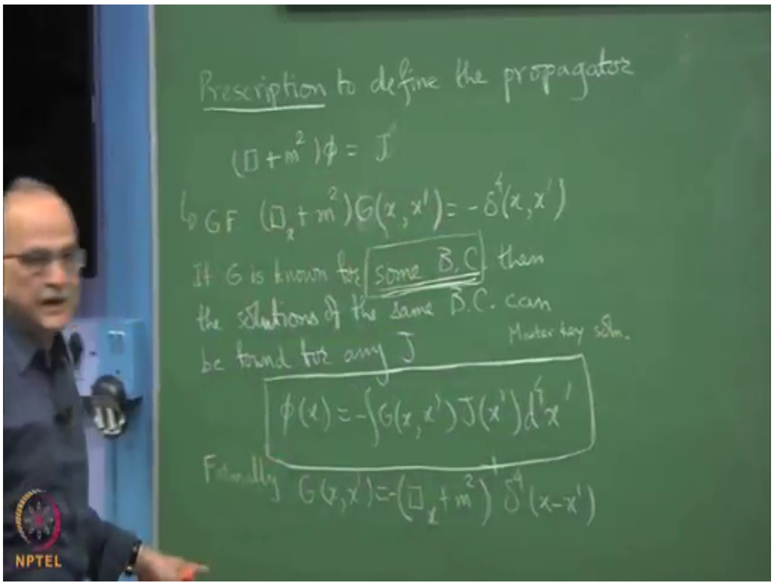


So, yeah, the prescription; so, basically the propagator is inverse of this operator we have $[\Box + m^2]\phi = J$ and in the notation of where it has some external function J. So, we convert it into a Green's function problem. The Green function problem says and since we are going to call it that let me or let us put G at first $[\Box_x + m^2]G(x,x') = -\delta^4(x-x')$.

So, you are familiar with the Green function technology right. So, if you solve for this Green's function for some boundary conditions then the solutions of same boundary condition can be found for any J simply as $\phi(x) = -\int G(x, x')J(x')d^4x'$ probably with a minus sign yeah because there is this minus sign there I do not know why, but these are conventions.

So, we can always find anything like this. So, the question is of putting the boundary condition right and formally G is simply equal to $G(x,x') = -[\Box_x + m^2]^{-1}\delta^4(x-x')$ So, it is so formally it is inverse of this operator. This can now be defined in momentum space. It becomes simpler.

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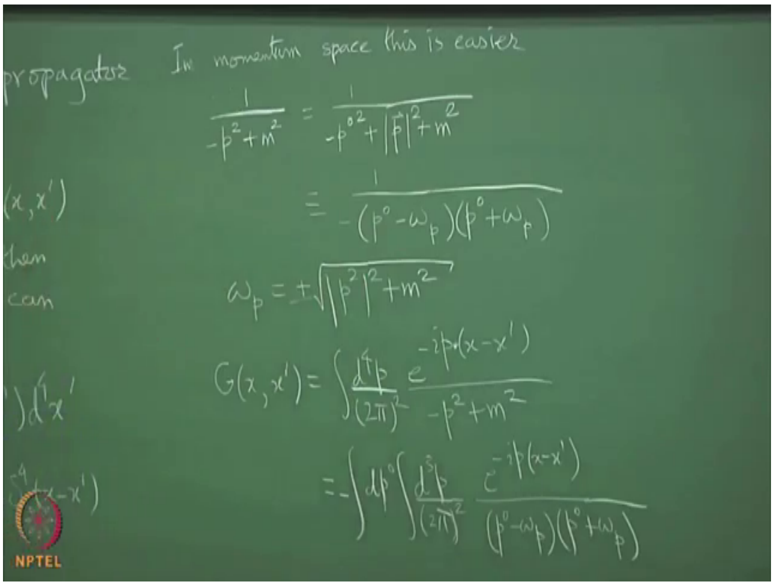


Because we can write it simply as equal to

$$\frac{1}{-p^2+m^2} = \frac{1}{-(p^0)^2+(|\vec{p}|)^2+m^2} = \frac{1}{-(p^0-\omega_p)(p^0+\omega_p)}$$

where $\omega_p = +\sqrt{(|\vec{p}|)^2+m^2}$ .

