**Path Integral and Functional Methods in Quantum Field Theory**
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**Lecture - 11**
**Generating Function in Field Theory – I**

So, let me write down the sequence of things we have been doing and then you can see.

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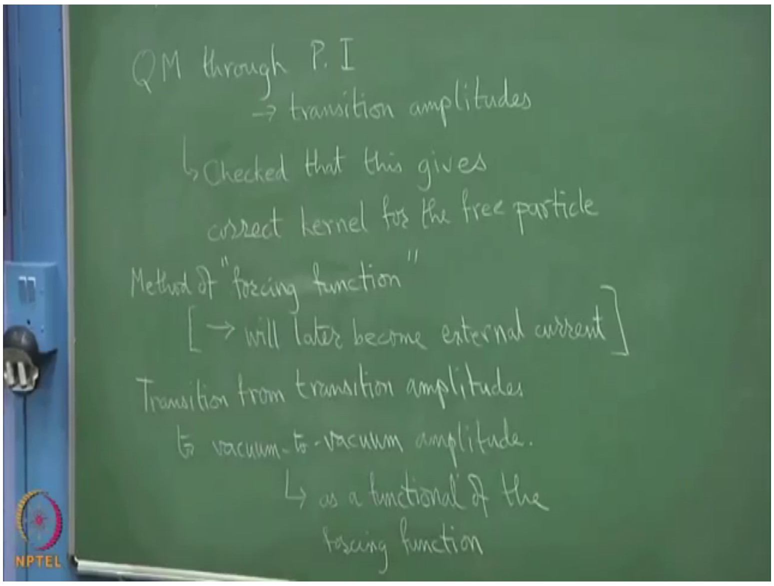


So far is that we defined the quantum mechanics through path integral which actually meant through transition amplitudes. But, we basically checked that this gives correct kernel for the free particle. Now, from this point on we introduce this method of forcing function and this is a method, there is no real forcing function.

So, this is method of forcing function which will later become external current. This is Schwinger's way of thinking of it. We also made a transition from transition amplitude to vacuum to vacuum amplitude ok, again a concept essentially due to Schwinger. Vacuum to vacuum amplitude is apparently a fake think because what would you learn from going from vacuum to vacuum it is like being back to square one.

But, the point is that in fact, the vacuum to vacuum amplitude is done in the presence of the forcing function. So, you get a vacuum to vacuum amplitude as a function of this auxiliary variable and then by varying this variable you can obtain all the information back ok. So, but this is as a functional of the forcing function and that is the key thing that you obtain the vacuum to vacuum amplitude in terms of the forcing function.

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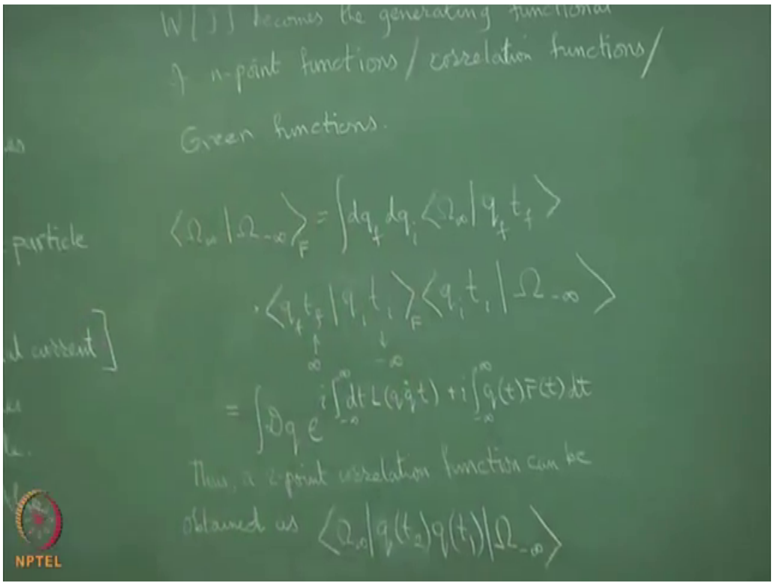


