Path Integral and Functional Methods in Quantum Field Theory

Prof. Urjit A.Yajnik
Department of Physics
Indian Institute of Technology, Bombay

Lecture – 01
Quantum Theory Fundamental Quantisation- I

So, you are all here to study very advanced stuff, but I am going to start with somewhat elementary stuff so that the language is uniformized. To this and I want to lay down what we considered to be the basics of Quantum mechanics.

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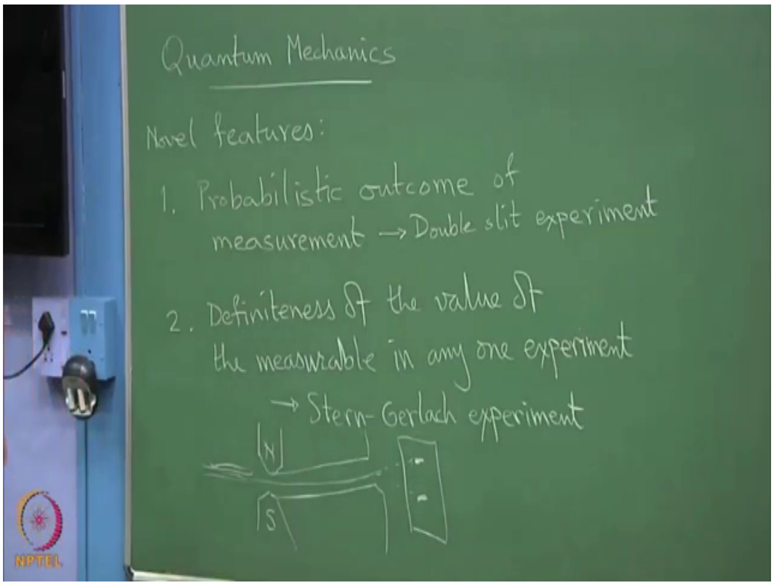

So, I will try to be fast so that you do not get totally bored, but I think that it will give you some food for thought and how the subject of quantum mechanics is organized. It looks like lots of formulae that you have, but effectively what it lies down to is the following.

So, the first things are the novel features; one is the probabilistic outcomes right, even if you have identical prepared systems or if you do repeat exact experiment the outcomes can be different. If you are an atom that is spontaneously decaying, it can emit a photon in this direction or in the direction. But if it is a let us say a S transition you would see isotropic photons only if you do enough experiments or if it was a p_y emission, you will see a p_z ; suppose you are oriented and spin correlated emissions. You will see the pattern of p wave only if you did lot of experiments. In any one experiment, you only have a probabilistic outcome. Actually, the related factors that definiteness of the value of the observable. So, probabilistic outcome about the observable, but the second point is definiteness of the value of the measurable in any one experiment.

So, another way of keeping these mind the emblemic experiments that correspond to this are double slit experiment, which in practice was some electron scattering experiment. The definiteness of values of the outcome is called Stern-Gerlach experiment. The double slit experiment, the photon could have gone through this organ through that. So, there is its probabilistic which side it came from.

But the definiteness of the value of the outcomes this Stern-Gerlach experiment which Feynman's spends a lot of time discussing in his third volume is like this you have magnets North Pole and South Pole and you send a beam of polarized particles through it. What happens here? You have a screen here. Because of the non uniform magnetic field, this beam will split. But it will split precisely into two patches. The spring will be either +1/2 or -1/2, it will not give in between answers.

Although, which answer it would give would be probabilistic. If you made the beam intensity sufficiently low so that only one particle was coming here in a given minimal measurable second time. You would find that you could get either this or that, but what you get is a precise value it is either +1/2 or -1/2, but which of the total come is probabilistic. So, those are two very hallmark features of quantum mechanics and I like to point out a little historical thing that Einstein later in his life kept expressing surprise about quantum mechanics.

But he was the guy who proposed photoelectric effect and if you think about it, everyone in his time thought that the light would come out like a wave which in then an atom emits, they would imagine some kind of a spherical wave coming out. Whereas, the Einstein was the guy who was saying it goes out like a bullet. If it does, then it violates isotropy of the process. It actually accord with these two things that were later to be confirmed by enumerable experiments. .