So, now, there are these 2 poles in this and now answer for G is the Fourier inverse transform of this as $G(x,x') = \int \frac{d^4p}{(2\pi)^2} \frac{e^{-ip.(x-x')}}{-p^2+m^2}$ This is the formal Green's function.

And, you have to give it some meaning because of these poles, how will you do the integration across these poles.

So, it is it split up into $G(x,x')=-\int dp^0 \int \frac{d^3p}{(2\pi)^2} \frac{e^{-ip.(x-x')}}{(p^0-\omega_p)(p^0+\omega_p)}$. So, now, the question is of how you negotiate the poles in the $p^0$ axis and there are several prescriptions you can give. So, the question is of the prescription, to provide to identify the Green's function that you need for your boundary condition.

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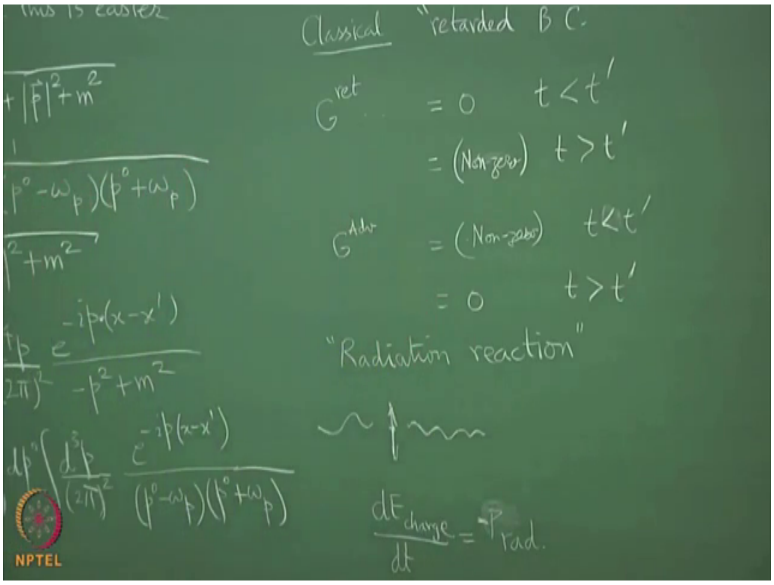


And, if you are in classical electrodynamics; as you will see in Jackson's book for classical radiation, you need retarded boundary condition which means that whatever $G^{ret}$ =0, if $t < t'$. So, the order of the arguments will then matter and it should propagate from $t'$ to t, but if $t < t'$ it will be 0 and is equal to something non-trivial; so for $t > t'$.

Now, whatever so, you have to give this boundary you have to create it like this. You have to specify some prescription for negotiating poles so that you get this and advanced is defined by $G^{adv}$ this equal to non something non-trivial and for $t > t'$, but is equal to 0 for $t < t'$, sorry. So, advanced means it is coming from the future. It will propagate things only backward in time.

It is quite a curious and interesting history that Feynman was actually fooling with these boundary conditions along with his advisor John Wheeler as a student because of a sorry, this is taking us on a historical trail, but if you will see Jackson’s book towards the end there is a whole topic of radiation reaction. So, radiation reaction is actually a crisis of classical field theory because what happens is that if you oscillate a charge it emits radiation. So, it has to slow down.

But, can you derive the slowing down from the equations of motion and there is a problem because so, if I wiggle the point charge radiation goes out. I can of course, calculate all the radiation going out and therefore, I can calculate the rate of change of energy, $\frac{dE_{charge}}{dt} = -P_{rad}$ equal to the radiation or just say energy carried away by power in the radiation right minus of the power carried away by radiation. This equation you can easily write. But, can you derive it from equation of motion of the charge?

