Theory of Group for Physics Applications
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Lecture – 06
Lagrange's Theorem and Cayley's Theorem - II

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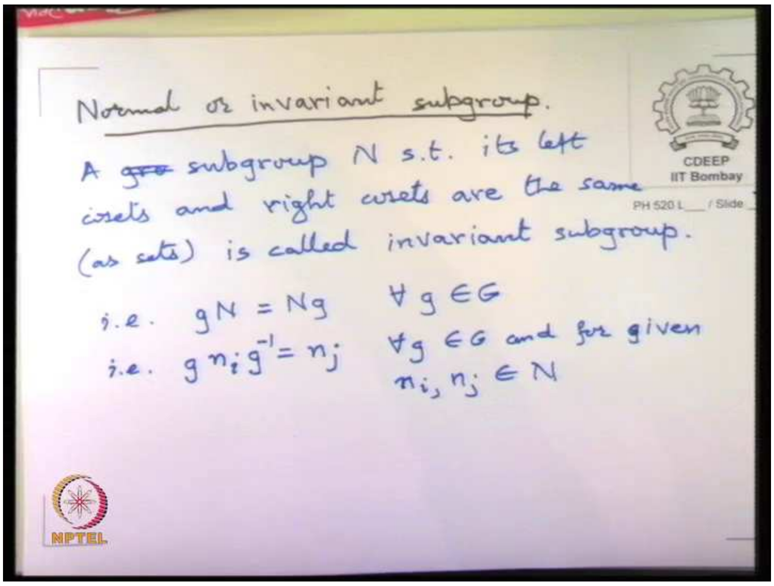

So, next we come to the idea of what is called normal subgroup, or invariant subgroup; both terms are used. This is the subgroup H or let us call it N such, that it is left cosets and right cosets are exactly same, a subgroup N such that it is left cosets and right cosets are the same as sets.

So, each set in the list of left cosets has a corresponding member from the right coset list, so that they are identical sets, so as sets they are identical. So a subgroup N such that there it is left coset and right cosets are the same is called normal or invariance subgroup. I think we will use invariance subgroup it is easy to quick to remember what it means. So what so I we can say $gN = Ng$ for all $g \in G$.

What this means is that, $gn_ig^{-1} = n_j$, for all $g \in G$ and for given n_i, $n_j \in N$. So if you pick some element of the normal subgroup and take any element of g do gng^{-1} you will get back element in the subgroup itself. So this is called normal subgroup ok. So, we will see examples of this a little later, we can try to see an example of this coset space, in fact the simplest example we have we will automatically get a normal subgroup. So let us see an example just to be understand what is going on.

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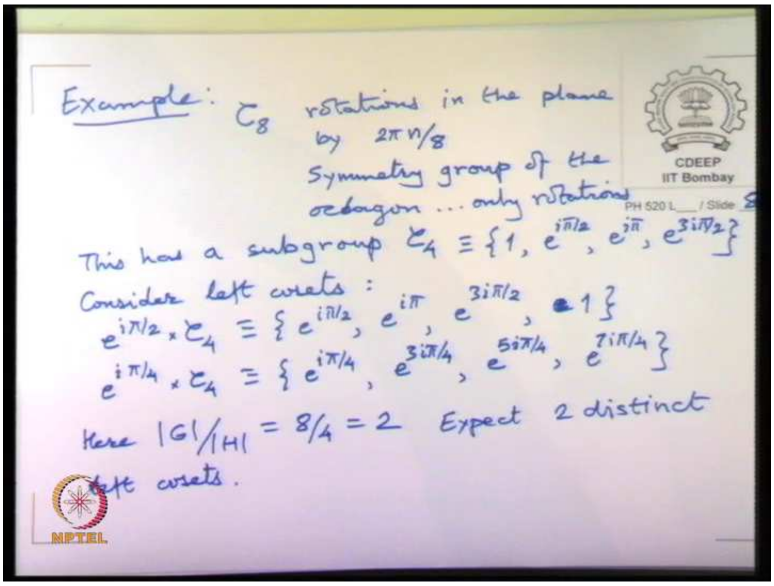

So, suppose we have C_8, this is 8 fold rotations by 2 /8 or in other words have an octagon the symmetric group of the octagon, but ignoring reflections and other things ok. This has a subgroup which is a C_4 $\{1, e^{i\pi/2}, e^{i\pi}, e^{3i\pi/2}\}$.

