**Path Integral and Functional Methods in Quantum Field Theory**
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**Lecture - 12**
**Generating Function in Field Theory – II**

So now, we come back to Functional Method and how to compute Green's Functions in Quantum Field Theory.

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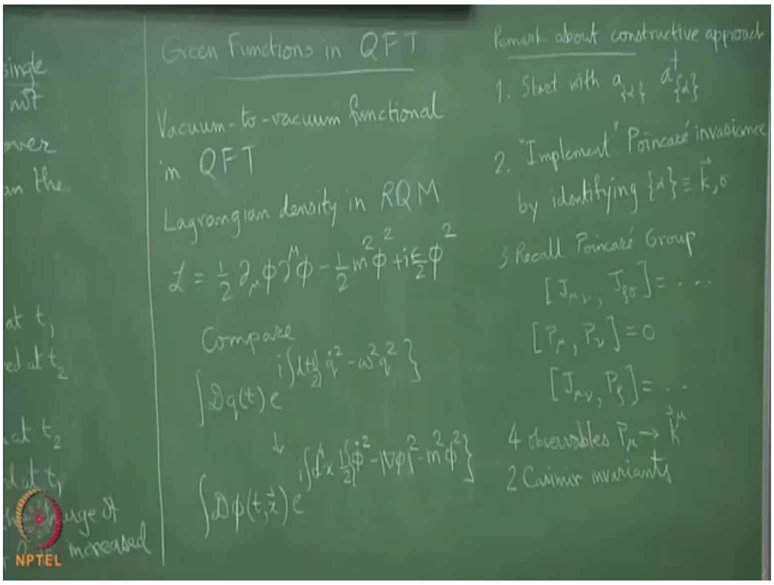


So, we begin by introducing the vacuum-to-vacuum functional in QFT. And, to that time we need the Lagrangian density in Relativistic Quantum Mechanics, which is

$$L=\frac{1}{2}\partial_{\mu}\phi\,\partial^{\mu}\phi-\frac{1}{2}m^{2}\phi^{2}+\frac{i\epsilon}{2}\phi^{2}$$

So, one analogy is that this is exactly analogous to the harmonic oscillator with enhancement of space and time coordinates. So, compare the fact that I have

$$\int Dq(t)\exp\left(i\int dt\,\frac{1}{2}\left((\dot{q})^{2}-\omega^{2}q^{2}\right)\right)$$

.

So, from this we are going to

$$\int D\phi(\vec{x},t)e^{\frac{i}{2}\int d^4x(\dot{\phi}^2-|\nabla\phi|^2-m^2\phi^2)}$$

So, instead of considering a one coordinate problem in one time direction, I consider a one coordinate problem, but as a function of 4 dimensional space time, but also with the Minkowski metric, that is the main difference. And, then the analogy is complete because the $\Omega^2$ term looks like the mass square term and all the methods will all the technical calculations will look identical, there is not much difference.

At this point I could also comment a little bit about this why we put the minkowski invariance, because we want Minkowski invariant S-matrix theory in the end of course. But if you go to the constructive approach of Weinberg, here what we do is we start with you remember that we argued that if your weekly couple system, it always boils down to introducing a and $a^\dagger$s corresponding to single particle set of Eigen values $\alpha$.

But, now we implement, the so called word “Implement” Poincare invariance by identifying this Eigen values as $\alpha$ with momentum k and $\sigma$ spin ok. This, these are single particle attributes; so single particle momentum and single particle spin. And, how does that connect? Well, recall in Poincare group we have $[J_{\mu\nu}, J_{\rho\sigma}]$, $[J_{\mu\nu}, P_\rho]$ commutators and $[P_\mu, P_\nu]=0$.

This $[J_{\mu\nu}, P_\rho]$ because $J_{\mu\nu}$ is rotation generator, it will only rotate the Ps into each other these are the 3-D part is the usual rotation group and thus is the Lorentz boost. So, this is the full algebra, this is algebra of the Poincare group. It has precisely two Casimir invariance. So, first of all it has exactly 4 mutually commuting observables, which are the P’s, which become the $k^\mu$, but remember that this will be using only the space like k because there are 2 Casimir invariance.

