Theory of Group for Physics Applications
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Lecture – 08
Factor Group Conjugacy Classes – II

Since we will be using the using the permutation group extensively; it is useful to define something from within permutation group first; which is called cycle structures. So, we are going towards refining Conjugacy Classes maybe we write it down first and then no, but it requires the proving and so on. So, we can do it in either order; let us do the conjugacy classes first; it is a very general idea and it requires some proving.

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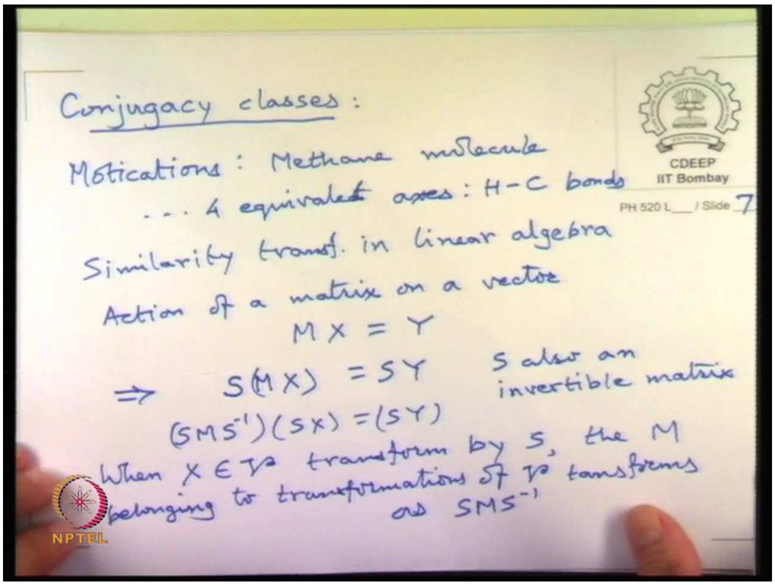

If you find this discussion to abstract then; we will immediately be seeing examples of this in terms of the permutation group. We can also motivate it in another way; so one motivation was like the which one did we say not ammonia methane ammonia has similar thing, but it has different structure ok. So methane molecule, there are 4 equivalent axes joining H and C.

The other way of thinking of conjugacy classes is what we call similarity transformation in linear algebra. Well, I apologize the bit here because, I am talking almost like a mathematicians. For mathematicians examples are more things from within mathematics. Ideally we should have some you know concrete thing to give in terms of, but let me say

I am drawing on what you may be already familiar with or what is little more familiar to us already; which is where a linear transformation X on a vector matrix . So action of a matrix on a vector which is a linear transformation on, so, suppose we have MX = Y and now we if we think of this geometrically then X has the notion of a vector which we can assign a physical meaning as like a position space and M is some kind of a rearrangement of it is components, but if it has to have a independent meaning then we can think in terms of the basis and a the action is like changing the basis, changing the basis in terms of which we were expressing this. So, to cut long story short let me just tell you what I have in mind; suppose we multiply it on the left by some transformation S which is also a matrix and we will say non degenerate invertible matrix; one that has an inverse.

Now, what we can do is we can rewrite this as (SMS^{-1}) (SX) = SY ok. So, if the vector space is transformed using S into from to '; I mean it is an it is a internal transformation automorphism sometimes called; then correspondingly the matrices have to be transformed by SMS^{-1} ok. So, transforms as SMS^{-1} that is the statement. Then the geometric meaning is preserved, it just that you went to some other frame which was S times the original, but physically what is happening is the same.

So, this is meant by a conjugacy relation. So, this is exactly what we are going to propose for group elements because group elements although we are writing them as if there an independent algebra; we know that secretly they are always going to be realized as operations on something. So, they are actually in the class M group elements. So, we proposed for group elements conjugacy relation.