Now, suppose we take some element of C_8 and multiply on the left by this, where I take $e^{i\pi/2}$, and multiply on the left. What will I get? I will get,

$$e^{i\pi/2} C_4 \equiv (e^{i\pi/2}, e^{i\pi}, e^{3i\pi/2}, 1)$$

So, this is 1 left coset, but because there is 2 n/8. I also have 2 n/8, which is so I should have 45 degree rotation as well right.

So, if I have $e^{i\pi/4}$ then we will have,

$$e^{i\pi/4} C_4 \equiv (e^{i\pi/4}, e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4})$$

and so you can draw an octagon and check what is happening. This is rotating it by /2 and then this is smaller rotation. This is just to demonstrate that if I take distinct elements of G, I may get distinct cosets, but the length of the cosets size to be the same and they are all I mean with distinct elements in it.

Now, clearly the order of C_8 is 8 and the order of C_4 is 4, the fraction we get is 2, and so we expect 2 distinct left cosets. And in fact, we are exhausted both of them here, if you now take anything else, so we took i /4 and we took i /2. Suppose you took 3i /4 ok. What is going to happen : $e^{3i\pi/4}C_4 \equiv (e^{3i\pi/4}, e^{5i\pi/4}, e^{7i\pi/4}, e^{i\pi/4})$. So basically you will repeat this list.

So, if you take any other element from the group G, you are basically just going to repeat either 1 of this or this. So we see that, that is we exactly get 2 cosets that are independently the cosets of this particular subgroup C_4. You could have played the game the other way round, suppose we pick the subgroup only {1, -1} because in the list of C_8, 1 as well as -1 are there.

Now, I have a subgroup of order 2, if I start multiplying from the left I will get 4 different cosets because I had only {1, -1}. If I multiply by say i /4 it will become the i /4 line, i /2 line, 3i /4 line but that is all, you will get 4 different subsets. The in other words it will be 8 divided 2 equal to 4 but you see that once you have a subgroup it is left cosets will form, the size of the left cosets is exactly a divisor of the whole group.

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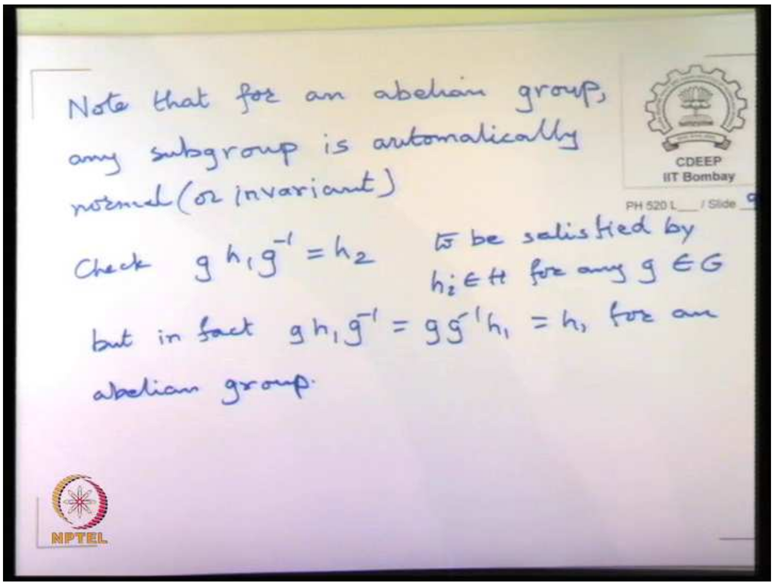

Now, because this is a this group also happens to be abelian group. So it also happens that the subgroup is invariant subgroup because, for an abelian group, any subgroup is automatically normal or invariant; why? Because need to check the condition that $gh_1g^{-1} = h_2$ right this is the condition to be satisfied by $h_i \in H$ for any $g \in G$. But this is always true because h_1, and g^{-1} commute it is an abelian group. So I can always bring g^{-1} to this side, so in fact it lays h_1, for an abelian group. So this can be checked in this particular case well it is rather trivial because they are just complex numbers whether you left multiplied or right multiplied it is going to be the same answer, so the left cosets are going to be same as right cosets.

So far an abelian group all the subgroups are automatically invariant or normal subgroups ok. So this is a property of groups which we will be using later ok. Then we do a few more things about the permutation group and let us see we can prove one important theorem. So, now we go to the notion of theory of permutation groups and we will see a few interesting things in it and hopefully if we cover all the required things and prove one interesting theorem.