So, if you did not have the translations, then you know that there will be two observables; one is the SU(2)×SU(2). So, there will be two independent observables one, you can say is your third axis rotation. But once you putting the $P^\mu$s, the $P^\mu$s never leave any of the $J_{\mu\nu}$ invariant. So, they only mutually commuting observables available in Poincare group of the $k^\mu$. However, there are 2 Casimir invariance, One is $P^\mu P_\mu$. This is one Casimir invariant, it will commute with everything, because it is a Lorentz invariant the J's will not touch it, and the P's will also of course, commute with it. So, this is one invariant, which we call $m^2$. What is the other casimir invariant for this?

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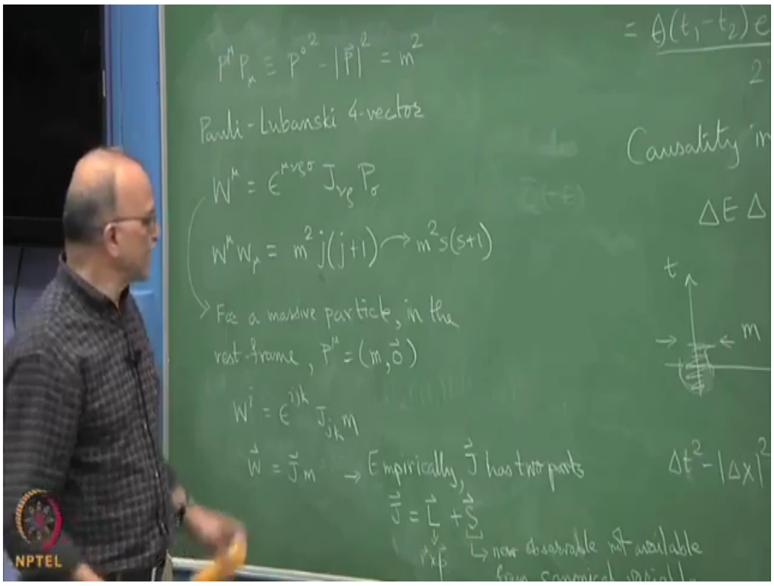


So, there is a 4 vector call Pauli Lubanski vector. So, it took a long time ok. It was like in mid-forties or sometime that and involve Pauli. So, you construct a

$$W^\mu = \epsilon^{\mu\nu\rho\sigma} J_{\mu\nu} J_{\rho\sigma} \quad .$$

So, we may as well break the field theory for a bit too just recover this well known facts of relativistic quantum mechanics that the one Casimir invariant comes simply from this and the other one is constructed out of this W. So, $W^\mu W_\mu = m^2 j(j+1)$. Where j is the would be casimir invariant of only the rotations ok.

The $j(j+1)$ is exactly the same as would have been there if you are not putting translations. Just to explain quickly what this W? $W^\mu$ is a real mysterious thing and you may wonder how the thought off of it. The point is if you look at it in the particles own frame of reference, so, for a massive particle at least in the rest frame $P^\mu = (m, 0)$. So, you choose the $\sigma=0$.

So, once we said this the $\sigma$ becomes has to be necessarily 0. Once, that is 0 and we have this completely antisymmetric vector this can never be 0 right. So, it will become $W^i = \epsilon^{ijk}$ m.

But, this is our friend, the angular momentum, I do not know what reasoning they used, but yeah it is very clever construction. So, the important and really the most important statement at this point is that there is nothing that instructs you to put spin in quantum mechanics. Lot of people think that since Dirac equation came from special relativity and it contains spin therefore, we have spin. So, these all completely wrong ideas. Because, you can write a perfectly valid relativistic wave equation, which is Klein Gordon equaation; there is no spin to be seen in it ok.

So, you can perfectly do relativity without any spin, but lot of people seem to think that Dirac proved that spin has to be there in relativity, without spin we could not have had a relativistic wave equation completely wrong. You have perfectly valid relativistic wave equations without any spin in them. In fact, there is nowhere that you learn from within the formalism that you need to put in the so called intrinsic spin. You are forced to assume due to observations in nature that J has two parts $\vec{J}=\vec{L}+\vec{S}$ , where L can be constructed from our postulate number 4 with said that all observables can be constructed out of the canonical ones.