And then this W[J] becomes the generating functional of n point functions or Green functions. So, that is what we have done so far. So, we can say what happens is that we do

$$\langle \Omega_{\infty} | \Omega_{-\infty} \rangle_F = \int dq_f \, dq_i \langle \Omega_{\infty} | q_f t_f \rangle \langle q_f t_f | q_i t_i \rangle_F \langle q_i t_i | \Omega_{-\infty} \rangle \quad .$$

So, vacuum to vacuum amplitude in the presence of F basically takes this form and then this becomes we saw in the limit that we take $t_f$ to $\infty$ and $t_i$ to $-\infty$, we basically recover

$$\int dQ \exp\left[i \int_{-\infty}^{\infty} dt \left(L(q, \dot{q}, t) + i \int q(t) \vec{F}(t)\right)\right] \quad .$$

Now, at this point itself we can observe that if we define correlation functions as

$$\langle \Omega_{\infty} | q(t_2) q(t_1) | \Omega_{-\infty} \rangle = \frac{1}{i \delta F(t_2)} \frac{1}{i \delta F(t_1)} \langle \Omega_{\infty} | \Omega_{-\infty} \rangle_{F=0} \quad .$$

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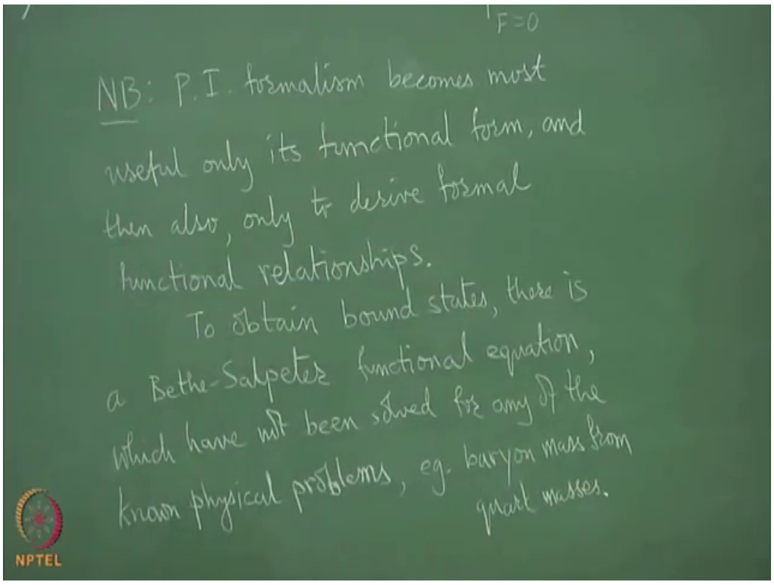


This is all we did so far effectively except for the calculation method of doing quadratic integrals to recast the path integral in various ways and of course, we will be using it again and again. We also use the stationary phase method to check that the path integral has to become stationary on the classical path. So, but other than that this is all there is and if you have n points then you put n of those and then you can recover the answer.

There is an illusion among people that path integral is a good thing to do quantum mechanics, this is completely wrong. The main use of path integral is only after you make transition to quantum field theory and then to derive relations between Green's functions. So, QFT again has been used primarily as an S-matrix theory. So, we tell everybody to get them excited that we are calculating n point function, but what we really having mind is calculating scattering of n particles.

So, there is a very formal procedure which then conversion n point function into the n particle S-matrix. So, we calculate those transition amplitudes the S-matrix and not necessarily stationary states. So, in fact, QFT fails completely, I should not say completely, but QFT has not proved to be very useful to compute any bound states nor this path integral very useful. I know that there are several textbooks entire text books written on how path integral is very useful in quantum mechanics.

Well, you can read them for their own value whatever they have, but I have never read them and I can vouch that no chemist will need them to calculate the many electron bound states. The chemist do use however the Green's function ideas. So, Green's function ideas make it a little bit more formal and peg it on a slightly different level, but path integral is not going to be. So, the only computation you can really do with path integral is a Gaussian integral and later we will see that it helps you to derive the so called diagrams, so called wick contraction at two-point function at a time.

