Theory of Group for Physics Applications
Prof. Urjit A. Yajnik
Department of Physics
Indian Institute of Technology, Bombay

Lecture - 07
Factor Group Conjugacy Classes - I

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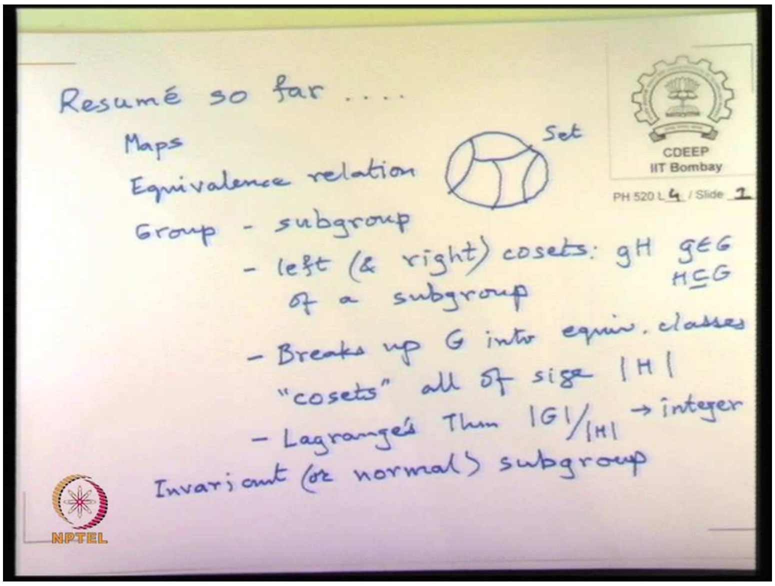

Resume, and so far we went over the idea some algebraic ideas of maps and equivalence classes or equivalence relation, then of course, the definition of the group it itself. Then the idea of subgroup and then the idea of equivalence class, the idea of left "cosets" and right cosets of a subgroup.

And we saw that this idea of cosets is really an equivalence relation. So, it actually partitions the whole group into so, equal this equivalence relation is the one of the cornerstone ideas, we will use in 2 or 3 very important theorems. And the implication of the equivalence relation was that any set S. So, at that stage there is no group theory or anything is just some set theoretic something related to maps and sets.

And, it breaks up any set into subsets, which are all disjoint they exhaust the whole there union exhaust the whole set, and they are then called equivalence classes each subset is called an equivalence.

Now, when we have a group and we take a subgroup of it and form left and right cosets by saying something like gH, you know where $g \in G$ and H is the subgroup of G. So, this is left coset. The fact that an element belongs to a particular coset is an equivalence relation and the way we express it is of course, we went though it last time by saying g in $g_2^{-1} g_1 \in H$ and so on.

So the presence of a sub group automatically allows us this kind of construction the coset construction. And that breaks up the group G into equivalence classes, which are all of size H. Which we called "cosets" all of size equal to the size of the group H or which is order of the group and subgroup H. And this implies intern Lagrange's theorem, that the size of H has to be a divisor of the size of the whole group G.

Then, we saw the idea of a invariant subgroup sometimes called normal subgroup this is the case when the left cosets are same as right cosets so, in general because of non-commutativity. So, what we will find because this division into cosets, the number of cosets has to be universal has to be the same whether you construct left or right because after all it is this number. That number of cosets is this ratio, but the subsets need not be the same all though the number of them is the same.

However, if it turns out that for any subgroup it is left and right cosets are the same then it is called a normal subgroup.

