Path Integral and Functional Methods in Quantum Field Theory
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Lecture - 10
Generating Functional, Forced Harmonic Oscillator – II

So, the point is this overall things are not that important what and the all the material is here.

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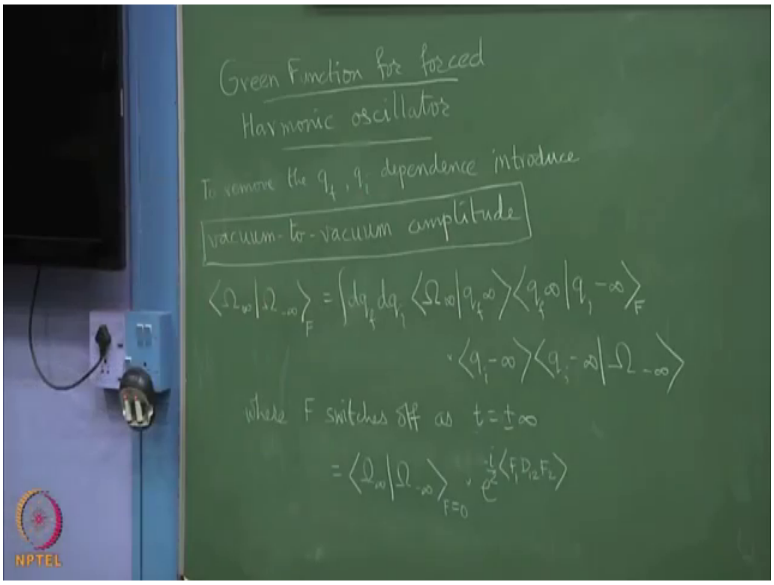

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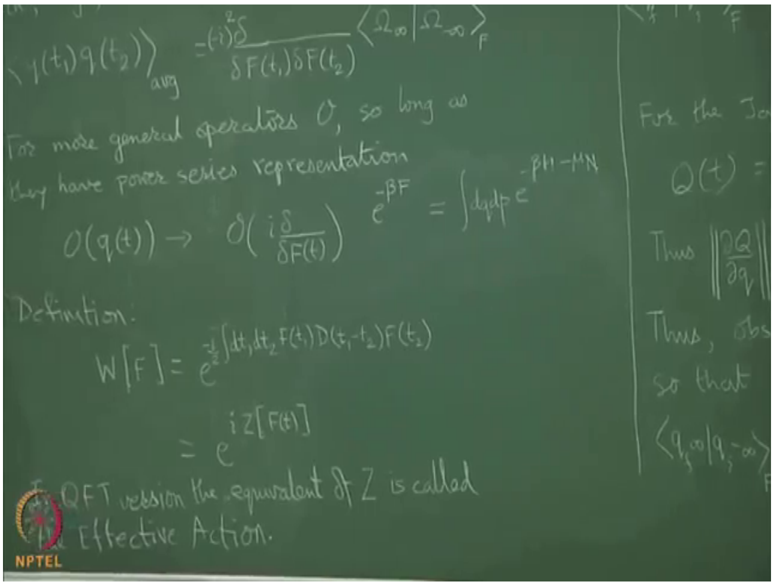

So, we define

$$W[F]=\exp[i\int dt_1 dt_2 F(t_1) D(t_1-t_2) F(t_2)]\ .$$

So, we call this expression the W[F], but now if you are astute an observant you notice that ultimately the path integral was supposed to give you this factor pulls out of everything else which is exactly what had happened, when we did stationary phase you said if it is stationary at the point where the F hits an extremum.

So, $\exp[iF(x)/\lambda]$ just came out and then the remaining term became Gaussian integral this is actually like that. What comes out is essentially what is the classical path and so, we think of this as essentially the effective classic collection ok. So, we define this to be equal to exp[iZ[F(t)]]. So, this Z[F] is actually what we call the effective action, but in terms of the current instead of in terms of the variables.