But, there is a the kind of power that you have in a partial differential equation which allows you to solve I mean hope for getting exact solutions for many different potentials that does not exist in quantum field theory. And in quantum field theory primary use of this method has been to calculate S-matrix elements.

So, just Bethe-Salpeter functional equation has been studied by lot of people and lot of work exist, but we were never taught that it calculates baryon and if it was then we would be not doing lattice gauge theory. So, that is the status, but we can also say the other use of the functional formalism is in fact, to implement this theory on the lattice. You can implement quantum theory on the lattice in this functional formalism.

So, for lattice gauge theory also it is a useful thing, but lattice gauge theory is an just a completely numerical calculation. It is Monte-Carlo calculation of that functional integral because the only exact calculation schemes available is the Gaussian integral of the so called steepest decent method in some approximation you can, it is like the stationary phase, we will see it. So, there are very few tricks available at the functional level that allow you any kind of coat answer. But, this trick does allow you to obtain functional relationships, the trick of partition in statistical mechanics also when use a something similar.

So, that is really the all there is to it ok, but it is extremely powerful for the purpose for which we are going to do it. The conversion from Green's function to S-matrix is itself quite a formal statement, but once you get over it you get used to the idea. It is not all that difficult. Hopefully, I will be able to do it if I have the time.

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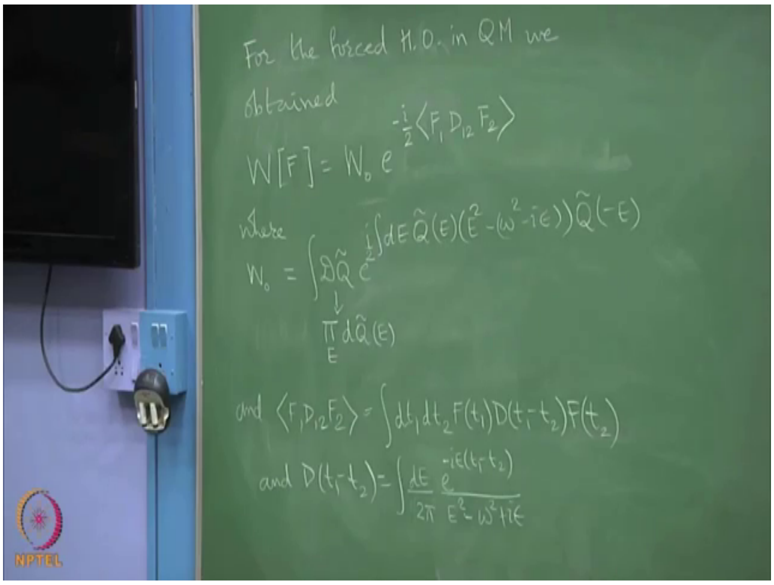


So, we have to interpret this as product over all E where you have to ordered the E, the energy spectrum because path integrals are always ordered and that was one thing I was going to comment here I forgot where we wrote this Green's function, we actually end up calculating only the time ordered product.

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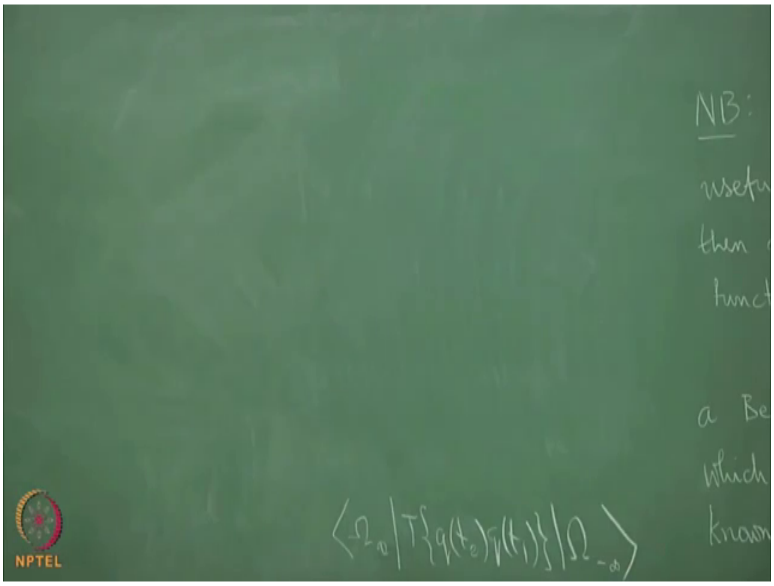