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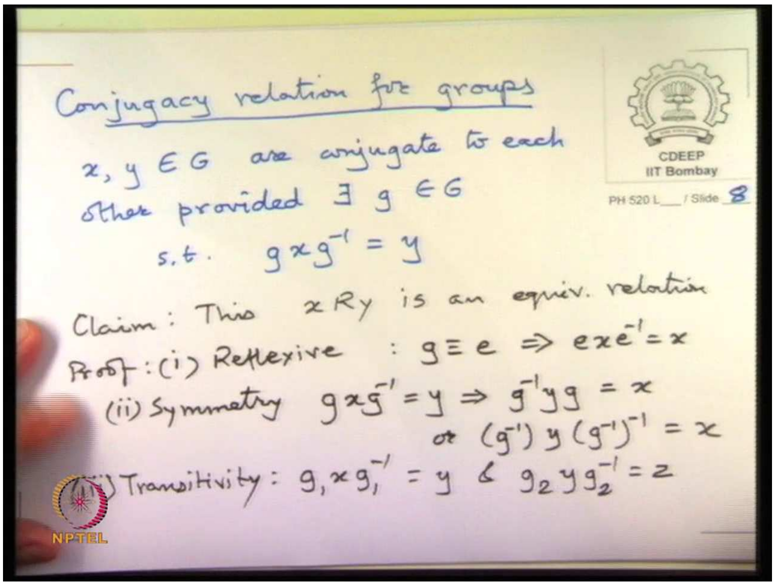

We say that x, y $\in$ G; I jump to some other symbols are related are conjugate to each other provided; there exist a g $\in$ G such that $gxg^{-1} = y$ and we can check that this requirement is a equivalence relation and I am sure you already started doing it in your head checking why this is true, can you write it down what are the things to be checked? So, we can check that Reflexivity holds $g \equiv e$. Symmetry, so g^{-1} is the required element and for Transitivity suppose $g_1 \, x \, g_1^{-1} = y$ and $g_2 \, x \, g_2^{-1} = z$; then we can see immediately that $g_2 \, g_1 \, x \, g_1^{-1} \, g_2^{-1} = g_2 \, y \, g_2^{-1}$ right because I this is nothing but y.

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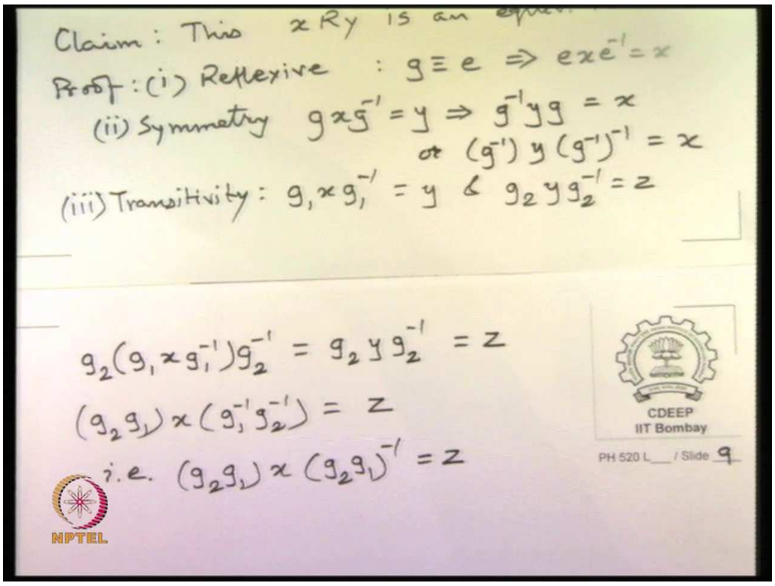

But then I rearrange this to write it as $(g_2 g_1)(g_2 g_1)^{-1} = z$. So, that is verifies transitivity. The beauty of this proof is that it uses up all the properties of the group G; group G has to have the identity element ok, it has to have inverse. So, that inverse of inverse is itself and associativity is very crucially used here, if I have g_2 times this then it becomes $g_2 g_1$.