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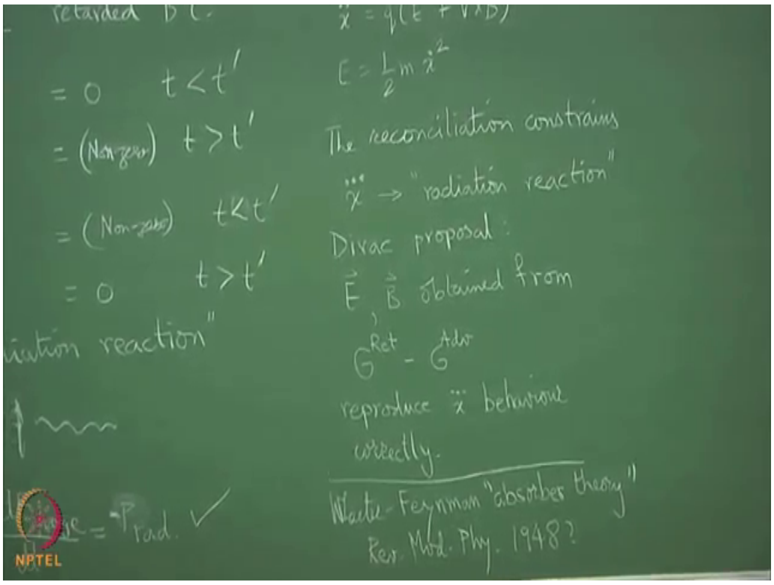


Because equation of the motion of the charge is $\ddot{x} = q(\vec{E} + \vec{v} \times \vec{B})$ right. Can you integrate this to get and we know that energy is equal to $E = \frac{1}{2} m\,\dot{x}^2$ for non-relativistic charge. Can you do a first integral of this motion to get this and get it is rate of change to match this rate of change? There is a serious problem because q is a point charge and what is the value of the electric and magnetic field at the site of the charge, there are no other charges in the universe this is the only charge oscillating.

So, it has to be it is own field, but if you put it is own field it is infinite. So, there is a conceptual difficulty trying to get this equation to be derived from what in the old literature was called ponderomotive equations for the charge; ponderomotive means the dynamical equations. So, not only that, this radiation formula is well known. So, this equation is well known what this has to be. So, can you reverse engineer to somehow get an expression for $\ddot{x}$ ? Yes.

Student: But the last equation should it be dE I mean all the energy inside that sphere is changing at the rate -$P_{rad}$. It is not just the energy of charge.

So, the radiation zone is only the 1/r zone which will be realized at infinity. All the remaining energy will sort of remain it is in the fields which remains confined. Yes, it confined it remains confined to some near zone, but it will not be lost, it will not be radiated to infinity.

Student: Right, but can we say $E=\frac{1}{2}m\dot{x}^2$ and expect that to change with period that.

Yeah, that is what we expect so.

Student: We also.

So, think instead of continuously forcing suppose which an impulse you and you have as you should have a spring of course, so that it oscillates. So, you are putting some external force into the problem and if the spring is conservative spring the radiation will take away the energy. So, you just pull their charge and let go. So, it has to come to rest. So, it has to undergo some rate of change of, the other stuff we will just kind, it is called reactive fields and they are not dissipative fields. They will not take away the energy they will keep reinforcing the motion in some way probably.

But, the big problem is that this is I when used to know the dependence or may not this dependence, but therefore, what happens is then that the only way you can get the answer is if you put so, it actually looks like you have to it determines. So, these conditions determine $\dddot{x}$ . So, this to record I will not be able to remember everything now, but the reconciliation constrains $\dddot{x}$ which is called the radiation reaction force. I can actually even remember why it would become $\dddot{x}$ if we remember the dipole formula, but you can say it is given in Jackson. Now, that is beyond Newtonian scheme because it is now $\dddot{x}$ .