So, it is automatic in path integral that you will get the functional method that you will get the tang ordered product. So, this is the $W_0$ and the W which is not of much interest anymore and the other part in detail is,

$$\langle F_1 D_{12} F_2 \rangle = \int dt_1 dt_2 F(t_1) D(t_1 - t_2) F(t_2)$$

where,

$$D(t_1 - t_2) = \int \frac{dE}{2\pi} \frac{e^{-iE(t_1 - t_2)}}{E^2 - \omega^2 + i\epsilon} \quad .$$

So, everyone knows all this and you know that this boils down to

$$\frac{1}{2i\omega}[\Theta(t_1 - t_2)e^{-i\omega t} + \Theta(t_2 - t_1)e^{i\omega t}] \quad .$$

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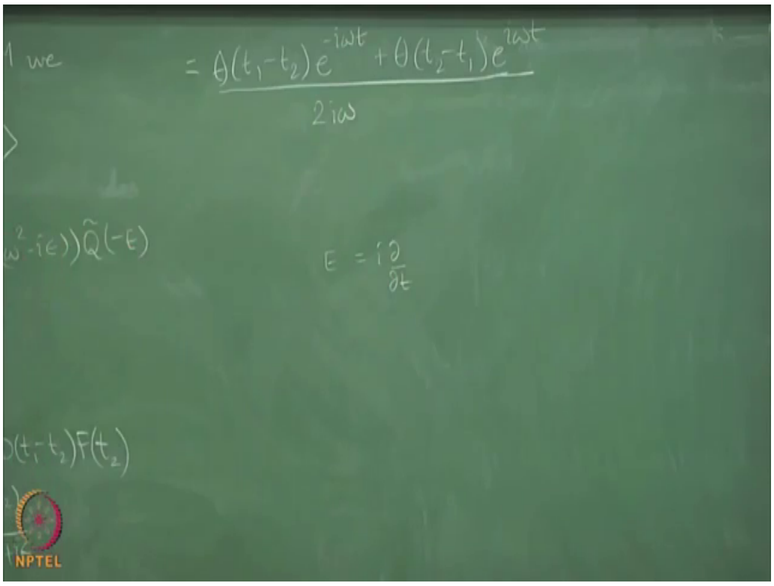

So, because it is square you will be taking (E+Ω) and (E-Ω) and each one is a pole. So, from each of the poles you get each of the pole irrelevant depending on whether this is greater that is greater than 0 and the value of the pole is that the 2π goes in the contour integral.

It was a great discovery when Feynman use this propagator that positive frequency particles go forward in time. So, ω is a positive number. So, it gets a minus sign which is the correct time evolution according to Schrodinger convention of setting energy operator to be equal to + id/dt. So, with the -i it gives correct ω. So, this is going forward in time, but this would give negative energy or would go backward in time and that is the interpretation.

If this has not being told you before I might as well spend a little time here telling you about is going forward and backward because this is at the heart of causality in quantum field theory.

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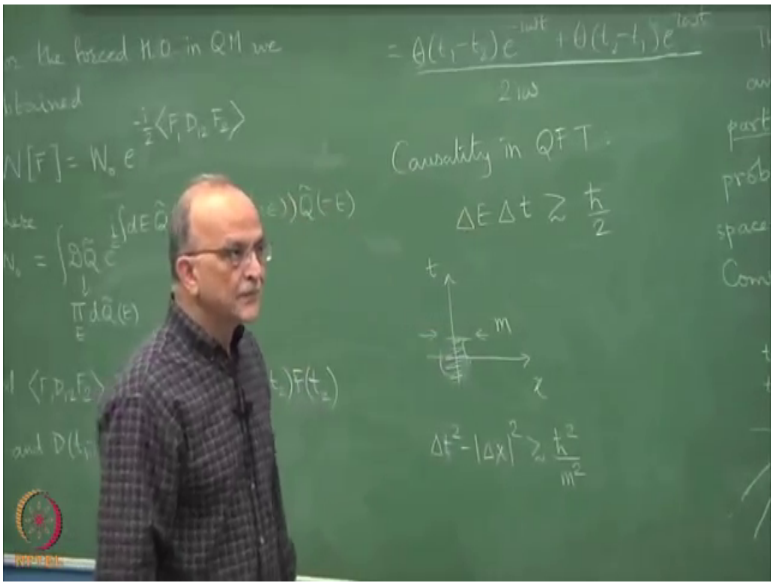