But Einstein's own first hypothesis, he premised on the fact that it can come at it can come out only in one direction. In any one emission process, it can only going one direction. So, it can only a precise value which violates the isotropy of the process, but you recover the isotropy in the sense that the probability of going any one direction is the same. So, these are the two slightly measure features of quantum mechanics which one has to digest, one has to learn to live with them.

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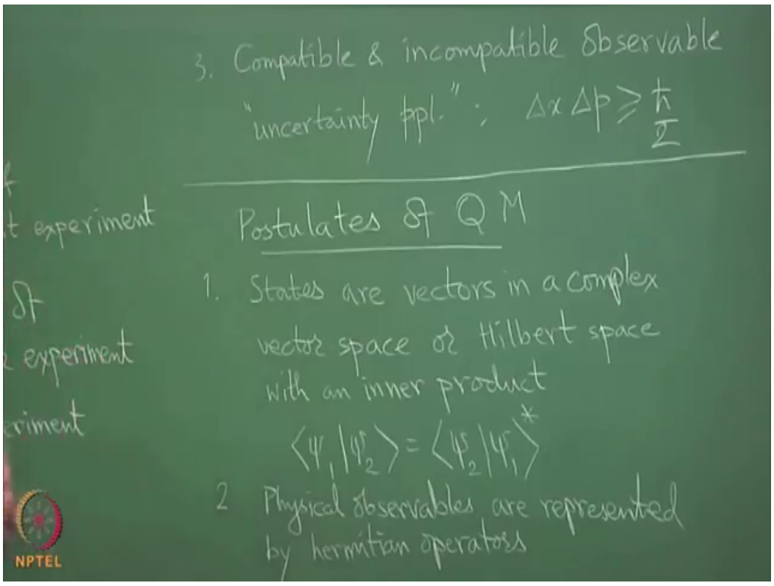

And the third thing is the mutual incompatibility. So, compatible and incompatible observables and this is what we popularly called as "uncertainty principle" because you have $\Delta x \Delta p \geq \hbar/2$. So, this basically says that x and p observations do not commute.

So, this is also a feature of quantum mechanics which probably is tied to this number 2 that any experiment that you do with x, will make it collapse into a particular value of x. So, maybe it in some way that were related, but not necessarily. So, these 3 are the general features. Next, I want to do is write down for you what I call postulates of quantum mechanics which we all imbibe just by going through the two courses, but nobody somehow less them out as simply an sequential as I am trying to do now partly.

Because it does require some mature vocabulary to do it, but let us put them down like this. The first is that and we can think of it as mathematical side and physics side ok. So, the states are vectors in a complex vector space, in fact in a Hilbert space. So, Hilbert space is when you go to the continuum of states, but otherwise it could be finite dimensional vector space like spin.

But they are essentially complex vectors in a complex vector space. What this means is that with inner in a product. So, there we laid down the notation quickly. The second thing is that all observables are represented by hermitian operators. So, again this is the physical side of it and the second half of the sentence is the mathematical side of it. It is a hermitian operator.

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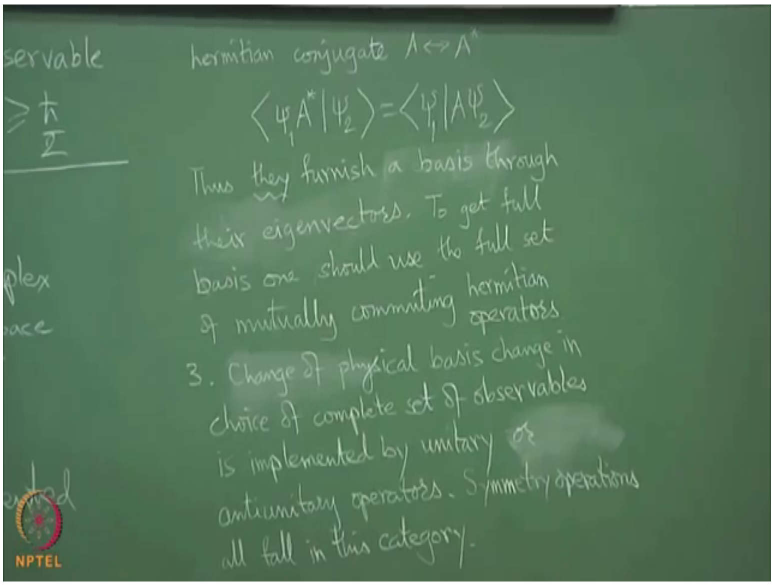