So, there were various rather exotic proposals. One of them was to say that actually there is some action at a distance between charges and but it is not dissipated in this detailed way and so on. Dirac being a clever guy observes that the $\dddot{x}$ formula comes out exactly correct. If you use this equation, but in the fields you do not put the cell field trivially, but you put retarded minus advance.

So, Dirac prescription and not to be mixed with that prescription, but Dirac proposal E, B obtained from $G^{ret}$ - $G^{adv}$, reproduce behavior correctly because only G by it itself would be divergent at the source. But, if you do $G^{ret}$ - $G^{adv}$ then it exactly cancels out the local fields and retains only the radiation zone fields ok. So, that correctly has any behavior of the radiation zone fields. Now, this tickled everybody's mind why it was retarded minus advanced and so on.

So, Wheeler and Feynman tried to create a system for this which is a absorber theory. It appears in reviews of modern physics sometime maybe 1948 or something like this. They come up with a completely crazy idea which says that actually in the classical physics all influences should be taken as retarded plus advanced, that is at least symmetric. So, you may ask why is it that you are forced to take $G^{ret}$ and there is no answer except that that is what you observe. You only see cause if you just like Maxwell's equations is a time symmetric.

So, why should you be taking only retarded? There is no real, no answer from within the theory you have to put it from observation. What they now did was to grab on this opportunity to say that actually it should be time symmetric and they claim to derive the retarded minus advanced prescription by a very crazy process. They say that all the charges in the universe are emitting retarded plus advanced.

But, now what happens is that if this guy is going to oscillate at some point, it is advanced potentials are advancing upon it, you have to picture what advanced potential means like retarded means a plane wave then I mean a waveform going out advanced means waveform closing in. So, if you are going to do it now 3 million light years ago from and Andromeda galaxy wave front is closing in on you so that it comes here right now ok.

Only if that happens will you. So, what they show is that if you take all the charges in the universe and consider the advanced potential is hitting all the other charges who then react to it and whose reaction then comes and converges on this charge. If you do this drama correctly which I don’t fully believe, but they claim that you exactly get retarded minus advanced at the site of the charge.

So, the charge is trying to emit retarded plus advanced, but because it is advanced potential influenced everybody else in the universe. Finally, the net result looks like it is retarded minus advanced at the sight of the charge in such a way that effectively on all the charges then you only have retarded happening ok. Now, they consider this a great trying also of solving the arrow of time problem.

So, you have a cosmological arrow of time expanding universe, you have entropic arrow of time, thermodynamic arrow of time. But, why should microscopic electromagnetism even when you do not have any entropic considerations you do a very simple dipole experiment that there is only retarded potentials.

So, they claim that they solved the array of time problem of classical electrodynamics by the absorber theory, but then this absorber theory was not found sorry Wheeler, absorber theory was rather gobbledygook and. So, Feynman was playing with these combinations even as a student and later on however, to do the relativistic electrodynamics and it is Green’s functions, he eventually had to pick it from Stuckelberg because the prescription was even more we have then retarded minus advanced. It split positive frequency is a negative frequencies making positive frequencies go forward in time and negative frequencies go backward in time. So, there was one more prescription which had been missing from all of these and that is the prescription we use.

