**Theory of Group for Physics Applications**

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**Lecture - 09**
**Cycle Structures & Molecular Notation - I**

So, we will do three things today, in three parts and you can remind me to stop a little bit after the third part. The third part will be the fun part where we will bring up the animations from this public website and look at some of the molecular motions. But I wanted to continue with. So, let me start by calling it regular representations or regular permutations.

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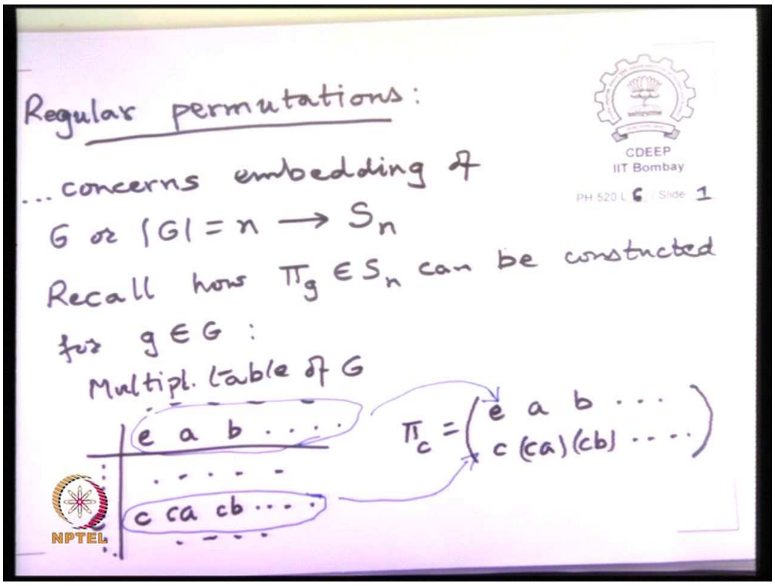


So, this actually I personally find the little difficult to register things just because you call something regular; does not suggest any property, but you just think about it and see why this is important. So, this has to do with embedding of the group of order n into $S_n$ ok. We said that this was a homomorphism. Remember, we can map the elements of any group of order n into the large permutation group $S_n$ .

So, it concerns embedding of G or $|G| = n \rightarrow S_n$; the symmetric group of order n the permutation group.

So, what does it concern? So, we so, remember the properties. So, recall; how we do this map, can be constructed for $g \in G$. And the process was, look at the multiplication table.

So, we look at this table which will look something like this e a b…..‥ And then there will be some element called c, so we have a some ….; c which of course under identity will remain c and then begins to do something to this. So, I will just write ca, cb ,…... That is what it does, right, it is a product of c and b that row c and a c and b.

So, we are looking at a and it can continue, of course, below. Now, the way to construct $\pi_c$ is simply to first fill the thank you; just for change of color, what we will do is write the top line to be this, we simply write out the unaffected ordering of elements here and then pick the line corresponding to the multiplication corresponding to the row of the element c in the multiplication table and put it as the second line.

That's all you have to do. So, it will become c ca cb, etcetera, of course, we are writing a composite symbol it will become a single symbol once you know what it is. So, the point is that essentially the element is constructed by lifting the row out of the multiplication table this endows this particular element of $S_n$ with certain properties which are special, that if you have a very general $S_n$ element, there may well be lot of elements which are not touched right, if I have permutation group of 15 elements, a particular permutation may affect only the first n and not affect the others.

So, things like that are possible in a $S_n$ group. So, it is affecting only some subclass of the elements in that group, but when you when we are looking at this particular sub group of $S_n$, every single element n is always touched because of the group property of G.

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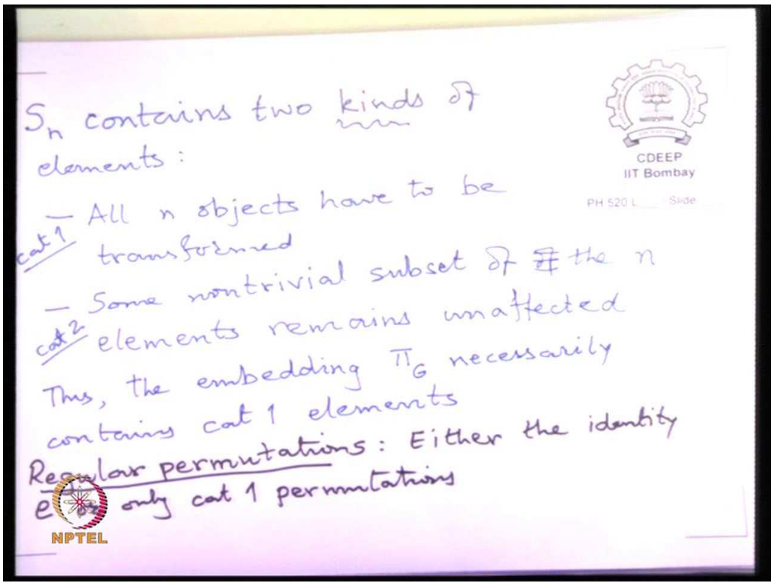