So, what do we mean by hermitian operator? So, the hermitian conjugate is defined with respect to the inner product ok. So, with respect to the inner product, the operators A, its hermitian conjugate is defined by whatever that is that if it is acted first on the left on the bra side. Then, it would give the same answer. So, I hope everyone knows that in this complex vector spaces, the multiplication is linear with respect to the ket side and anti linear with respect to bra side right.

If you multiply scalar here and try to take it out you will get complex conjugate of that. If you multiply this side by scalar and take it out, it remains real. So, this define hermitian conjugate and if it is self conjugate then it is hermitian and that is what observables are and of course, observables has specific eigenvalues. So, now, this thing leads to the interesting fact because from the mathematics side, we know that if they are hermitian; then their eigenvalues values are all real and their eigenvectors are a complete set.

So, now, here I have been little weighed, they means you actually have to find all the mutually commuting hermitian operators and that nails down the system completely. But we are coming to commutativity later, but let me just say that by they remain one should identify all the mutually commuting hermitian operators.

So, that goes back to this part compatible and incompatible. Any ways so there are some overlaps between these are not necessary logically completely sequential, but together they make a consistent set of statements. The third thing is that symmetries are implemented by unitary operators, change of physical basis or change in choice of complete set of observables.

So, if you change the set of observables through which you are going to specify the system like instead of angular momentum and energy lets say linear momentum energy something like this right. So, if you do a change of basis, then that is implemented by unitary operators. In particular actually unitary or anti unitary operators that is the term Wigner invented, so we are going to use it and in particular symmetry operations are also realized through unitary operation. So, this is a more general statement; the operation you are carrying out with not be a symmetry operation, but you are doing some operation on the system. Then, there will be a unitary operator that will connect the two ways of mapping the system, but further symmetry operations are all fall in this category.

So, the anti-unitary is like the time reversal. It involves the a complex conjugation as well. I am not getting into the detail of it right now, but that, okay. Then, the fourth is of course, the probabilistic interpretation or the this part that the definition of these two things together in a way are stated by saying that the outcome of any observation. So, you can say this postulate as collapse of the wave function.

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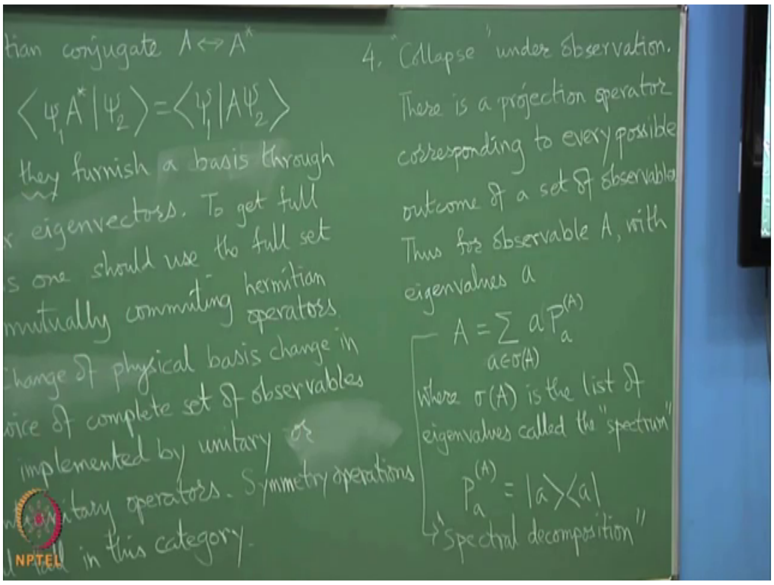