For the time being just remember that this is how this is the concepts we introduced. We call W[F] to be this, in field theory it will make more sense and however, this is effectively saying that I am also defining logW as some new quantity, but its logic is that it is whatever is in the exponent as if it was the result after you had integrated out all the dynamics and that is the dominant thing that is what you would see in the classical limit.

So, it is called effective action. Actually, you have seeing this in Stat-Mech. In statistical mechanics, say if you have the phase space thing $\int dq\, dp \exp[-\beta H-\mu N]$ then you get after you do the integral, you get the potential there. The thermodynamic potential appears in the exponent $\exp[-\beta F]$. So, things like this used in other contacts as well. I rubbed it out because I am not quite sure what it is better I am trying to give you the idea that this actually came out of doing a dq the big DQ, but after all that settles we just identify the leading term as the new effective action. So, it contains actually the information of all the quantum mechanics that is going on right because by varying it enough times, you can calculate all the correlation functions that is the point. At least for the harmonic oscillator case, the D is in fact this or this is equal to D because look at it. This is the answer in front of you.

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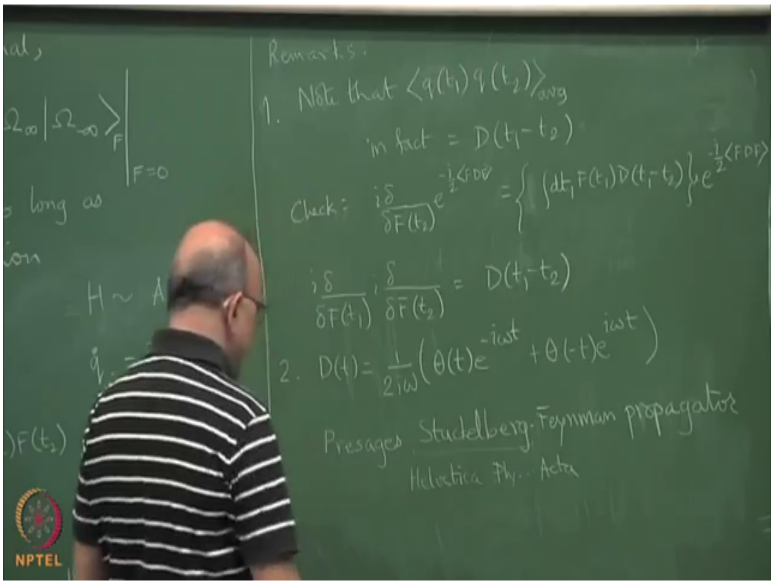

So, two remarks, one is that $\langle q(t_1)q(t_2)\rangle = D(t_1\text{-}t_2)$ because

$$\frac{i\delta}{\delta F(t_2)}\exp-[\frac{i}{2}\langle FDF\rangle]=[\int dt_1 F(t_1)D(t_1-t_2)]\exp-[\frac{i}{2}\langle FDF\rangle]$$

$$\frac{i\delta}{\delta F(t_1)}\frac{i\delta}{\delta F(t_2)}=D(t_1-t_2).$$

The averaging prescription says that you vary and then, set the auxiliary variable to 0. So, then you will get that, some details need to be check, but I think this is correct. So, this is one thing, the second thing is that this D(t) is very nicely and analogue of the Feynman propagator. So,

$$D(t)=\frac{1}{2i\omega}(\Theta(t)e^{-i\omega t}+\Theta(-t)e^{i\omega t}).$$

This expression you should memorize by heart although I have not done it. The thing is that you know the proof of this right in quantum III we have done this many time. So, you know this, you have when t is greater than 0, you can close in the upper half; if is less than 0, you can close in the lower half plane.

So, if Θ(t), then that minus sign will be inherited from this minus sign; when t is positive you closing the upper half plane, when t is negative you closing the lower half plane, so, you get this. So, this is very nicely what the Stueckelberg Feynman propagator looks like.