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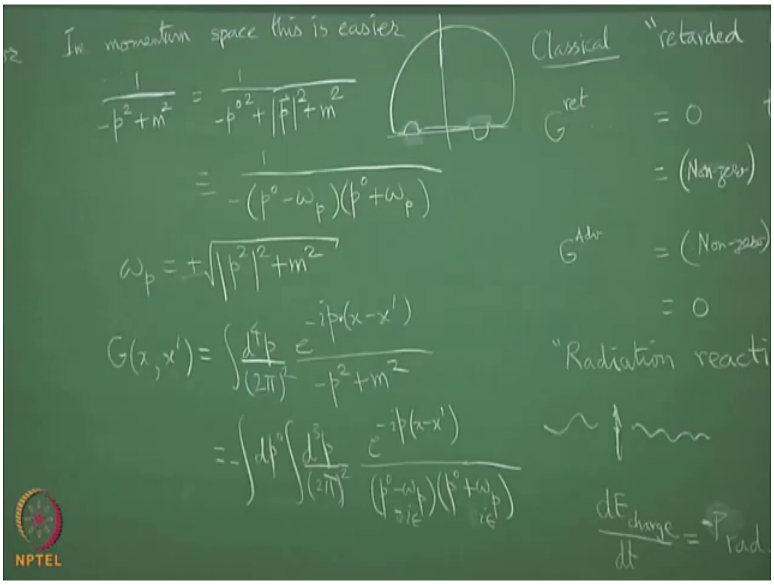


So, there is a very simple. So, to get retarded, all you have to do is to make sure that both the poles are below ok. So, when you close in the upper half plane you get sorry that would be the advanced. So, if you do it like this then when time is positive you get contribution of the poles, but if time is negative then you do not get contribution of the poles which can be arranged simply by shifting the poles down. So, to both of these

$$G(x,x')=-\int dp^0 \int \frac{d^3p}{(2\pi)^2} \frac{e^{-ip.(x-x')}}{(p^0-\omega_p+i\epsilon)(p^0+\omega_p+i\epsilon)}.$$

So, that is one prescription, but if you cross multiply it you will get an answer proportional to $p^0$ times i**ϵ**. But, $p^0$ is not relativistically covariant ok. So, you will be start with a non-covariant propagator and so on. Mr. Stuckelberg for whatever reason he used figured out that the correct thing to do is what we know now which amounts to putting one up and one down. So, it amounts to doing this.

So, for positive time you want let us say this 2 for positive frequencies to contribute. So, this is correct, but for negative frequencies. It should contribute when it is negative time.

So, at the negative pole it is $$G(x,x')=-\int dp^0 \int \frac{d^3p}{(2\pi)^2} \frac{e^{-ip.(x-x')}}{(p^0-\omega_p-i\epsilon)(p^0+\omega_p+i\epsilon)}$$ . So, it comes out epsilon times a positive number and that uniquely fixes the. So, it is a prescription that one has to put to get that propagator.

So, you have to interpret the Green's function in some way the formal expression is simply inverse of the differential operator which you can find in momentum space like you can do in electrical engineering where you have both quadratic and first. So, this is all second derivatives, but if you had first derivatives as well then you need a Laplace transform instead of Fourier transform and you know that problems become algebraic instead of having to solve differential equations.

Here just to I may have forgotten to emphasise, but the point of this machinery is that it is like a master key solution because it converts a differential partial differential equation problem to an integral equation with the boundary condition already built-in. So, you do not have to carefully remember to put in any boundary condition. You just feed it into this machine and integration is always the psychological easier than difference then solving the partial differential equation because integration just requires you to put some you know do a Simpson integrals even on multi-dimensional space. It is not a conceptually difficult thing and you can implement it on computers etc very easily.

So, the philosophy of Green's function method is that kernel as we have been calling it is that it converts partial differential equations into integral equations; except that you also have to remember to pack in the boundary conditions in that Green's function. Green's function essentially comes from differential equations theory. You did not have any quantum field theory at that. Feynman did not do any quantum field theory at all. He just put the propagator there and integrated. So, but how to get the propagator from quantized field theory that is where this time ordered product of quantum fields and vacuum to vacuum thing comes.