To the possibility of a particular outcome of eigenvalues, you know there corresponds a projection operator. So, we talked about hermitian operators, unitary operators and projection operators. So, there is a projection operator corresponding to every possible outcome for a set of observables. For example, if you have an observable A with eigenvalues a or the observable values a, it can be written as $A = \sum_{a \in \sigma(A)} aP$. I will write down what is σ(A) and here, we say projection operator on to the state with eigenvalues a and because P is a generic thing if you want to be sure what you are talking about you give you tell about which operator you are dealing with and where σ(A) is the spectrum.

So, when you go to advanced notions in quantum mechanics that is when you go to the continuum and the more certain issues of the Hilbert space, this is called a spectral decomposition that an operator can be written out like this; so, $P_a^{(A)} = |a\rangle\langle a|$. So, P essentially is a projection operator that this representation for an operator works has to do with the fact that of course its eigenvectors are complete and it is watertight in the case of finite dimensional vector spaces.

But when you go to continuum, then the availability of this kind of spectral decomposition is not always guaranteed ok. So, this is called spectral decomposition. The point is that if you have operator like x or p one-dimension, then their unbounded operator. Their eigenvalues list is going straddling infinity to infinity. So, then what this kind of sum means etcetera all becomes complicated, but roughly speaking this is what it is and so the fact that this projection operators are available, we can then write the statement about the probability.

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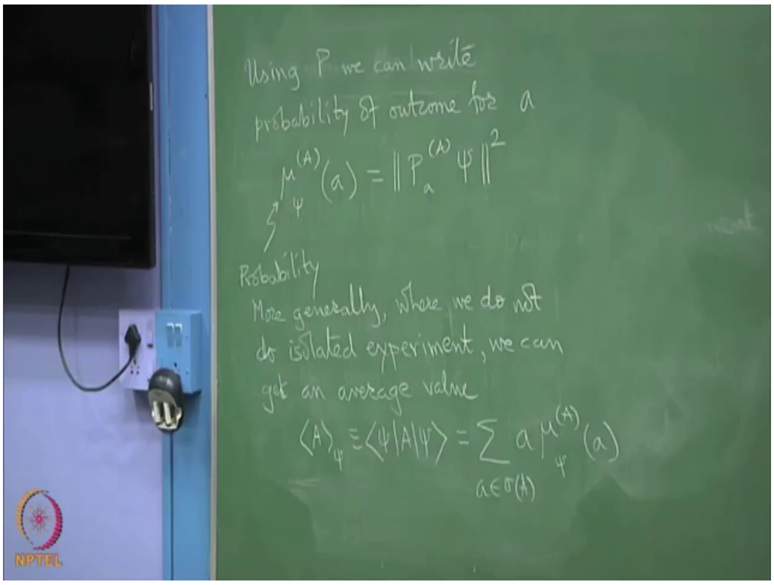

We can write familiar things. Probability of outcomes for eigenvalue a to be $\mu_{\psi}^{(A)}(a) = \| P_a^{(A)} \psi \|^2$. So, what is says the μ is the probability and μ is borrowed from measure theory. So, that is what the outcome probability of outcome of a particular eigenvalue is and in general, where we do not get; then we get a average value $\langle A \rangle_{\psi} = \langle \psi | A | \psi \rangle = 0$.

So, this symbol just help us to nail down what we are talking about, they are not any mathematics you can start computing from, but it is just a more precise way of stating what we are trying to say all stating it in symbols ok. So, we had this thing of the states they are hermitian operators, they are unitary of operators, they are projection operators which capture the essence of observation process and then we come to the more interesting parts. So, this is just the setup. So, now, we come to postulate number 5 which is quantum kinematics.