So, all the closure associativity, identity and inverse are all used for proving this equivalence relation. The result therefore, is what you would have expected; this conjugacy relation is an equivalence relation and therefore, it divides the group G into conjugacy classes.

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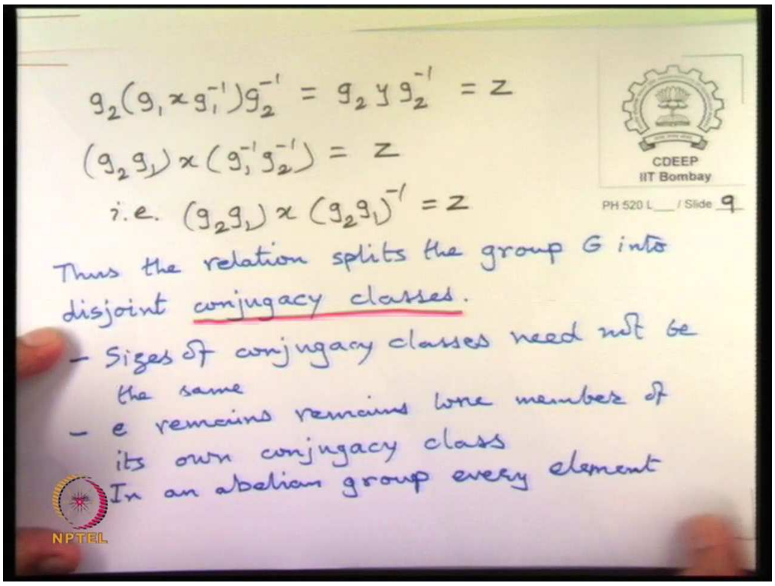

Now note that the conjugacy classes are not of the same size; they are all different, but of course you can recover the whole of group G as union of the conjugacy classes. The element identity is in it is own equivalence class, is it is own conjugacy class because it is it will never get related to anything else, e will always related to itself; if you put geg^{-1} you will get back e regardless of what g you used.

So, e can never be converted into anything else by this kind of transformation and so e will always constitutes it is own. So, size of conjugacy classes need not be same; size is what is what I mean. Sizes of the different conjugacy classes need not be the same, identity element remains in it is own conjugacy class.

More generally if you have an abelian group then in an abelian group every element remains it is in it is own conjugacy class, because in an abelian group any operation like this everything commutes. So, x will remain x regardless of what g you use. So, in an abelian group every element is its own conjugacy class.

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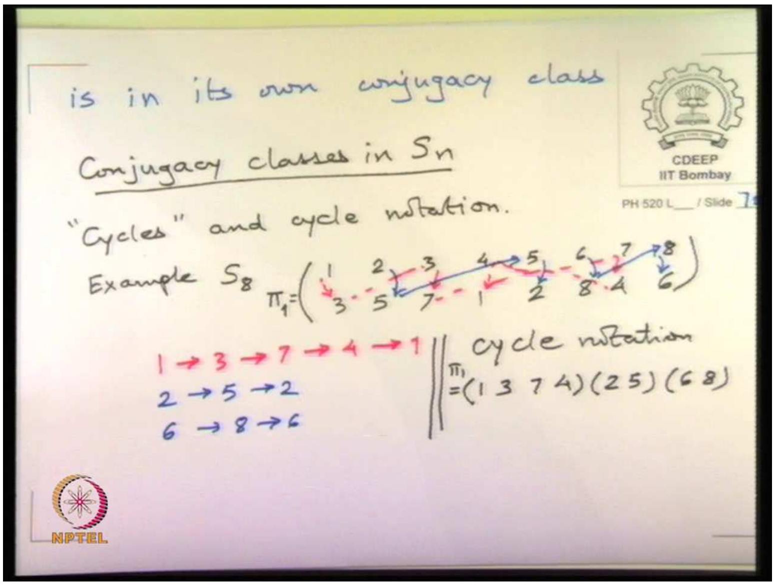

But we will see that; for the non abelian groups the conjugacy class proves to be a very powerful idea because you can in fact, think of only the members from each. So, you can think of much fewer elements instead of having to think of all the elements, all the elements belonging to the same conjugacy class you can treat as essentially 1. 1 representative from it will be enough for thinking whatever you are trying to think; also we will see some very useful technical things which have to do with the representation; when they get represented also there is a uniformity in the representation.