So, due to so, $S_n$ can be thought of as containing. So, contains two kinds of elements, we define what kind one where all n objects have to be changed or transformed and one where some subset of this is the career space we are talking about. So, some subset of the career space n remains unaffected. So, what we observe is that our attempt to represent a specific group of order $n \rightarrow S_n$ , necessarily brings out those elements from $S_n$ where all elements have to be necessarily touched.

The embedding $\pi_G$ necessarily contains category 1. So, let us call this category 1 and category 2 do you agree with this is because you remember that if you have a group multiplication table of group of order n, then any row of this has to necessarily shift things around, it cannot say it cannot give cb = b, because then c would be identity element. So, if c is a non trivial element its multiplication with e will be itself and it is necessarily going to change every other element into some other element. And therefore, this kind of permutation necessarily has all the elements changed all the n elements $S_n$ is some; so n elements of the career space on which you are doing the operations. So, the big group $S_n$ therefore, can contain large number of elements you are not interested in where some subset of n remains unaffected.

In fact, those will be elements of $S \supset M$ where M is some smaller permutation group right, if I have 15 elements, but I permit only 10, I am effectively looking at $S_{10}$ not $S_{15}$. So, it will contain all kinds of other elements, but the particular embedding of a group of size n when reflected as $S_n$ acting on career space of size n, it necessarily can fix out those elements from $S_n$ which necessarily are category 1. So, now, we define what is called regular representation or regular permutations I am sorry are this category, so either the identity e with changes nothing at all or only category 1 permutations.

In which every element has to be touched and transformed into something else in the career space this kind of permutations have one important property which it sort of. So, this is a sufficient condition not a necessary one, but the sufficient condition that these obey is that; their cycle structure will come out to have cycles of exactly same size.

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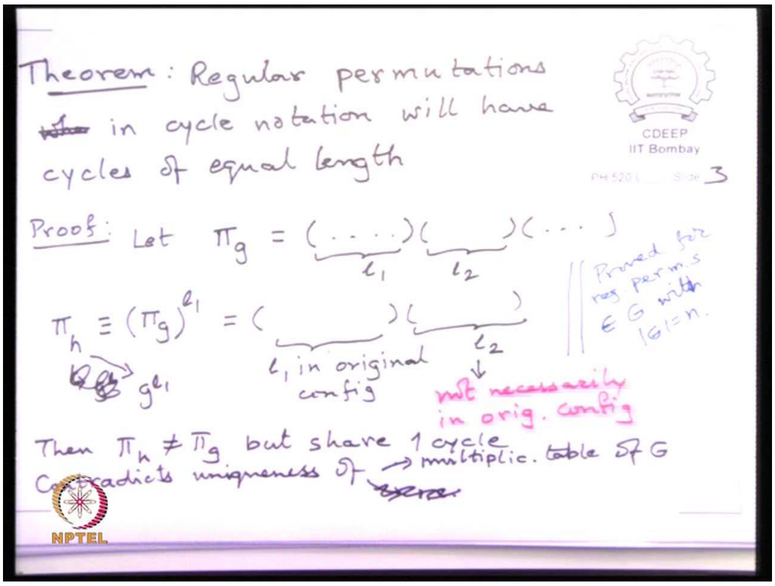


So, this is the theorem, we are now trying to prove regular permutations in cycle notation will have cycles of equal length. So, this is an interesting theorem to prove and the proof goes something like this, suppose I have some group element g. So, it is now of course, represented through this $\pi$ suppose $\pi_G$ contains one cycle of length, $l_1$ . And another cycle of length $l_2$ and then whatever else there is, but suppose, it contains two cycles of unequal length. But now the point is if I raise this $(\pi_g)^{l_1}$ , then what will happen is that this cycle will have cycle through $(l_1+1)$ times and will have return to the original configuration.

So, it is equal to this the first $l_1$ in original config, but what happens to the others? So, if $l_2$ is an in compatible number or a larger number. So, if $l_1$ is 8, if $l_2$ is 4, well, you are lucky because it will also a cycle through enough times, but if $l_2$ is 5 or is 59, then after doing power $l_1l_2$ is not going to return to its original form. So, you have reach the contradiction that you have two elements in the set $(\pi_g)$ and $(\pi_g)^{l_1}$ such that the share one cycle, but we already know that this is not possible unless the two elements are identical, you cannot have some elements unchanged and some not changed and have two different elements right because each element each so this, suppose we call this $(\pi_h)$ ok; some new element which is where h is actually $g^l$. So, it will be represented by .