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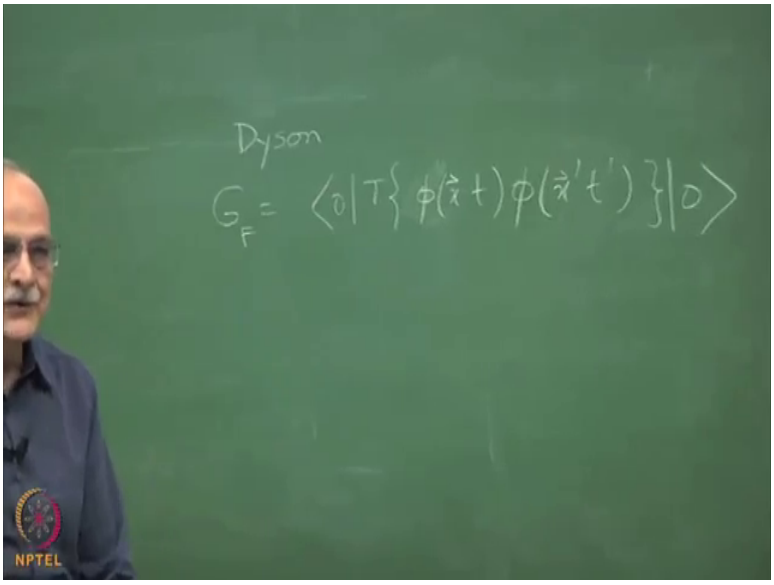


In quantum field theory, it becomes this was elucidated by Dyson right. This is due to Dyson, that only if you do time ordered product of the quantized fields you will automatically get this Stuckelberg propagator. But, then that you need Stuckelberg propagator simply has to do with having a covariant propagator. The other boundary conditions will give you non-covariant propagators. If you do purely retarded then you have a particular choice of time slice and if you do Lorentz transformations it will not remain covariant and so, you can see the explicit covariantness of it in the expression as well because it will be $p^2 - m^2 + i\epsilon$ and if everything is Lorentz covariant.

So, what is quite miraculous is that that time ordering which in Dyson prescription looks something artificial because you have to specify it; in path integral or the functional method is automatic because of the interpretation of that functional integral as a time sliced integral.

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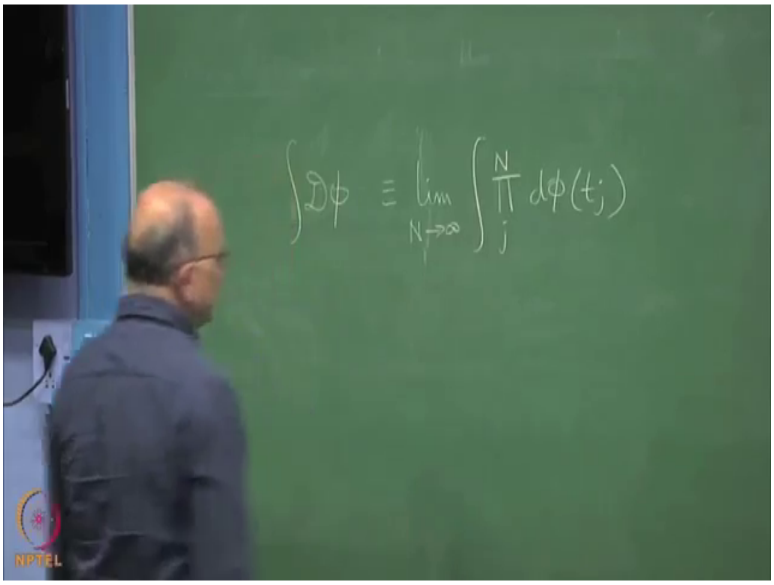


And, on each slice you integrate over the whole configuration space of the scalar field or whichever field it is and that is what it is. So, now just to reorient you again, what let me write our emblemic formula we have been writing for a few days now.