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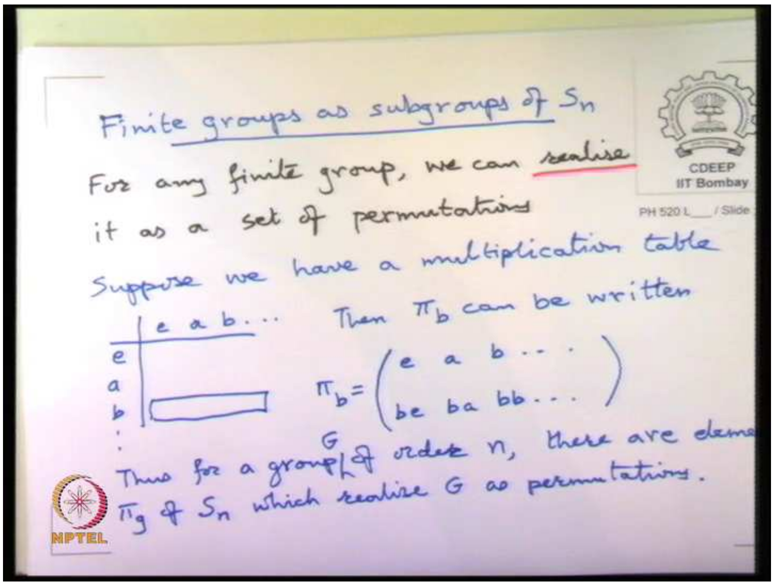

So, let us start by saying representing, so finite groups as subgroups of S_n for some n, given any finite group what we are now trying to show is that actually it is a subgroup of S_n ok. First we can show that we can write out the elements of any finite group as a permutation ok. So the first statement is we can realize it as a set of permutations. Note the word I am using realize we had said last time that you can have an abstract group which is just specified by some table, and when you produce an example of it you say that you have a realization of the group, you where you supply some set of objects on which operations can be carried out and then you check that the operations carried out obey the multiplication rule given in the table then you have the a realization of the group.

So, suppose somebody hands you a multiplication table, where I have now we have been writing like this and we have some e and how shall we label the things a, b something a, b, c, suppose we take one particular row in this. Now what is what does this row represent? It is equivalent to this element b acting on this top initial row of elements that is what this row is producing. So we can write a permutation corresponding to b like this, in the notation we introduced last time, right we introduced this notation for permutation group that we write a top row and then write the effect that the permutation has in the second row.

So, we can represent the element b like this. Thus all and remember that each of the in this row all of this element I mean this realize on the fact that it very crucial realize on the group properties that the group multiplication give unique answers, you cannot have ab = c and af = c. So the effect of b multiplication here is exactly just a permutation right of the elements of this. So thus, for a group of order n, there are elements $_g$ of S_n. Which realize so group G of order and which realize G as permutations.

So, this looks like a clever trick and what one has to remember is that n is a some number, but S_n it is whole sizes n! . So S_n is a gigantic group, but by paying the heavy price of considering members of some huge group you at least represent any possible group of size N as some permutation. So every possible finite group can be realized as a subgroup of S_n, where n is the order of the original group. So firstly I have said, so far I actually said subset right because all I proved to you that there exists and element in S_n for every element of g. Now we have to check the group property that it actually forms a subgroup of S_n needs to be proved.

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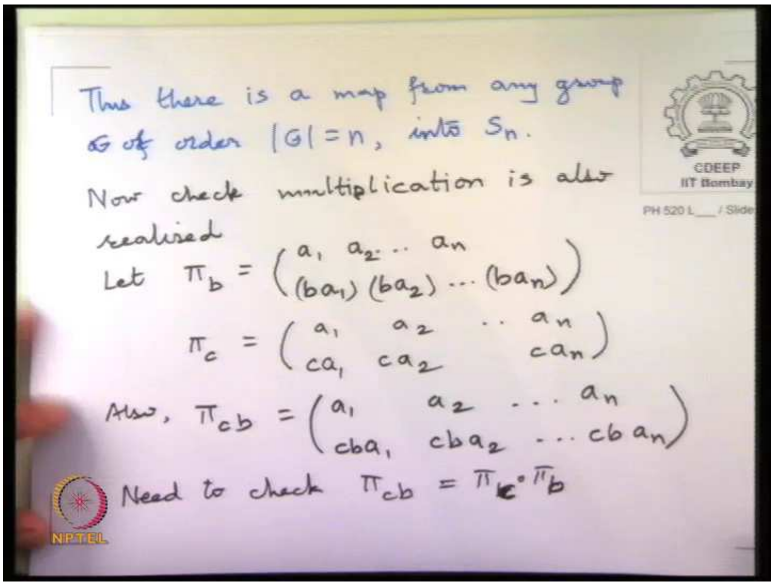

There is a map from any group of group G of order into S_n. So, we have verify that there will exists some element in S_n which will realize whatever and g does. But now we need to check that it actually forms a subgroup when realized as a permutation ok, to check the multiplication rule multiplication is also realized. So what do we need to do? So suppose we realize $_b$ and $_c$ because we are going to use a_1's here.