So, L is of course, constructed as $r \times p$, but this S cannot be constructed from any canonical variable you have to take it on face value, you have to put it out of your pocket ok. So, this has to a put by hand, but once you put it and there is the beautiful plus sign whatever the plus means; once you put it the algebra is exactly same as that of J and we know in fact, that the algebra of J is complex and allows for both integer and half integer point values. And so, that fortunately matches what we observe in nature it allows you to introduce spin half, but that is all there is. And, the square of this $W^{\mu}W_{\mu} = m^2 j^2$. And so, that is the other casimir invariant.

So, you have to take the other casimir invariant and factor out the m. So, whatever this j is in the particles own rest frame, it will of course, we come just S, only the intrinsic spin will remain. You set at the location of the particle in it is own rest frame, there will be no orbital angular momentum, what remains is by definition what is spin ok. And so, the other casimir invariant has value $m^2 s(s+1)$ and that is what we put. The auxiliary in not canonical you obtainable, but observed in nature observable is what is spin.

So, in the constructive approach then it goes like this and you choose the a, $a^\dagger$ to be like this. And, at this point you argue that you need to construct fields. Why we got on to this point was this relativistically invariant derivatives we put because if we just went by the analogy, then the other coordinates that we introduced should also be just that square, but instead we put this, because we are trying to implement the Poincare group. And, there you need to reorganize.

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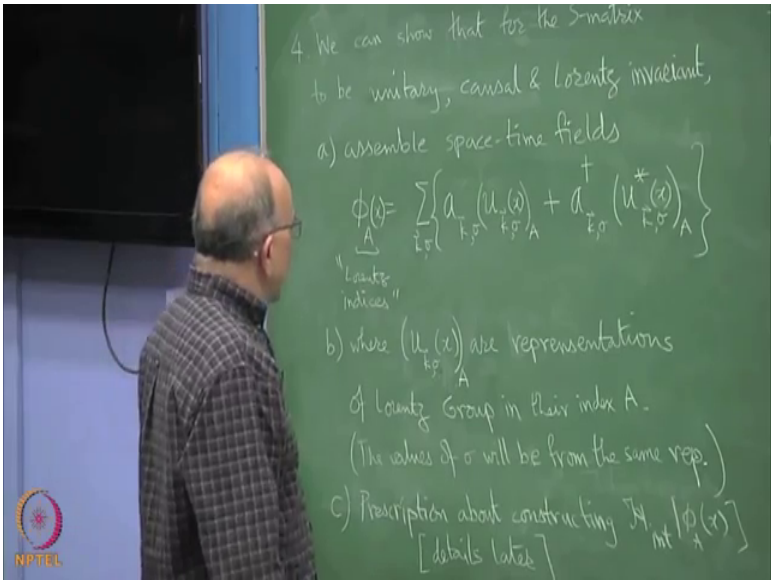


We can show that for athe S-matrix to be unitarity causality and Lorentz invariance, we need to assemble space-time fields,

$$\phi_A(x)=\sum_{k,\sigma}\left[a_{\vec{k},\sigma}\left(u_{\vec{k},\sigma}(x)\right)_A+a^{+}_{\vec{k},\sigma}\left(u^{*}_{\vec{k},\sigma}(x)\right)_A\right].$$

where $\left(u_{\vec{k},\sigma}(x)\right)_A$ are the representations of Lorentz group in their index A.

So, certainly A will be related to the value of this $\sigma$ ok. So, $\sigma$ s will be related to it will be some other representation, but it will be a representation, but same matching it. And, the recipe C is to construct interaction Hamiltonian, which we will see later. So, if you will remind me I will tell you when we get to the point, but the point is that in this we have doing things you never write the free wave equation.

The reason why I am telling you is that historically because Dirac pulled his equation like out of the hat, it left peoples stand for almost 20 years, you know people could not work; they could not sleep. Because, they thought how do we now write equations for higher spin.

So, the causality statement is that only few assemble them like this will you get that support on the light cone right if you take their commutators. If, you calculate the Feynman propagator only if they have been assembled in this form, I should not say Feynman propagators. So, actually it is the two point function you calculate their commutator. That commutator will be relativistically covariant only if you have assembled them a space time fields like this with both this and it is conjugate part included.