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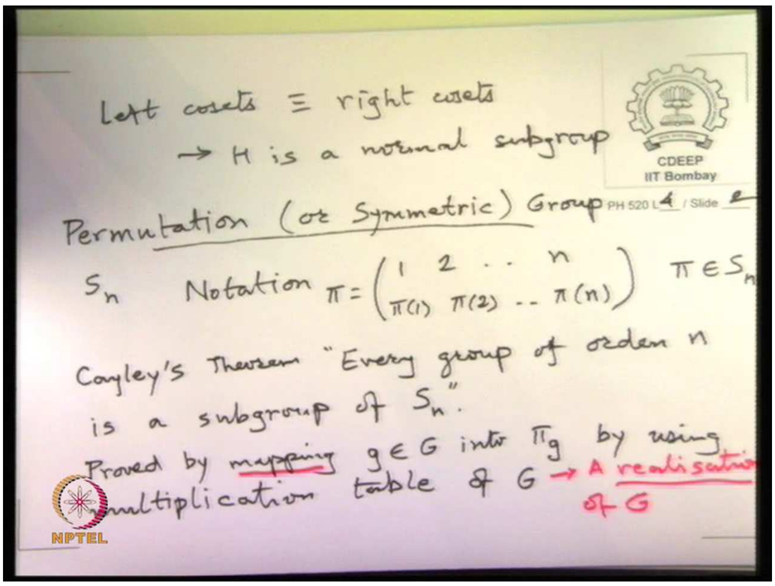

Identical to right cosets then H is a normal subgroup ok. So, that was the generalities about subgroups and Lagrange's theorem. The next thing we covered was permutation groups, you are again there is an alternative word symmetric groups, the nice thing is that most of mathematics develop between the 2 or 3 countries France, Germany and England.

So, occasionally if lot of development happened in Germany, that to later translate and then they pick some word to translate whatever was being used in German. So, the symmetric group S_n it is denoted S_n we introduce the notation for it, because it is a permutation group. So, $\pi = \begin{bmatrix} 1 & 2 & .. & .. & n \\ \pi(1) & \pi(2) & .. & .. & \pi(n) \end{bmatrix}$.

For an element $\pi \in S_n$ we write something like this well we write this particular way we used this one particular way of representing, but there are other ways of representing S_n as we will see soon. And, then we proved what is called Cayley's theorem; Cayley was British the theorem says that every group of order n is a subgroup of S_n.

Now, this may at some in some sense seem a little extra wagon because S_n is a gigantic group with n! elements and your group has only n elements, but the fact that you can embed like that is certainly an advantage, because you know that if you explode all the properties of S_n groups. Then you are effectively learn a lot about all other groups as well of course, the smaller groups may have other special properties.

But, many general properties maybe just inherited from the fact that their sub groups of S_n also you can write every group of order n therefore, by in this notation. It will be encompassing only a some subset of the elements of the gigantic group S_n, but certainly you can write them out in this particular form. And in fact, we relied on the mapping this is done by mapping.

So, can we proved by mapping a $g \in G$ into a π_g by using the multiplication table of g, all you do is take the multiplication table of g. And to represent π_g all you do is pick the row corresponding to g and apply that fill the lower row of this notation by taking the effect of g on the all the elements right.

Because, as we know the multiplication table of g will be $n \times n$ table and if you look at the specific element g, it will have permuted the original list of groups group elements into a new set, but that that is effectively a permutation. So, innocence in our terminology of which I forgot to write in our resume we said that a group is realized on a career space. So, you have some space on which you carry out some operations; like you have a solid you rotate it or you have a lattice, which you rotate or any object that you rotate.

So, the rotation is a action or you have a set of objects and you permute them o the set of objects is what we call carrier space and the action on it is a realization of the group. So, abstract group can be written algebraically simply as some multiplication table, but it is realized in a particular way. Now, what we have done here is innocence realize that strike group G in terms of permutations ok. If you consider permutation and more concrete than the abstract group, certainly permutation conceptual is actually like shifting things around. So, it is a particular realization of so, it is one realization.

So, this map is a particular realization of G. So, since we have reached this point where we can also note that so, that is roughly the summary, but now that I mentioned this let me point out some more algebraic terminology. This map that we made of group G into the large group S_n is called an endomorphism.