So, Feynman try to apply the path integral to electrodynamics, but the boundary condition was still a problem and if you are any nice guy, you will say well you would put same boundary conditions as on the electro dynamic Green's function. So, you must have done Jackson chapter 13. So, where you computes the Green's function for classical electromagnetic field, it is just quadratic; it is just $\Box \quad = 0$.

So, that $\Box$ becomes $1/p^2$, but if you just want classical electrodynamics, then you want influences to propagate only forward in time. So, the prescription you put is not a iε, but simply put both the poles above the axis. So, that you get contribution from both the poles; everything goes forward in time, nothing goes backward in time. That is what Feynman tried and he was not getting the answer. So, Feynman's where he does the calculation correctly, he sites this paper by Stueckelberg which was published in Helvetica Physica Acta, I forget the spelling, some Physica Acta. This was Helvetia as you know is Switzerland. So, now, we go to field theory.

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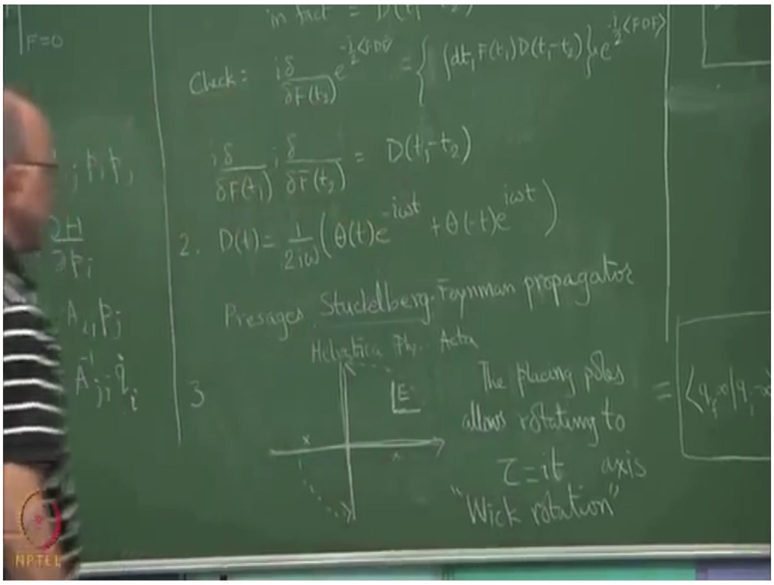

One last comment as third remark is that this particular fact that the poles are above and below right like this on the complex E plane. So, this is my notation for the complex plane of the variable. This means that this path of integration for E could actually as well have been rotated to this path and instead of integrating like this, you could as well integrate this and according to complex analysis nothing would change. Because so long as there are no poles here, this contours which are at infinity should not give anything and so long as the signs are correct. So, even if the value of t is tiny so long as there is a damping exponent you can always close it and so this contour is equivalent to that contour and that is equivalent to just euclideanising the time, instead of t you are using a new variable τ which is equal to it, is rotated by 90 degrees.

So, comment number 3 is that the placing of poles rotating to τ=it. This is a trick used quite a lot in quantum field theory and this is called Wick rotation. So, you can have some solos of a genuinely convergent Gaussian instead of having to have that fake Gaussian with an i stuck in it and so, last time somebody was asking me this question what is all this trickery, you know you go to imaginary access you do this and everything is otherwise mathematically ill-defined.

Well, the point is that we really want to express the functional dependence on these F's, this is all that matters, because this functional dependence will always correctly reproduce what these averages are and again whatever this averaging expression is, what we are interested in is the dependence on t_1 and t_2. We want to eventually calculate a correlate or a end point function in which we are interested in the parametric dependence on t_1, t_2,......,t_n and everything else is really a scaffolding.

So, in it if there are these impressionistic flaws from the point of view your mathematics, the point is we are not actually trying to compute the value of it. We are trying to use this as a generating function for a formula, this is the formula. This formula will correctly generate t_1 parametric dependence of this average value on t_1 and t_2 by doing this. In the end answer should come out right and so long as that part is right that is all you care. And so, these are ways of remembering how to generate it out of the full thing.