So, this is $g^{l_1}$ , therefore, in the $\pi$ presentation, it will be represented by $(\pi_g)^{l_1}$ because of homomorphism multiplication property, right. So, you understand that then these two elements $\pi_g$ and $\pi_h$ will share one cycle, but not share the other cycles, but this is not possible because it will mean that there multiplication table row, we will share some elements, but not share some other elements, but this is not possible because every row in this is unique.

So, if there is a contradiction. So, this is not necessarily in the same form. So, then there are two elements $\pi_h \neq \pi_g$ , but share one cycle. So, these contradicts uniqueness of or the property or the category 1 property of uniqueness of G multiplication table of G, you can think of it in two ways or you can think more correctly that it contradicts the category 1 property that all objects have to be transformed and yeah the uniqueness is required you do need the uniqueness property ok.

So, i.e I will just say contradicts it does not contradict category 1 property category 1 is slightly more general thing as we said that is a sufficient condition. So, what I have not fixed is the fact that all regular permutations a set of regular this is actually claimed for any regular permutation without checking that; it is member of a group, but at least I have proved when there permutation is a member of group G of size n, it is certainly true. So, in any case this proof works, even if I made a mistake in writing $l_1$ instead of $l_1+1$ the fact remains that if the other cycles are not compatible with that length, then this, you will get a non unique representation.

So, what; this is a very powerful theorem which says that when you embed an element of size n a group of size n into the permutation group or symmetric group you will automatically get groups represented by elements represented by cycles of equal length and the smallest example of this group of size four the $C_4$ in $C_4$.

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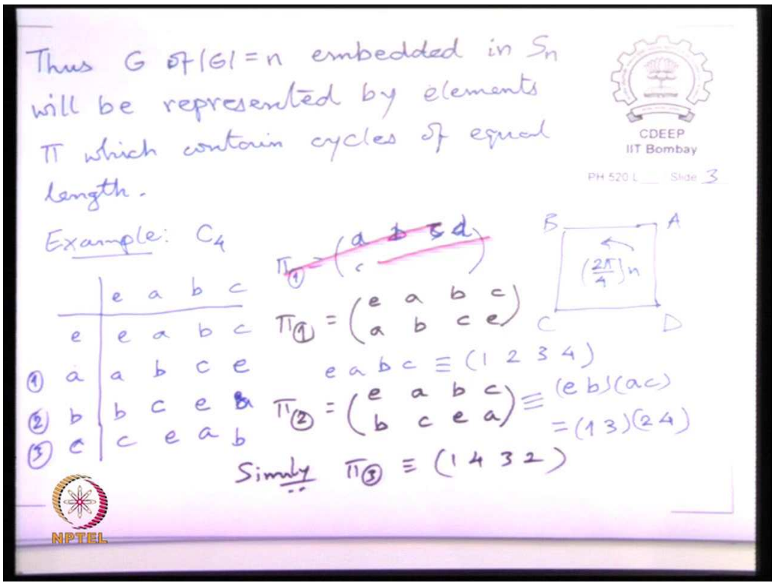


So, considered $C_4$ did not we call it $C_4$ s. So, I have a square ab c d and I am basically considering $\pi/2$ rotations at a time. So, $2n\pi/4$ is the possible rotations.

So, let us draw the multiplication table that is the simplest way one can see probably geometrically, but I just find it easier to do this like this here just now and this was the case in which basically b. So, a is the $\pi/2$ rotation and b is the $\pi$ rotation. So, our rotation was true, its square was equal to identity, but the others did not square to identity. So, $a^2$ actually became b you know $2\pi$ rotations becomes $2\pi/2$ rotations becomes a $\pi$ rotation. So, aa is actually b, similarly, cc also becomes b because $3\pi/2$ rotation and another $3\pi/2$ rotation.

So, it becomes b. So, this is essentially what the multiplication table is. So, we got ab, then I have to fill here c and I have to fill here e because a and c are inverses of each other $3\pi/2$ and $\pi/2$. So, e here and a; so, here I have to put the c and here, i have to put a and a. Now, let us write this out a cycle. So, this of course, is the identity element this is written as. So now, if we represent this call this 1, 2 and 3. So, let us write $\pi_1$ according to rule, we have devised and then we look at the row of 1 with says a b c e.