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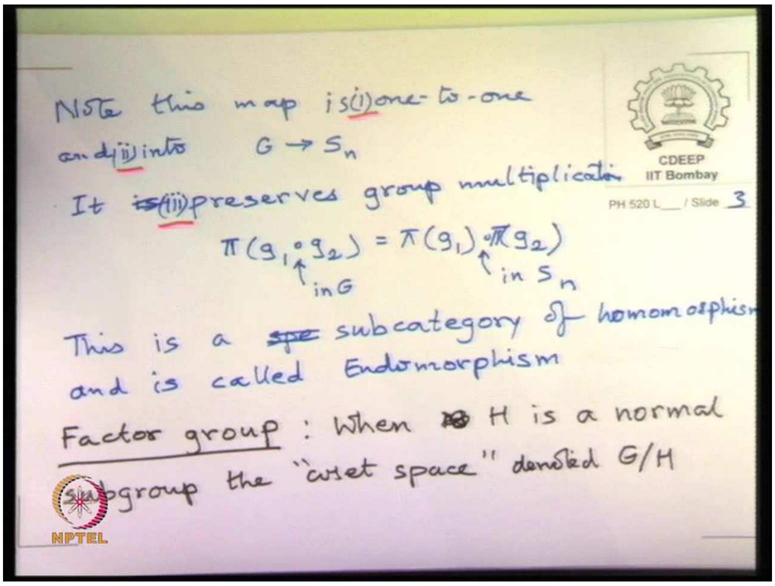

Because, note that this map is 1 to 1 and into right this map which is from G into S_n is 1 to 1, because to every element of G, we associated 1 elements of the permutation group and it is into it does not exhaust the range of the map, but it is also algebra preserving.

So, we can say (i) 1 to 1, (ii) into and (iii) it preserves group multiplication. The algebraic structure on our sets, that is to say $\pi(g_1 \bullet g_2) = \pi(g_1) \bullet \pi(g_2)$, where this $\bullet$ is in G and this $\bullet$ in S_n right. Because, once you map g into $\pi(g)$ you are you are looking at an element of S_n. So, the multiplication on this side is in S_n whereas, the multiplication here inside is in g, but the map preserves this in this is what we called preserving the multiplication.

This kind of a map is a homomorphism, this is a special kind of homomorphism or we can say sub class of subcategory of homomorphism and is called endomorphism. So, next let us say a few things that arise out of our definition of coset spaces, one is that this when we have invariant or normal subgroup. So, we now define something called factor group.

So, when H is a normal subgroup the coset space, by "cosets space" we simply we mean the set of all the cosets. And it is written denoted G/H. We saw that |G| or number of elements in G divided by number of elements in H is an integer natural number there it was usual division. This / is not a division, but is used as a free form symbol to denote a coset space it.

When you right G/H it means that you are using an equivalence relation based on H to subdivide G ok. So, this is called a coset space and is denoted G/H. So, when H is a normal subgroup the coset space can be also endowed with a group property.

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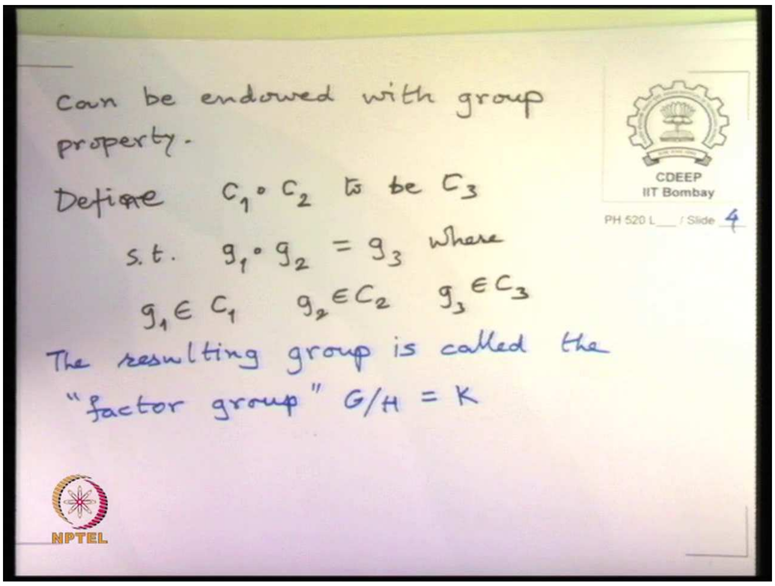

So, maybe we will leave the detail to exercise, but the point is that since the G/H this coset space is unique in left and right multiplication. All you have to do is define the multiplication between elements of the coset space simply by multiplication of that corresponding representative's ok.