So, if you think of it that way, then you stop worrying about the mathematical precision of the process the procedure looks quite shaky, but this is why you do not worry because in the end it is not that you are computing a value out of it, but you are computing a function out of it, dependence of a function on its arguments. So, we can begin with how this appears in quantum field theory.

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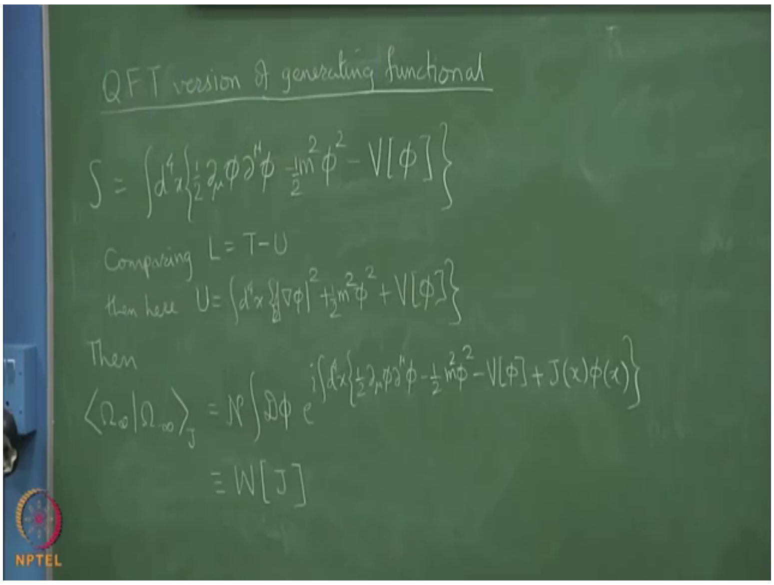

Now, the curious point is that the non-relativistic harmonic oscillator serves as the motive for the free particle in relativistic field theory. So, here we will have action which is equal to

$$S = \int d^4x \left(\frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V[\phi]\right).$$

So, the kinetic energy is just $\frac{1}{2}\dot{\phi}^2$.

So, if you think of L= T - U, then here the U expression is

$$U = \int d^4x \left[\frac{1}{2}|\vec{\nabla}\phi^2| + \frac{1}{2} m^2 \phi^2 + V[\phi]\right].$$

So, the so called potential energy part is this ok. Now if we take this and then do the usual thing, then the vacuum to vacuum amplitude can be written as

$$\langle \Omega_\infty | \Omega_{-\infty} \rangle_J = N \int D\phi \exp\left[i \int d^4x \left(\frac{1}{2}\partial_\mu \phi\, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V[\phi] + J(x)\phi(x)\right)\right].$$

So, this is how we define it and this is already in configuration space because of the reason that this is anyway quadratic in the velocities, so, does not matter. Now most of the steps go through as before and you can consider a Euclidean version by doing τ=it, so, you can always introduce a Euclidean one if you do not like this one.

We now say that this expression we referred to as W[J]. Some warning if you read Itzykson and Zuber, then the W and the Z are exchanged Isakson Jewberg call this the Z and the log of it W and I think Greiner's book also does that; somehow Raymond used in the wrong notation, but once I am reading that book, I can change the notation. So, and anyway it is a matter of symbols. So, this W[J] can be shown to be equal to,