So, the a by themselves are just nuts and bolts right. In fact, Feynman is supposed to have commented somewhere how can you have a creation operator for an electron? Because, you are violating electric charge creating charge out of the vacuum, well I am not anybody to correct Feynman, but it is just the calculation device. It does not actually create anything ok. So, by themselves they are rather bare, they begin to acquire a physical or more compatible physical appearance only if you assemble them like this that is the statement.

And, there is very beautiful reason of how you we assemble the $H_{int}$ and how the unitarity follows, but that is a much longer story and it is a different path. So, I just wanted to say the most important remark is that this bit. The so called free field theory is not required in that constructive approach, because it is already solved for, you already put these quantum numbers.

So, you only put the k, because it is on shell, but we showed that there are two casimir invariants. And 4 mutually commuting observables, out of the two casimir invariants one is here I mean square root of, the other casimir invariant got used up in reducing from this 4 to 3 it is on shell ok. So, that is how these are listed.

So, we exhausted all there is to this beautiful differential equation the Klein Gordon equation over which often half of the quantum III courses spent trying to tell you that it has bad interpretation, it gives you negative probability. All this is not really required at all it is not part of quantum theory at all you do not have to worry.

If, you start quantum theory right this is the process. Implement classically observed symmetries on the Hilbert space by constructing the operators, that have the same commutation relations as the classical ones have the Lie algebra, that is the realization. And, with an advanced assumption that we will recover these operators out of the canonical variables of the system, which we eventually do, we can recover them out of the $\phi$ and the canonical conjugates $\pi$. But, the point is that the free equation is already solved. And there is no further content to Klein Gordon equation, then imposing the energy momentum relation that is all the really is.

All the rest there is the spin content, which you have to carefully construct into the construction of the functions u, they are either the spinner functions or the polarization vectors of the electromagnetic, you know ε and of course, $e^{ik}$, which is solution of that so, already built in.

So, the free theory is already solved there is nothing to be done about it. And, that is where we have a bit of a contrast, because here the free propagator appears and it has the free wave equation etcetera.

So, for the quantized field all everything goes through exactly as before.

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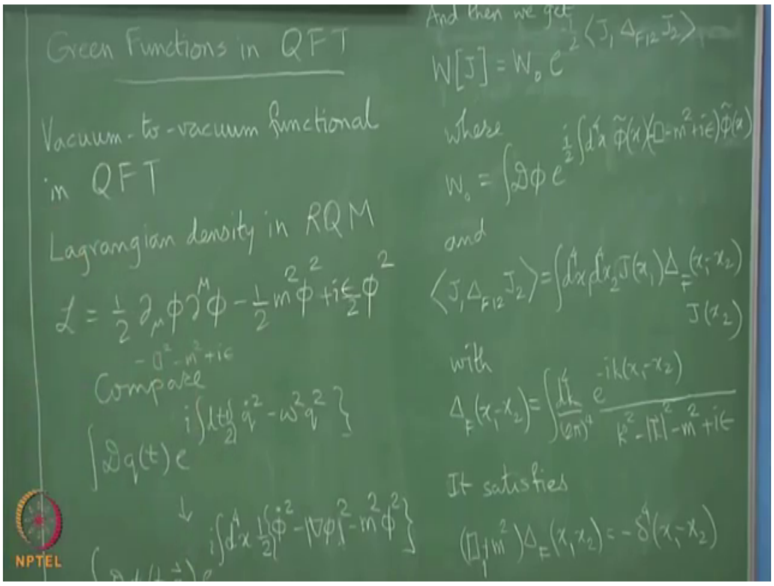


So, I can hurry up a little bit and say that therefore, we write the W[J] now. Now, we switch to the notation J current, it will

$$W[J] = W_0 e^{-\frac{i}{2}\langle J_1 \Delta_{F_{12}} J_2 \rangle}.$$

So, let us just specify how this is with respect to the Klein Gordon equation. It is equal to $(\Box + m^2)\triangle_F (x_1-x_2) = -\delta^4(x_1-x_2)$.