Now, how do we write this in cycle structure e → a, a → b, b → c and c → e. So, it is of the form e a b c, but which is same as what we would have written in the 1 2 3 4 notation, it is this next let us write $\pi/2$. So, what do we get with $\pi_2$ it splits up into 2 cycles to 2 cycles e → b, but b →e. So, there is an (e b) cycle and then e → c, c → a. So, there is another cycle. So, in our note, we can write this is equivalent to first visible (e b) and then write (a c), but it is also equal to in our usual notation, this we would have called 3 and 2 with 4.

So, something like this. So, it is now broken up into two cycles, but again two cycles of same size. Now a example of size 4 is not a very general one, but we do see that this is what happens is there we get cycle of size 4 or we get cycle 2 cycles of size 2 ok. So, we can just quickly check what happened can we check what happens to $\pi_3$, right. So, it should be of this 4 cycle form. So, similarly $\pi_3$ is also of what is the cycle (1 4 3 2). So, we got cycles of equal size. Now this is the very important. Similarly, I can tell you right away, if you do the other order four group.

The Klein group which if you remember had e along the whole diagonal Klein group will produce all pairs 2 cycle pairs, it will have no 4 cycles in it. So, you can check that let us try to write it in table form and then see what we get.

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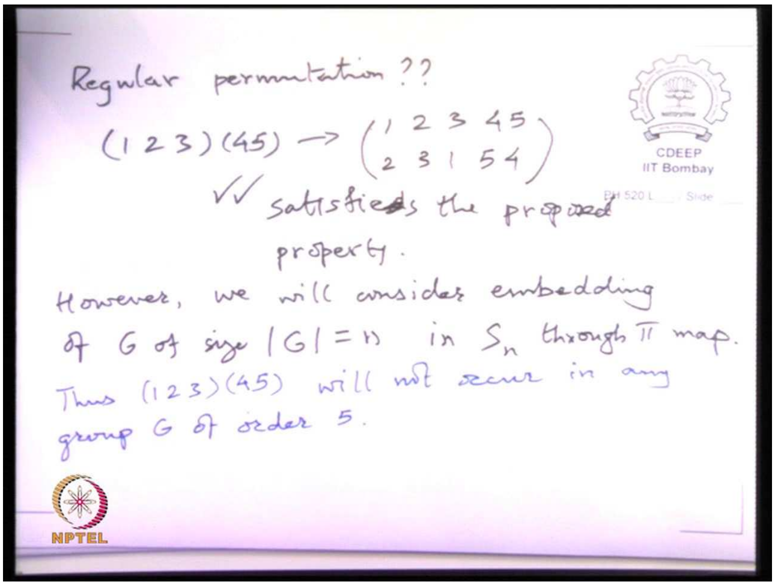


So, check this particular. So, regular permutation I am not completely sure. So, I will just check it in front of you. So, I have (1 2 3) and then (4 5) in the usual notation, we will write it out as $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{bmatrix}$. So, I do not think there is anything wrong with this satisfies the proposed property. So, I think I have to restrict myself to we will consider embedding of a group size n → $S_n$ through $\pi$.

Then for that this facts are correct as you know I had 2 here, I was about to write, then I didn't write. So, this is verified for all the regular permutations that occur in representing a group. So, at least check for approved for which belong to a group G with |G| = n and because if G has to, if the set of these regular permutations have to become elements of G, then this multiplication table imp restriction is a powerful one; that each row has to be unique. So, what will happen is that this particular thing that we checked will not square to the will not represent any group element, we will not occurring any group of order 5 that much I can guarantee you because you take any group of order 5 and try to embedded in this, it will make this will necessarily this argument will then apply.

That I will I have incompatible cycles. So, if I took this and just squared it I do this and then I do it once again, this these two will come back to themselves, but this (1 2 3) cycle will not have completed. So, it will amount to having two distinct elements in this table whose two of the elements remain same while the other 3 are not same. So, it will not be a unique representation and here the argument is quite water tight that you know from basic axioms of group theory, it is impossible to have any repeated element in any row and it is impossible to have any two rows having any identical element in any column because then it means one of the elements is identity.

The way we defined by just requiring that every element is touched is a little to generic that is the broader property, but conversely a group of size n and when embedded in $\pi_n$ will necessarily be represented by regular representations, but it does not mean that every regular representation will occur in a group ok. As I said, the word regular by itself does not register as what it what are the property is it captures. So, there are nomenclature problems.

But as for you so long as you try to embed a group of size and into $S_n$ you will find that these properties are correct that it has to break up into any cycle representation will be cycles of equal numbers. And in fact, that brings us to the interesting point that at least for small size groups where you do not have some n! to worry about, but up to 5-6 size groups. You can in fact deduce all possible groups by this requirement in reverse, because if you want to write a group of size 5 you are force to have only one group, because you cannot break up 5 into equal cycles.