So, this is very general argument and the way Weinberg puts it that look at the uncertainty principles $\Delta E \Delta t \geqslant \frac{\hbar}{2}$ , but in relativity it is not Δt and Δx that really matter because one persons t is another persons mixture of t and x and actually Weinberg writes so called uncertainty principle which is written like this

$$\Delta t^2 - |\Delta x^2| \geqslant \frac{\hbar^2}{m^2} \quad .$$

So, it basically says that the space time interval between two observations has to remain greater than the Compton wavelength ok. So, this is the form in which it is written in his book and what this is saying is that you can have single particle interpretation only provided you do not probe the object in space time intervals that has smaller than the Compton wavelength ok.

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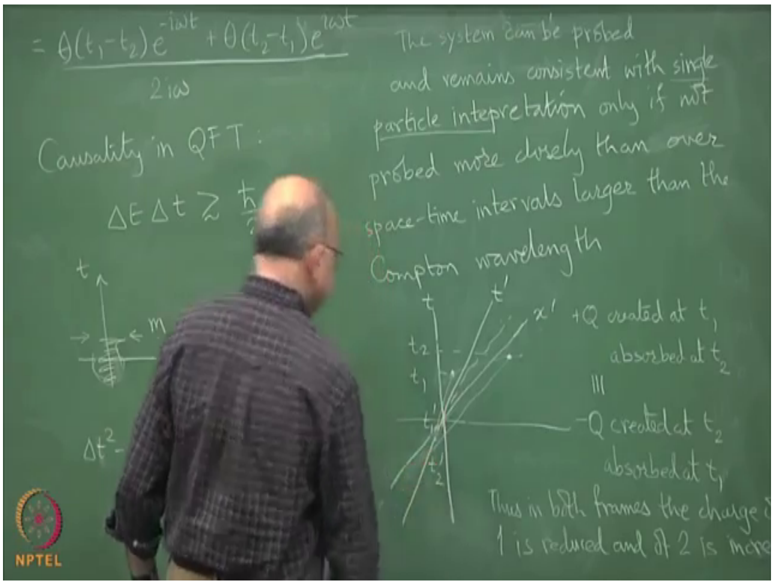


But, now in quantum mechanics you could always probe the system more closely and then you will lose the single particle interpretation as in most relativistic quantum mechanics courses emphasize that you will create particles. If you probe at this land scales shorter than this then the value of $\Delta$E will have to be larger than the mass scale of the particle and you will end up creating more particles and the single particle interpretation will be lost.

But, we also have a more specific statement. Suppose that I have creation of a particle. So, now, we draw this space time diagram and the light cone, normally if you create a particle here it will be later found here right. so, this is $t_1$ and $t_2$. So, it will propagate from this to this, but quantum mechanics only tells you some inequalities. It does not say $\Delta$x cannot be less than something or $\Delta$t all that you have to is do is maintain this, but if this happens you could also have a situation where $t_1$ is here and $t_2$ is here ok.

This may not be forbidden by this relation because all have to do is adjust that the $\Delta x^2$ will be negative in that case. But, because I do not have control in quantum mechanics it may very well happened that I create a particle here, but destroy over there. This is the also say Bell inequality thing you do something here and it determines something over there.

So, the point is that we recover the causality correctly in this case because here the events are space like separated. So, it is always possible to at least reverse the time ok.