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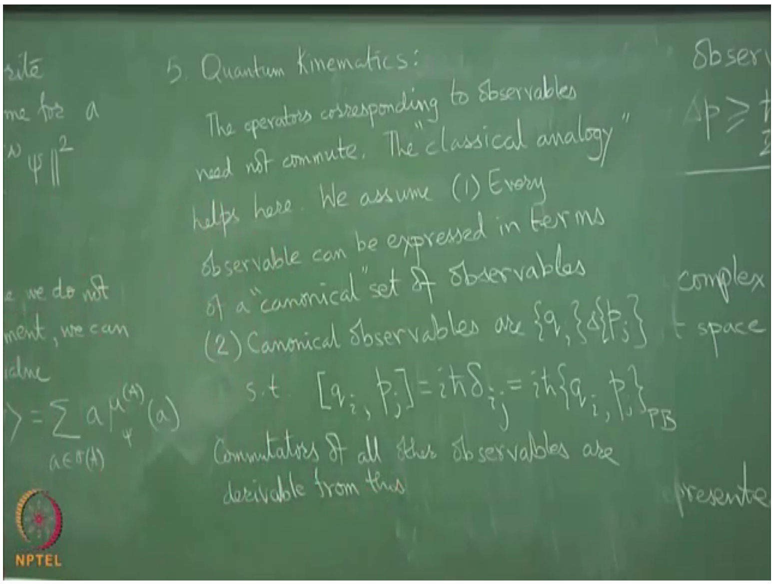

In first year physics, you know what kinematics means it is an nothing much is happening; you define things like velocity and acceleration and it is all kinematic if nothing is no energy is being added to it and a free particles are essential is what is described by kinematics. In the case of quantum mechanics, there are more subtleties because we are living in some Hilbert's space and this is where we come to the mutual incompatible observables as well, but the point is that we have operators for every observable.

So, the quantum kinematics part essentially is the free commutators is what I want to talk about, but what I want to be specific about is the fact that we do some simplification here; a very big simplification. So, operators corresponding to observables need not commute. It would then mean that for every pair of incompatible observables you will have to specify what their commutator is ok.

The grand simplification is that we assume that it is enough to specify the commutators of q and p and that all other observables are expressible algebraically in terms of them and that their commutators therefore can be worked out knowing those of p and q. So, the p, q the canonical variables. So, we work under two assumptions.

So, we assumed the availability of such canonical set in terms of each all other observables can be expressed. So, this as you remember where we call it classical analogy is that this is same as Poisson bracket, the corresponding classical Poisson bracket ok. So, the commutators of all other observables are derived from this.

Now this itself is a slightly big assumption because firstly, the ability to represent. So, there are ambiguities in this process because of the operator ordering. So, if you are some opera like angular momentum, if you which involves r × p, but we know that the r and p's do not commutate the indices are same. So, there is always a slight doubt has to how you define such quantities and that is called operator ordering ambiguity.

But what I want to emphasize is that technically all you had to do was gather enough experience about the commutation relations of all the individual operators and then, the ambiguity arises not because of some problem with quantum mechanics, but it is because of the grand convenience you try to derive from the availability of observables that in classical limit will become the classical cannonical variables ok. So, we are now approaching the end of the list of the postulates. The next is the dynamics which is either Schrodinger or Heisenberg picture and the last part is the most important part.

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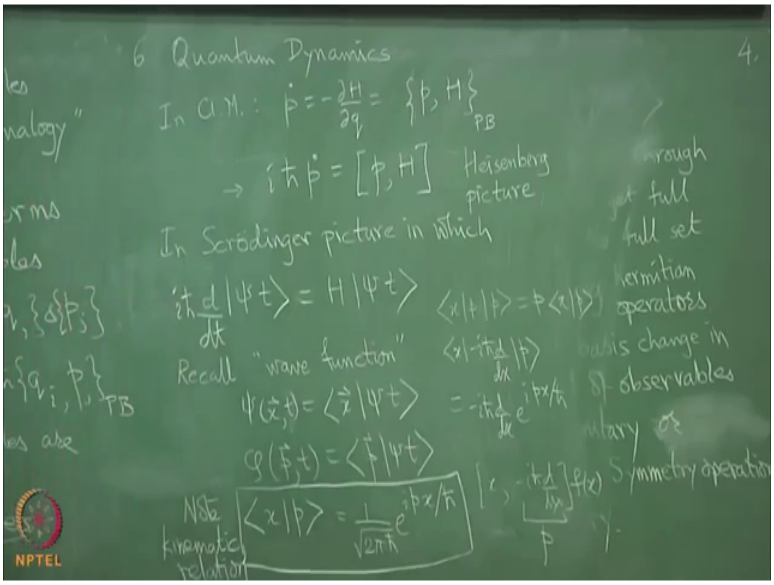