So, let $\pi_b = \begin{bmatrix} a_1 & a_2 & \cdots & \cdots & a_n \\ ba_1 & ba_2 & \cdots & \cdots & ba_n \end{bmatrix}$. So this representation gives you how b is represent as a permutation and let $\pi_c = \begin{bmatrix} a_1 & a_2 & \cdots & \cdots & a_n \\ ca_1 & ca_2 & \cdots & \cdots & ca_n \end{bmatrix}$.

Now, how are we going to compose the two? Well, so also note $\pi_{cb} = \begin{bmatrix} a_1 & a_2 & \cdots & \cdots & a_n \\ cba_1 & cba_2 & \cdots & \cdots & cba_n \end{bmatrix}$. Let us check that $\pi_{cb} = \pi_c \cdot \pi_b$.

So, b will act first then c will act so $_{cb}$, we should have to multiply $\pi_c \cdot \pi_b$. So b will have to act first and then c then we should reproduce the action of cb on this. Now to do this we have to use the trick we had introduced last time for multiplication of permutations.

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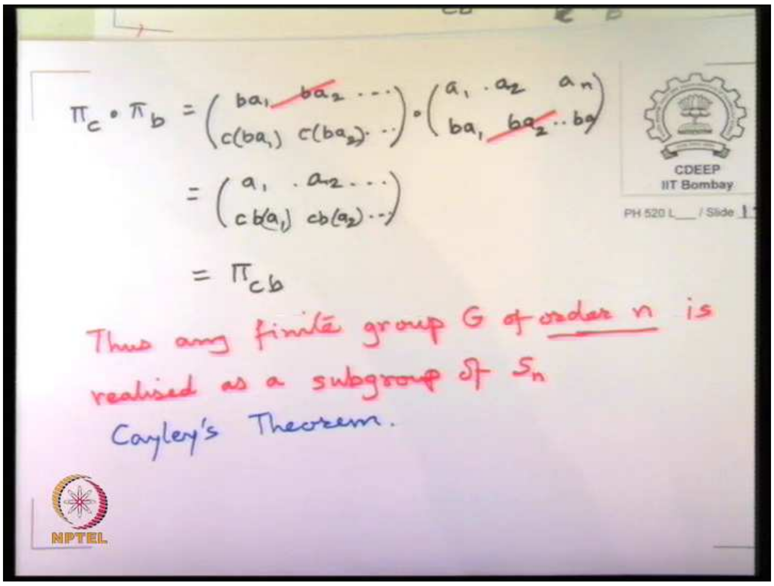

So to calculate $\pi_c \cdot \pi_b$, we will write $_c$ in the form ba_1, ba_2 ,…... Suppose I write the top row like this then the by our rules of representation it means that the lower will be equal to $c(ba_1)$, $c(ba_2)$,….. right, it is the same permutation as $_c$ written here. So this $_c$ what I do is I multiply the top row by b, but do the same thing to the bottom then the permutation has remained the same. So this represents $_c$ as much as this and times I write $_b$ as before which is here, but here I have ba_1, ba_2 ,…...

Now, remember the rule we had made that if you have if you can rearrange things like this, where the bottom row of right element is same as top row of the left element we can quote cancel these 2 like we would like to do in high school and so it reproduces a n times $c(ba_1)$, and now because of associativity c on (ba_1) is same as cb on a_1 etc.

So, this is same as our $_{cb}$ right, it is exactly the same thing. So one can reproduce the entire group property of any finite group by realizing it as a set of permutations and then carrying out multiplications as you would do for any permutation. So thus of order n, this is very crucial of order n is realized as a subgroup of S_n, the permutation group S_n which is a huge group, it has n! elements where as our g had only n element, but we manage to represent it ok.