So, now we will see an example. Unfortunately as I say a technical example of conjugacy classes in the case of permutation group. So, let us return to the symmetric groups; for this purpose we first develop the idea of cycles in a permutation group. Quite simply put we let us start with an example. Suppose we have S_8 and I pick $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 7 & 1 & 2 & 8 & 4 & 6 \end{bmatrix}$. So, this is some element of S_8; it says the original order is permitted into this order.

Now, we see what happens here is that 1 → 3, but 3 → 7 and 7 → 4 and 4 → 1. So, we have this sequence 1 → 3 → 7 → 4 → 1. So, this is like a sub permutation; permutation on a subset and this thing did not involve any of the other elements. What you can see is that if you square this element, take square of this element multiplied by itself all it will do is it at slided 1 → 3 first, now it will slide 1 → 7 and it will slide 3 → 4 ok, but it will basically slide these among themselves. If you take higher powers of this particular element of S_8 all that will happen to this cycle is that the cycle will continue within itself ok. It did not touch any of the other element so, let us see the fate of the others, so 2 → 5 and 5 → 2.

So, 2 → 5, but 5 → 2. So, that is another cycle and now we have covered 1 2 3 4 5 and only 6 and 8 are remaining and obligingly they are going into each other. So, 6 → 8 and 8 → 6. So, we have one more cycle which is 6 → 8 and 8 → 6.

So, we write the cycle notation then is to identify such cycles and then only write (1 3 7 4) (2 5) (6 8). This is what we write; so the same element if you want to call it let us call this $_1$. So, $_1$ of course, the standard way of writing is this long way, but we can also write it out simply by its cycle structure. It is the element that is going to send 1 → 3, 3 → 7 and other ok. So, this is another way of writing it.

And you can see that if I square this element what will happen? It will just slide this among themselves again; 2 → 5, 5 → 2, again 2 → 5, 5 → 2. So, higher powers of this will maintain the cycle structure, it will not alter the cycle structure. So in fact, that is the I mean I wrote is an example, but the more formal way of defining cycle structure is to define it like this.

You say that take an element of the permutation group and if you find that a to the power some and look at one of the objects it is action on one of the objects and then look at the action of higher powers of on that same object, because the whole set as is exhaustible it is finite, at some point ^{r} must return that element to itself; that is called the size of the cycle. This is called cycle of the then contains 1 cycle of size r and then you take another element which is not in any in your first cycle, again take higher powers of in it will find a shorter or longer another cycle.

So, the more formal way of defining actually relays on the fact that; the same element raised to higher power is going to only cycle the elements of 1 cycle among themselves and therefore, bring them back to original order at some at some power r and that power r is actually the size of the cycle.

So, if you do $_1^2$; it is going to bring (2 5) → (2 5). So, this is cycle of size 2 because $_1^2$ returns this pair or this list of elements to itself. This is cycle of size 4 because you will have to raise $_1^4$ to restore this set of elements to their original order. So, this is how actually the a cycle is formally defined.