So, number 6 and this works in an analogy with classical mechanics, but in classical mechanics we have $\dot{p} = \frac{-\partial H}{\partial q} = \{p, H\}_{PB}$. So, we have to do the same thing here accept that we replace the {p, H}, Poisson bracket by the commutator bracket and divide by $i\hbar$. So, $i\hbar p = [p, H]$. So, this is Heisenberg picture as you know and there is the

Schrodinger picture in which in which we write $|\psi\ t\rangle$. So, I am using the notation that dirac uses in his book.

So, ψ is the generic state and you show its stand dependence for putting a t. So, $i\hbar\ d/dt\ |\psi\ t\rangle = H\ |\psi\ t\rangle$. So, this is a Schrodinger picture state and. So, just to explain you the notation recall that "wave function" $\psi(x, t) = \langle x|\psi\ t\rangle$ is to take the generic state $|\psi\ t\rangle$ and then, project it onto the x space. So, this is how direct delineates what this mysterious thing is, it is actually just that we are looking that it in the x basis.

Often you have seen also $\phi(p, t) = \langle p|\psi\ t\rangle$. If the simplest system of point particle in one dimensions; there are only to observables position and momentum. Of course, you can construct the energy operator which will be p^2. So, no ambiguity is there, but and we know the that x and p are mutually incompatible and we know that if you list all the momentum eigenstates, then you can list the system completely because energy is just dependent variable. So, you can have either x basis or p basis for a point particle in one dimensions and these are the corresponding wave functions and incidentally this also reminds me that instead of writing this q, p, we can also write $\langle x|p\rangle = \dfrac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}$.

So, this relation is equivalent to this canonical commutator what in the kinematic statement of in terms of commutator is exactly in terms of states. It turns out to be this because you can see that

$$\langle x|p|p\rangle = p\langle x|p\rangle\ ,\ \langle x|-i\hbar\ d/dx|p\rangle = -i\hbar\ d/dx\ e^{ipx/\hbar}.$$

So, you recover this relation from setting the x, p overlap to be this and this will reproduce the x, p commutator correctly. Let it act on some function of x. Then, this represented this is correct representation for p in x space and that is why this overlap basic or so this is as fundamental and statement as saying that the canonical commutators are these, they are equivalent. So, all this is well known step, so, I am just going fast and the last thing which I personally believe should be always taught along with all quantum mechanics.

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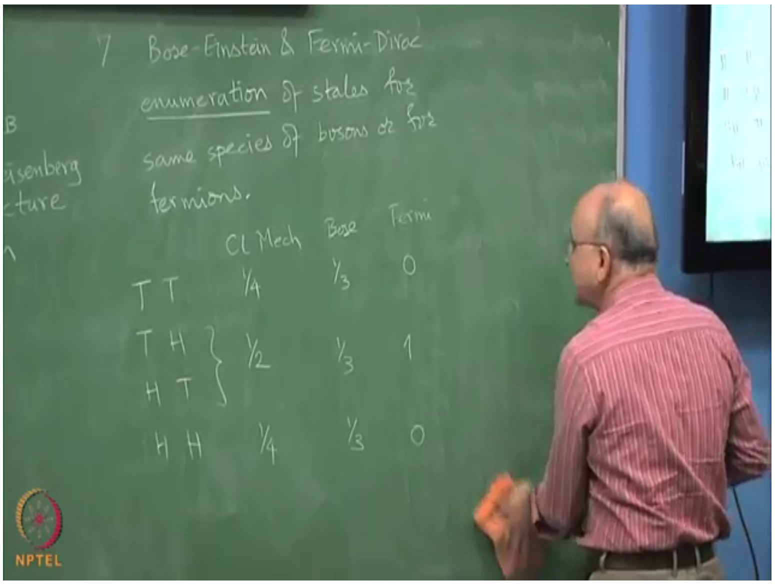