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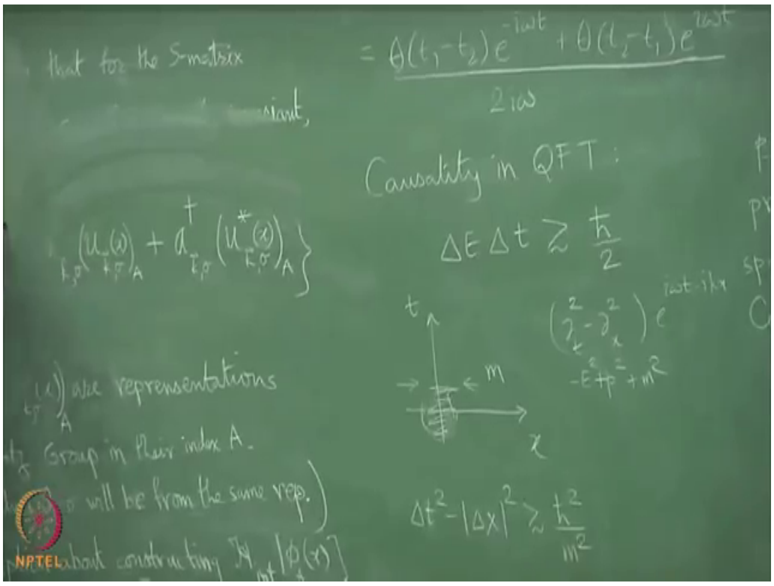


So, so far so good and as a trailer of next time, let me tell you that if you now use your machinery of varying with respect to external current,

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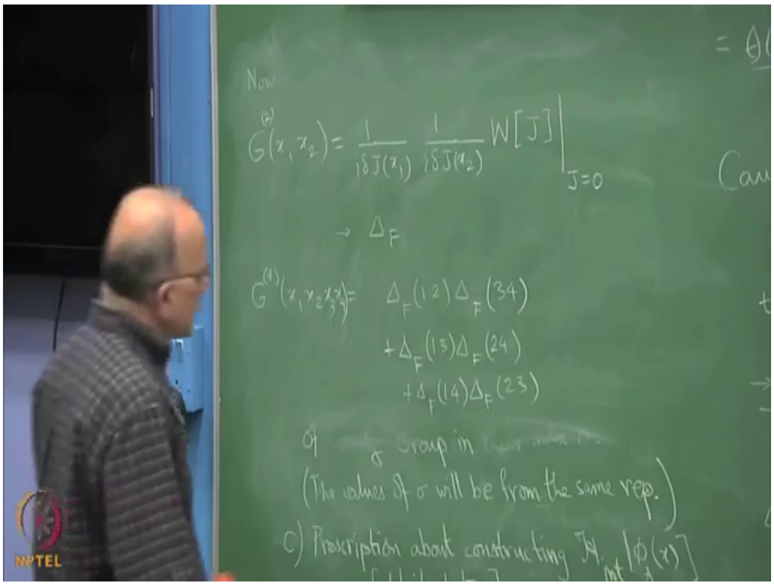


$$G^{(2)}(x_1, x_2) = \frac{1}{i\delta J(x_1)} \frac{1}{i\delta J(x_2)} W[J]_{J=0}.$$

You will find Feynman propagator here and if you do it for $G^{(4)}$, so, you will find that the odd ones are always vanishing, because remember that after differentiation you have to set J equal to 0. So, if you differentiate and this expression is quadratic in J.

So, if you are varied J ones this thing would have come down from the exponent with the J sitting in front. So, it will become 0 when you said J equal to 0. So, only the even number Green's functions are non-zero. And, if you try to do this it will become

$$G^{(4)}(x_1, x_2, x_3, x_4) = \Delta_F(12)\Delta_F(34) + \Delta_F(13)\Delta_F(24) + \Delta_F(14)\Delta_F(23) .$$

So, we will check it in more detail next time. So, the 4 point function will just boil down to a sum of terms each of which is again product of pair wise two point functions. And, this is how eventually the particle interpretation emerges. Because, you always have this 2 point functions appearing everywhere. So, something propagating from 1 point to the other, you can draw the Feynman propagator the line.

And, so, eventually those are the building blocks. So, the building blocks of greens functions eventually are these tics and blobs which are vertices that is what we will show in the next 1 or 2 lecture.