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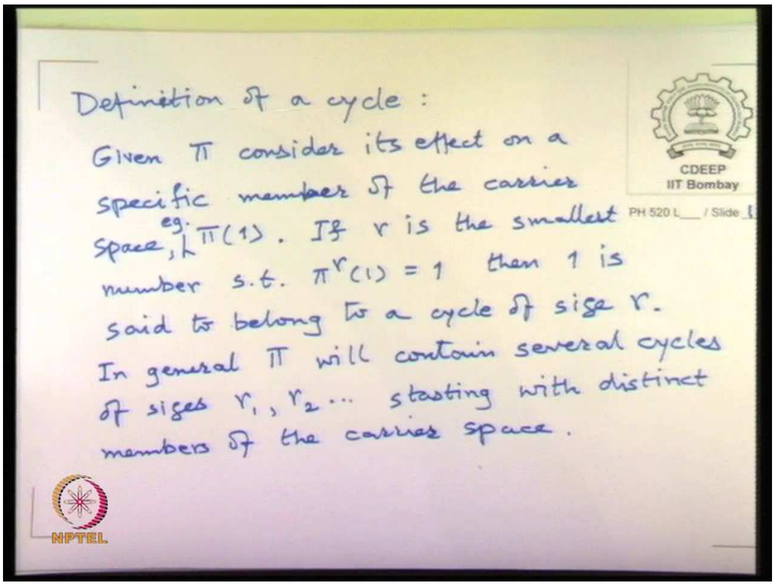

So, we already have an intuitive idea what a cycle is, but what we say is that given consider its effect on a specific member of the carrier space of let us say one for example (1). Then if r is the smallest number; such that $^r(1) = 1$, then 1 is said to belong to a cycle of size r. So, in general will split up into contain several cycles of sizes r_1, r_2 etc.

Starting with distinct members of the carrier space; so the important fact we are moving towards proving is that, once you represent elements of the permutation group as in the cycle notation what you will find is that under conjugate transformation the cycle structure does not change. So, let me write it ok; so we shall see that.

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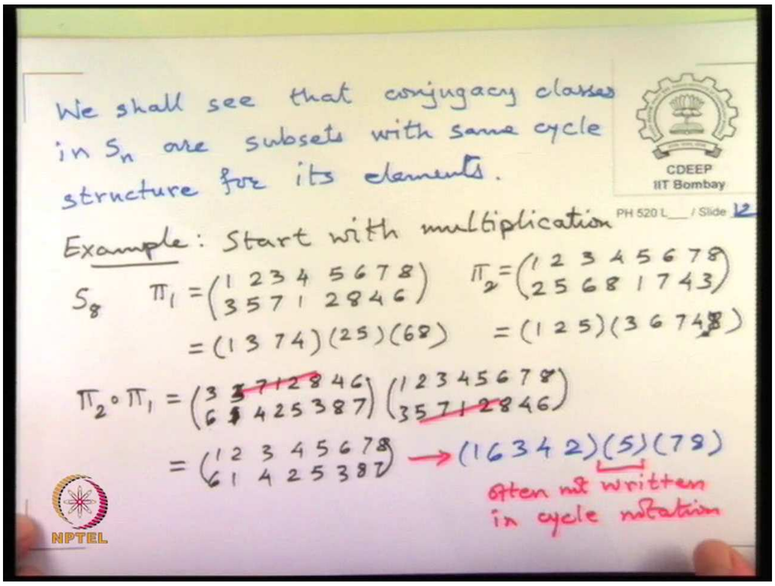

Conjugacy classes in S_n are subsets with same cycle structure for its elements. So, to begin with and just to get a little familiar with this whole thing; let us take an example where we are not actually going to do conjugacy relation, but just multiplication. So, the our example is S_8 as,

$$\pi_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 7 & 1 & 2 & 8 & 4 & 6 \end{bmatrix}, \pi_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 6 & 8 & 1 & 7 & 4 & 3 \end{bmatrix}$$

So, first what are the cycle structures? So rewrite the elements in cycle notation, (3 6 7 4 8) in fact, this is the long cycle and 8 → 3 so that is what. So, it this is a element with 3 cycles and this has only 2 cycles; there is 1 long cycle in that. Now suppose we take ${}_2$ • ${}_1$ first; how do we do this? We want to be able to cancel this with this. So, ${}_1$ I write over here; now ${}_2$ is here what I do is I write out ${}_2$ in such a way that the top row is exactly in same order as this row and transposing the corresponding bottom element with it.