So, we define $C_1 \bullet C_2$ to be C_3 such that $g_1 \bullet g_2 = g_3$, where $g_1 \in C_1$, $g_2 \in C_2$ and $g_3 \in C_3$, because the factorization is unique into this subsets, it will automatically happened that if you multiply it 2. Firstly, we know that it has to be completed disjoint. And so, if I pick 1 so, representative from one coset and another from another coset and multiply them as necessarily get something for which is not in either of the 2 original one otherwise there would be a relation.

And so, you will uniquely get some other and you identified with the new coset you got. So, this is the way that you so, it needs a little bit of checking I am just telling you that this is how it works you need to think about it and check, but we can actually endowed this coset space with group structure and in this case it is called "factor group".

What should we call it G/H = K ok. All this may be becoming some abstract overdose, but we will get to some specific in after a little while. So, there is one more general definition which arises from statement like this, which is of a product group or when you can represent the whole group as a product of it is subgroups.

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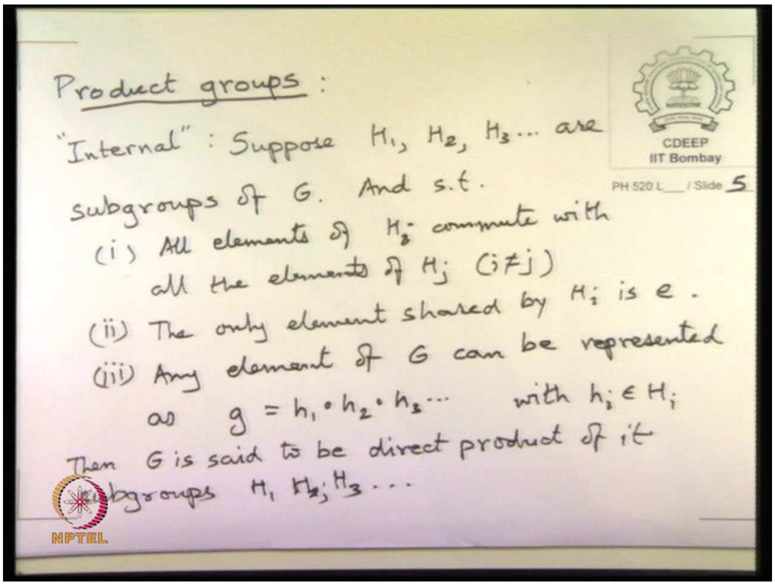

So, first we considered the case of what is called "Internal". Suppose we have some subgroups H_1, H_2, H_3,... of G and we have the properties that (I) all of them commute among themselves i.e. all the elements of H_i commute with all the elements of H_j of course, $i \neq j$ and in so, there is subgroups mutually are just commutative.

(ii) the only element they share is identity and (iii) this is important you should be able to recover the whole group out of products of this then, G is said to be direct product of it subgroups H_1, H_2, H_3,..., H_n whatever there.

So, here a group G gets represented as product of it is subgroups, you can imagine that if you took any one of these say H_3 then you will find that it is coset space will be product of the remaining groups write $G/H_1 = H_2, H_3,..., H_n$.

Now, of so, if you do not agree with it right now do not worry it can be checked. The second where therefore, of thinking of product groups is if I had G_1 and G_2, can I construct a product group out of them. So, that is the external definition.