But is somehow postponed indefinitely to tail end of quantum 2 courses is the Bose or Fermi statistics and the correct thing to say is not the word statistics strikes fear in the mind of most people who like precise science, so suddenly the thing there statistics it is there is nothing statistical about it; it is correct enumeration ok. So, Bose Einstein and Fermi Dirac, it is more an enumeration process than any kind of statistics.

Now, I personally think that unless this is done upfront from the beginning it does not bring out quantum mechanics correctly. You will find that most discussions of quantum mechanics only deal with 1 to 6 ok. Especially all the mathematicians spend a lot of time worrying much more about the how you go to the continuum and how the unbounded operators make sense and all that. But I think the heart of real quantum theory lies here. It is this bizarre enumeration of states that one does we changed our perspective completely and the reason why the word statistics comes is because it was done in the context of thermodynamics first.

So, this is completely elementary statement. But excuse me for repeating it, but it is important enough that we say it. Suppose you toss 2 identical coins, then you have 4 outcomes. In classical mechanics, we have the probability for this is a 1/4, probability of this is 1/4 and this together is the a 1/2 because all though they are indistinguishable that number of times this will turn up is half of the times.

But if you have Bose statistics, then it is exactly 1/3, 1/3 and 1/3 right and this is what is the strange thing about really the strange thing about quantum mechanics that the two are quote indistinguishable they are so indistinguishable that actually the word indistinguishable is a misnomer. If there is in principle possibility of distinguishing something, then you can later say there in distinguishable. But that putting a negative indistinguishable is actually wrong; what it says that the always have only completely symmetrize states and it is impossible to tell them apart, tell one photon apart from another ok.

So, identical particles is in some sense better than saying they are indistinguishable because indistinguishable means that they are identical billiard balls, but my manufacturer was so clever that I cannot observe them. But this is a matter of principal, it is not that somebody really polish them very well. So, that they look the same they actually are such that they produce only one state together and fermion is even more blizzard because it is 0 and 0 and 1 right; only they cannot be in a identical.

So, assuming this is only quantum number. So, if the two of them cannot have the same quantum number and they will have to be in this state and they will obviously, be in the anti-symmetrized state. So, it is this counting this the way to enumerate the states that makes quantum mechanics so completely different and I think that most quantum mechanics is lost upon you, if you keep worrying about this and all the general public in the general world worry about. Of course, they have got to this by through the socalled Einstein Podolsky Rosen Paradox which I think is daily life thing for a quantum system. But it has to do with this as you know if you.

The system remains correlated even over a space like distance because the the problem with that is that they think that there are two distinct photons there which are become indistinguishably entangled, but there is no such thing. From the Hilbert space point of view, there is one state. The number 2 is a psychological thing; the number 1 is the correct enumeration of state. But since you think that there are two now we can play many mind games say I observe this one and then, this collapsed and this all this is really good amusement.

It is a practical way of thinking, but I think that there is nothing fundamentally paradoxical about it. That is how the enumeration rules of quantum mechanics are. But it has become a very big issue because you do have states that are spread over space. So, the point is in quantum mechanics, if you think in terms of quantum physics in space and time, a quantum state is intrinsically and always delocalized. In classical mechanics somebody says there is a ball or there is a particle, it is in some point in space time. Quantum mechanics is not like that ever that is the main lesson. You have a system; it will be spread over the space over the basis states enumerated by its set of observables.

Here the observable being location. Generic state will always be a linear combination of a lot of special eigenvalues, eigenvectors and therefore, it looks like its delocalized system, but whatever that non locality is valid, it is actually physically correct. It does not violate relativity principle and is the postulate number 7 of quantum mechanics. So, you can either call it a postulate or you can call it a paradox. The second one will provide more amusement than the first. So, that is the end of my summary of quantum postulates.