So, I need 3 first. 3 has 6 below it, then I need 2; 2 has 5 below it, let me write it out here 1 2 8 4 6 ok. So I write this row exactly over here and then here I pick the corresponding columns. So, 3 → 6, 5 → 1, 7 → 4, 1 → 2, 2 → 5, 8 → 3, 4 → 8 and 6 → 7 right. Now we can cancel out the upper row here with the bottom row here and it produces the equivalent permutation which is $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 1 & 4 & 2 & 5 & 3 & 8 & 7 \end{bmatrix}$.

Now what is this in cycle notation? Well it is 1→ 6, 6 → 3, 3 → 4, 4 →2 and 2 → 1. So, that is 1 cycle and now 1 2 3 4 have been exhausted so 5, 5 remains 5 then 6 is already used up and 7 and 8 get exchanged. Sometimes in cycle notation this 5 is not written, what is not been changed is sometimes not written. So in fact often not written in cycle notation; so they will write only the nontrivial cycles whatever is not in the list of listed cycles is not changing that is all it means I also wanted to make a comment here that I introduce transpositions by just saying you exchange pair wise I have to do lot of things like this.

Now, that we have different cycles a little more formally I can tell you that, transposition is essentially cycle of size 2 ok. So, it makes it a little more rigorous what we meant by transposition and we can check that every possible cycle of any size can be written as a product of transposition; say products of 2 cycles. So, those things tie up with what we have been talking about earlier. Now well if we spend a few minutes more then we will see something useful which is do the multiplication in reverse order.

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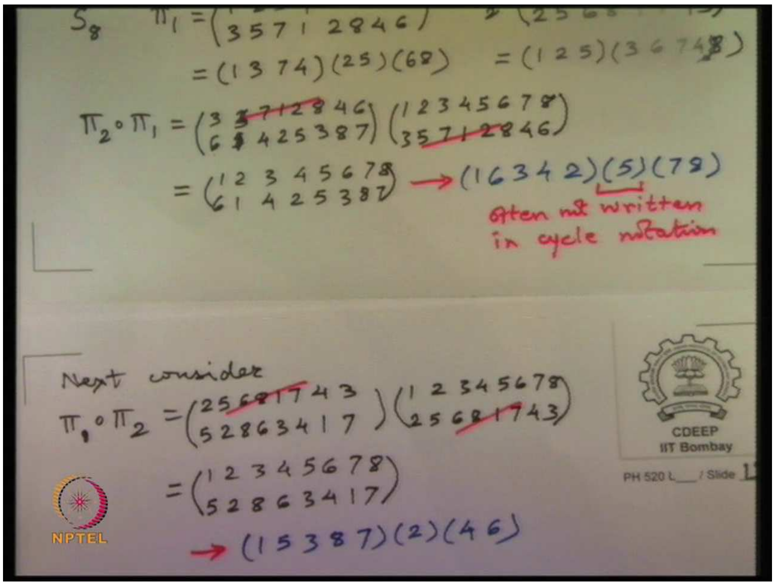

So, next consider ${}_1$• ${}_2$ so I hope you are enjoying this game. So, ${}_2$ we write out as is now ${}_1$ has to be written, but before writing it I put this at the top and then from ${}_1$ read off the corresponding columns. So 5 → 2, 6 → 8, 8→ 6, 1 → 3, 7 → 4, 4 → 1, 3 → 7.

Now, we cancel out the commons layout of the permutation and so we get the answer (1 2 3 4 5 6 7 8) actually the 8 is pitch so that it does not if you take smaller I mean you can work with 5 also, but sometimes it looks like accidental and you begin to repeat things. So, it is better to have lot of elements around. So it becomes (5 2 8 6 3 4 1 7) and what does that become in cycle notation? (1 5 3 8 7) that is 1 cycle, 2 → 2 and 4 → 6 and 6 → 4.

Now, what we observe is that of course, $_2\ _1$ is not same as $_1\ _2$ they are different elements, but the cycle structures are identical. So, the order of multiplication produce the same cycle structure. So, this is going to be useful for proving that conjugacy classes all have exactly same cycle structure ok. So, we will continue next time.