So, this is a very important thing to know because you do not have to struggle you do not have to worry. So if we are just said that a group means a multiplication table that obeys associativity that is why you can say right enclosure that every element is somehow has produces a unique product and it is inverse exists, so I have a table. You might worry that at order n I might have some very large number of groups and whether I have listed them all and so on, but now the point is that every possible group at order n is going to be some subgroup of a c of the S_n and therefore, some of the properties will be anyway inherited from S_n as well.

So, this is a very reassuring thing and this is called Cayley's theorem. So I think we have seen quite a few generalities of groups and subgroups normal subgroup, so we will end today by saying a few more things about permutation groups, which is yeah so one thing is about. So we are now actually going so far we said something very general about finite groups and that there subgroups of the permutation group, but now we gone to say something's about permutation groups themselves.

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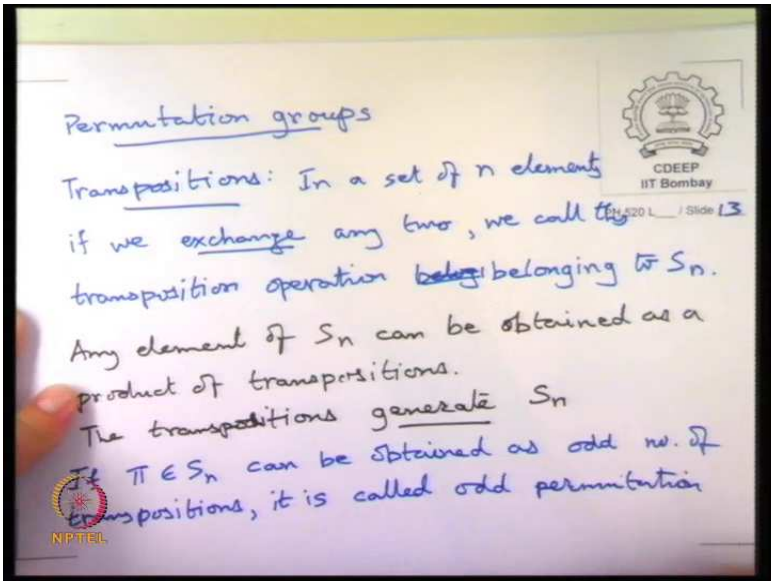

Well they are also called symmetric groups. What is transposition? If I just exchange 2 elements in a set. In a set of n elements if we exchange any 2, we call this transposition operation belonging to belonging to called to S_n ok. If we exchange it is a permutation this, but it is an operation on the n elements, this we call a transpositions. Our claim now is that transpositions generate all possible permutations given any permutation I can break it up into a series of transpositions.

And therefore, the set of transpositions which how many elements do we have ${}^nC_2$, right, I to have to choose any pair. So the ${}^nC_2$ transpositions generate all the n! elements of the

S_n the and as you ${}^nC_2$ is n! divided by 2 I mean ${}^nC_2$ formula gives you a number smaller than n! and that produces n!. The transpositions generate S_n because if you can think of any permutation you can do it in sequence of exchanging pairs.

Now given a particular permutation if you start exchanging you might sometimes end up taking a longer route you may do something back and forth once in a while you could have taken a shorter set of permutations, but the number of permutations you will take to reach a given number of transpositions you will take to reach a particular permutation will be either odd or even and that number will that oddness or evenness will remain the same regardless of if you do some redundant transpositions; $\in S_n$ can be obtained as odd number of transpositions it is called odd permutation and likewise even permutation.

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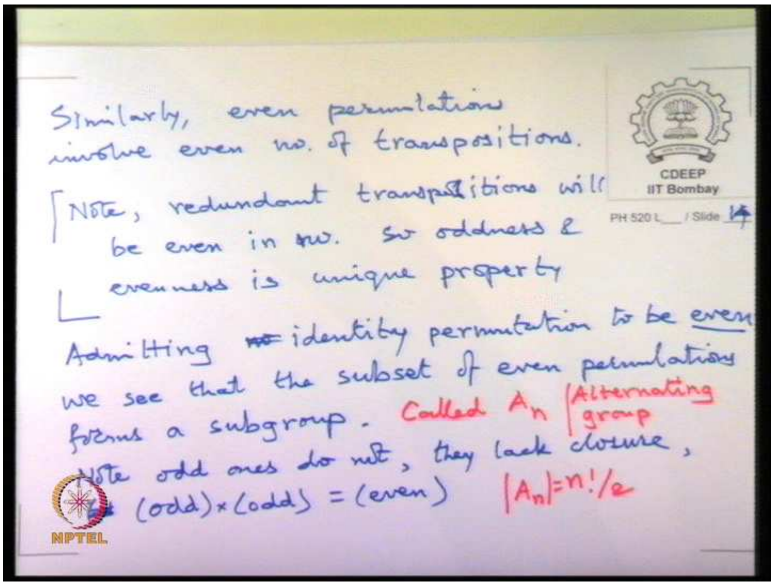

Note that if there are redundant transpositions you ended up making there will always be even in number. So oddness and evenness will remain the same. From this we arrive at one interesting subclass of the S_n group.