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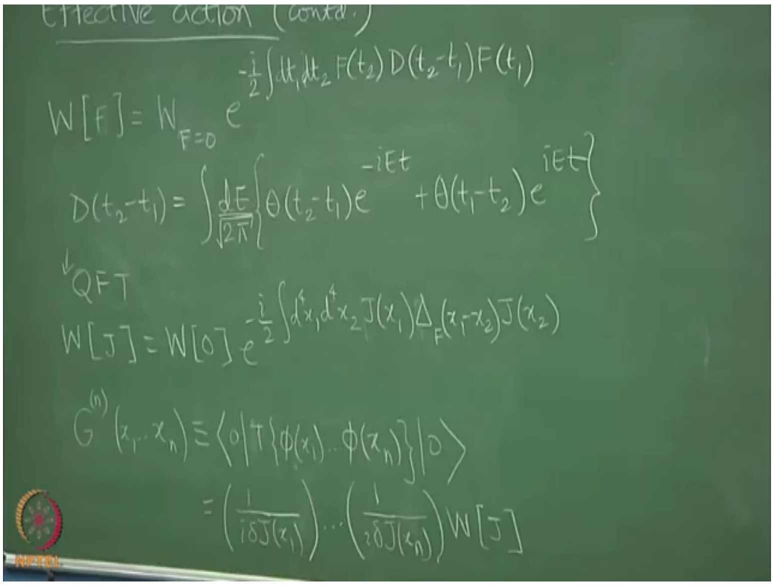


And, we have this formula $W[F]=W_{F=0}\,e^{\frac{-i}{2}\int dt_1 dt_2 F(t_2) D(t_2-t_1) F(t_1)}$ right now with $D(t_2-t_1)=\int \frac{dE}{\sqrt{2\pi}}[\theta(t_2-t_1)e^{-iEt}+\theta(t_1-t_2)e^{iEt}]$ , it is for the oscillator. But, in field theory this becomes $W[J]=W[0]e^{\frac{-i}{2}\int d^4x_1 d^4x_2 J(x_1)\Delta_F(x_1-x_2)J(x_2)}$. And, we will learn this technology that you can obtain G n-point functions which are defined to be

$$G(x_1,\ldots,x_n)=\langle 0|T[\phi_1.....\phi_n]|0\rangle=\left[\left(\frac{\delta}{i\delta J_1}\right)\ldots\left(\frac{\delta}{i\delta J_n}\right)W[J]\right]_{J=0}$$

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So, this object then would be calculated from that this formalism by taking variations with respect to J because each variation would bring down a ϕ and put it in the. So, this then basically just acts sorry going back. The original action function acts as a weight function with respect to which you take averages and they can be computed like this. Now it so, this is where this is called the generating function.

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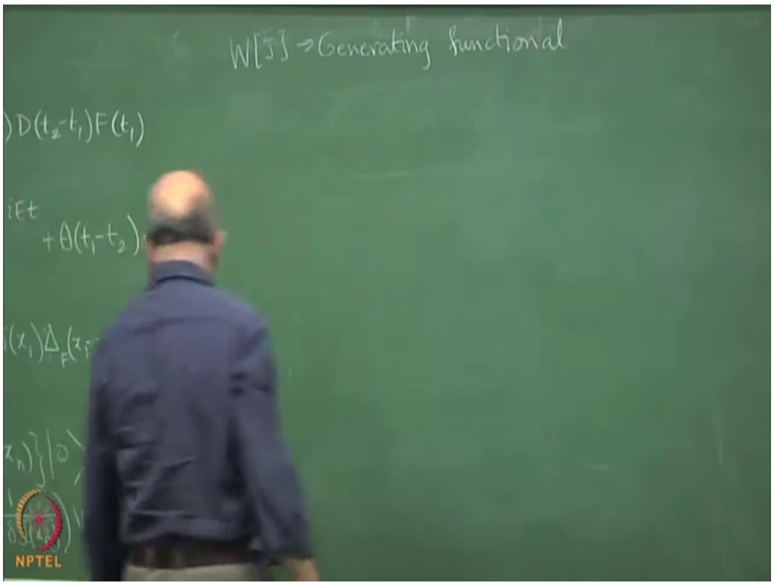


So, W is called the generating function of Green’s functions and evaluated at J equal to 0 ok. So, you vary and then set J equal to 0. So, that W[J] is called generating functional.