For space like separated events it is possible to re-orient. This is a little let us see how does one recover I would have to really tilt it a lot until the projection onto that access reverses the directions of $t_1$ and $t_2$ right. If it is a space like separated interval, then if I choose my new axis to be like this then now I have to do a projection parallel to this axis.

So, I do this and this. So, this is $t_1$ and $t_2$, but now and this as a slope like this. So, I do draw a new choice of axis which is highly relativistic. So, it is approaching the light cone, very thin close to light cone. If I now project these, but drawing lines parallel to this they will go and hit the time axis in the reverse order and this one from here to draw like this until we hit this axis, already there. So, if you will do this carefully in your notebook, this is the new $x'$ axis, the projection lines parallel to $x'$ axis which go away and hit $t'$ axis, the $t_1$ and $t_2$ are reversed and these well known result and you can find the algebraic expression for the Lorentz boost required for this to happen.

So, now there is a question of causality that you emit a particle here, but absorb it here which is at a later time in this frame of reference. In the other frame of reference it look as if it got emitted first and got absorbed later. This problem is solved by quantum field theory because of this, because in the other frame of reference it will look like an opposite charged particle when backward in time that is what the interpretation is.

So, for a charged system +Q created at $t_1$ and absorbed at $t_2$ is equivalent to -Q created at $t_2$ and absorbed at $t_1$. So, this amounts to the $t_1$ location reducing its charge in both the things. What is same is thus both the charge of 1 is reduced and that of 2 is increased. In the person who is observing frame of reference as the clock is ticking he considers time sequences going forward in time strictly and if the two events due to the this uncertainty this is what I meant to suggest.

In some space time region well actually it will not be a circle, but some kind of hyperbola in. So long as this is of the order of m anything can happen here in particular particle can get produced and annihilated at space like separated points. And if that happens with a particular time sequence and if it is space like separated in another one it will look like creation event is happening after the destruction event, well that is not true because it will be in the other observer's frame of reference it will be interpreted as -Q sequentially got created at $t_2'$ which in this frame of reference occurs before, $t_1'$.

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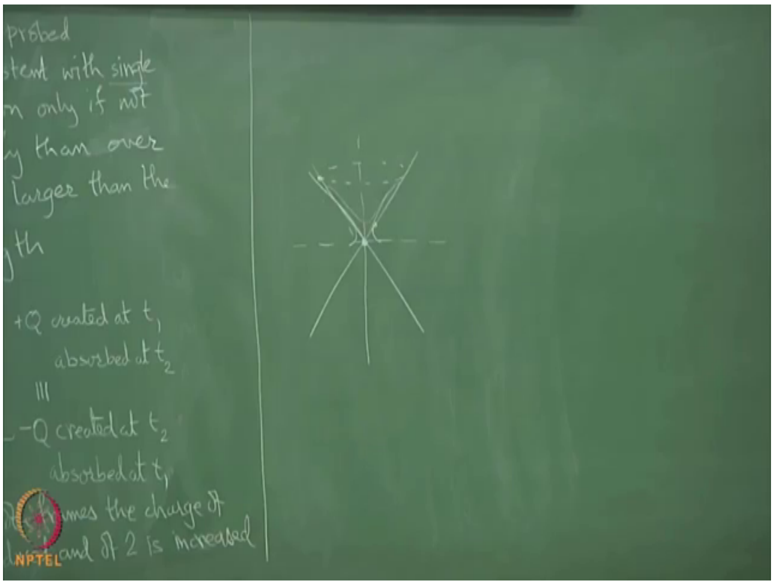


We will find that if you compute for a massive field if. So, for massless fields we will find that the support is only on the light cone. So, if you start with the origin as the reference the Green's function the Feynman propagator will have support only on the light cone only on these points out side of that it vanishes.

But, if you do it for a massive particle then you find some slightly different function which has of course, higher support here, but it also has a little tail outside. It is not strictly on the light cone. It is a exponentially dying tell just like in a barrier problem; barrier problem the wave function penetrates under the barrier. So, in quantum field theory the two-point Green's function will actually protrude into the classical relativity forbidden zone, but exactly of the order of the Compton wavelength and not more. And if things happen in that region so, if you see creation destruction events in that region they will be resolved by this explanation.