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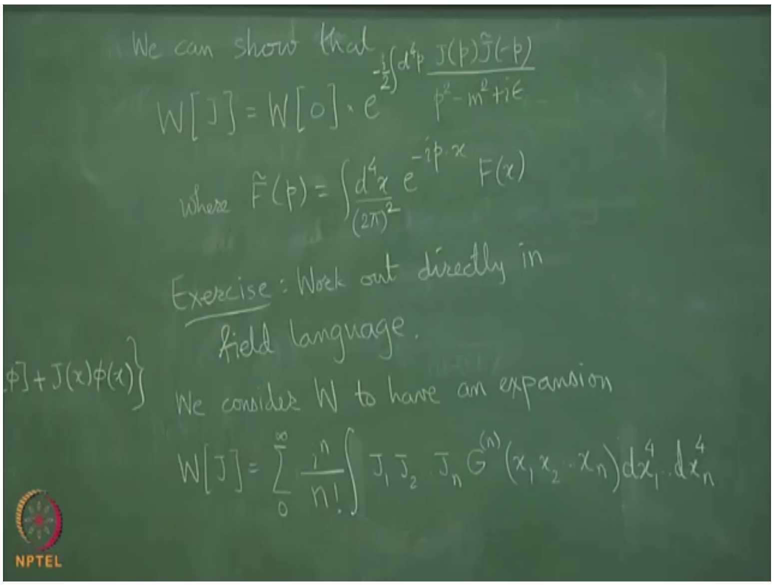

$$W[J]=W[0]\exp[i\int d^4p\,\frac{J(p)\tilde{J}(-p)}{p^2-m^2+i\epsilon}]$$

with $\tilde{F}(p)=\int \frac{d^4x}{(2\pi)^2}e^{-ip.x}F(x)$.

So, then you get this answer. You will not be too surprised because exactly like we got for the $\dot{q}^2$ if you start with $\dot{\phi}^2$ and have dt, then it will become $\Omega\tilde{\phi}(-\Omega)$, it will become product like that. So, eventually you will just and then you will complete the squares etcetera, so, you will get this. So, you can try to do it for your own benefit ok.

So, from here on starts the story of Green's functions and we can as well stop here by just noting that, we shall say that

$$W[J] = \sum_{0}^{\infty} \frac{i^n}{n!} \int J_1 J_2 \ldots\ldots J_n G^{(n)}(x_1, x_2, \ldots\ldots x_n)\, dx_1^4 \ldots\ldots\ldots dx_n^4$$

So, you can always write some kind of a power series for W as a functional of J and like multivariate calculus, this is what you would do as a expansion. The coefficients is what we call Green's functions or simply correlation function.

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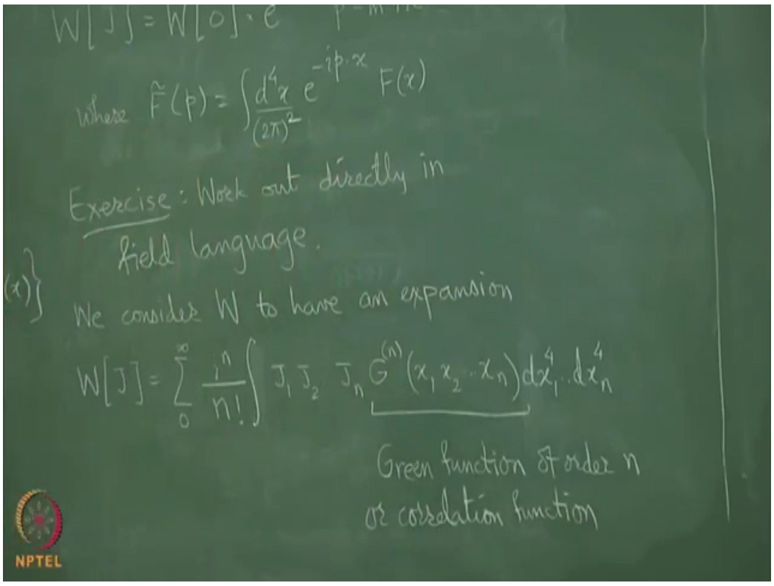

So, we will see next time that this correlation function is a bit of an overkill and can also have redundant pieces in it, but if you take log of this W like that Z that was define, then you get really the connected pieces. Right now, this would not makes sense, but let me just stop at that ok. So, we will stop with this.