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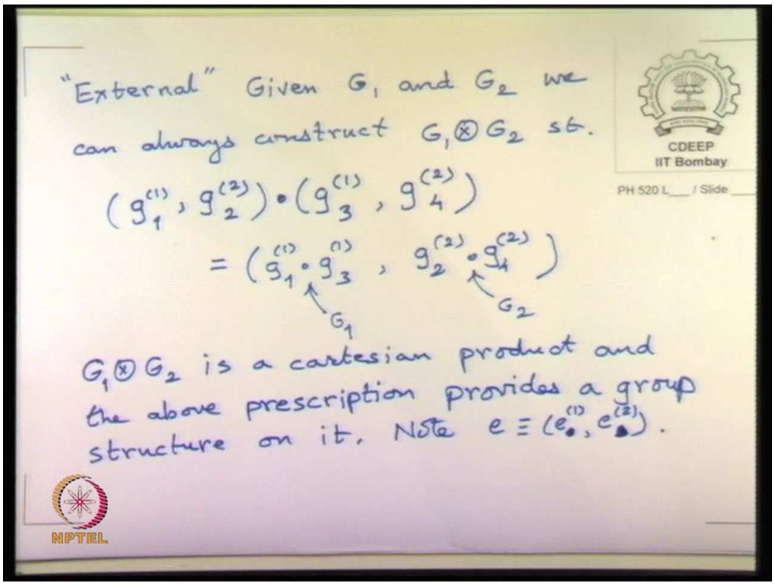

Now, for the internal case we it was quite restrictive you cannot just declare any group as product of it is subgroups. In fact, there will be several groups which will not be expressible as product of that subgroups.

However, given any 2 groups G_1 and G_2 you can always construct a grand group of size $G_1 \otimes G_2$. So, that it is actually a product group. So, given G_1 and G_2 we can always construct such, that all we have to do is provide a multiplication rule on this large group $G_1 \otimes G_2$ set that it satisfies the group axioms right. So, we provide the rule as $g_1^{(1)}$. So, we need some simplifying notation to write this out, what we do basically is that if I have a pair and if I have to multiply it by value and here we go so, still in 1, but I have 3 and I have right.

So, the upper level tells you which group it belongs to lower is just some indexing it does not matter. Then all we do is we say that this is equal to $g_1^{(1)} \bullet g_3^{(1)}$, which of course, both belong to group G_1. So, this is the new dot new multiplication table this is belonging to G_1 and this belongs to G_2.

But, then we have defined new multiplication out of them all we do is construct ordered pairs. So, sometimes in set theory this is called Cartesian product you take any one set and you take any other set, if you create ordered pairs then you call it a Cartesian product. Because this is like the real line and you take another copy of the real line and then you create ordered pairs you get the plane. So, the plane R^2 is a Cartesian product of endorse coordinates are called Cartesian coordinates. So, this is called a Cartesian product. So, $G_1 \otimes G_2$ is a Cartesian product and the above prescription provides a group structure on it.

Note. So, one has to verify all the 4 properties closure associativity, which looks quite easy to check right closure associativity existence of identity and, existence of inverse note that, the identity element of the product group is essentially the even e has 2 element : $e \equiv (e^{(1)}, e^{(2)})$ because that is the label for which group they come from so, the those that ordered pair which is ordered pair of identity elements from each of them is the identity of the whole group.

So, one can check the group axioms ok. So, the next thing we are going to do is in 2 parts. So, this notion of the coset space and coset space was one of the important concepts and it gave Lagrange's theorem the factor group is another important conceptual thing out of it. The next major conceptual thing is the notion of conjugacy classes ok.

So, think of ammonia molecule ok. Which is one nitrogen and then there are 4 hydrogen symmetrically placed around it, if you take any one of the hydrogen's you can draw a vertical line through the carbon and then you have a threefold symmetry rotation ok, but then there are 4 such hydrogen's. So, each of the hydrogen's will produce it is own threefold rotation symmetry.

So, these are actually just like conjugate to each other, these are copies of the same thing that you are doing with respect to anyone axis. So, these are called conjugacy classes and they are going to be very crucial in analyzing groups in general. And so, we are moving towards defining and specifying their properties.