So, if we do not do any permutation at all we treat it as even permutation because there is zero, no exchanges. Admitting identity permutation to be even we see that the subset of even permutations forms a subgroup right because product of 2 even permutations is always even that you can check because of the number of transpositions involved remains even, but product of 2 odd permutations becomes even because odd plus odd is an even number. And therefore, product of 2 odd permutations does not generate an odd permutation, but product of 2 even permutations does and we include identity in that list, so then the even permutations generate a subgroup, do not because the lack closure, odd into odd is even.

But even into even remains even and we include the identity element. So certainly this is true. Inverse is not so much of a problem and odd permutation the inverse would also be odd, but the closure itself is not realized. So the odd ones do not, but the even once by themselves form a subgroup. This subgroup is sometimes called A_n; why sometimes I mean this is the standard terminology A_n. Again through some strange reason called the alternating group A_n. So even permutations of a set of size n forms also a group, it is called A_n and it is the one of the biggest subgroups of S_n. It has how many elements; n!/2.

The next idea is the idea of cycles which let me just introduce and then we will continue next time. So you have a set of elements, but what we are talking about the operation on the elements right. So if I have particular order and then they appear in another order, how many things did I have to exchange to get from here to there? So we can see an example. So we can try to check suppose I have something likes this $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{bmatrix}$.

This is an element of S_5 right. Now can we obtain this as a series of permutations?

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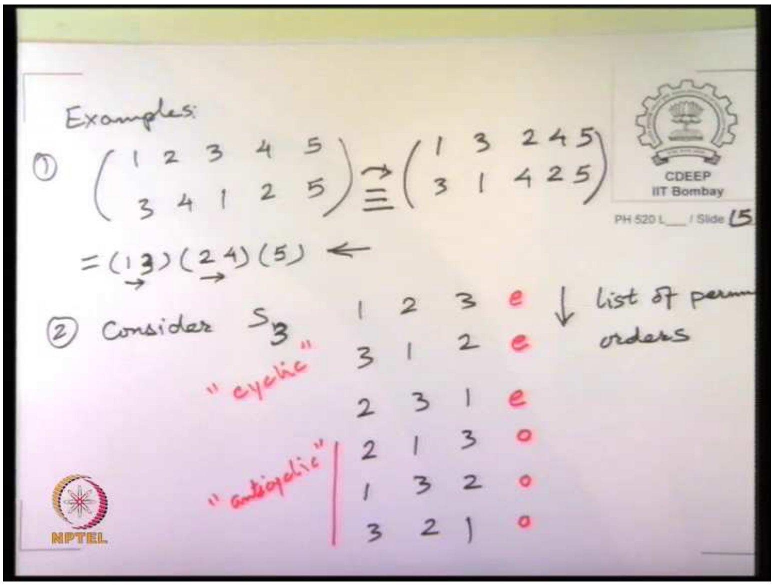

Then according to our rules of representing permutations we are supposed to right this out as like this, whatever is below 3 goes here, whatever is below 2 goes here, whatever is below 4 goes here and whatever is below this. These are both same. So it does not matter what order you started with, but what you are going to perform on it, how many operations. This is the statement about realizing a group, there is a base space or a career space and there are operations on that career space, we have just little square or octagon and we are rotating it.

So, we could have already rotated a b c d into something d c e f g, but the operation of /2 rotation will still remain /2 rotation. So we can check 2 things here I mean, does anyone doubt that this can be represented as set of permutations as a transpositions pair wise exchanges? How we will we get from this to this? So if I take so 3 and 1 are of course, pair wise actually the way it is written it is already of the form 1 and 3 are exchanged and 2 and 4 are exchanged and 5 is not exchanged 5 remains 5. So this is a pair wise exchange representation of that same permutation.

But now we are writing it like this. This is a notation in which this is a different notation in which 1 goes to 3, 2 goes to 4. Suppose we have suppose we take the group of permutation of just 3 elements ok, consider S_3. We know that we get (1 2 3) the cyclic permutations are (3 1 2) and (2 3 1) and then there are so called anti cyclic permutations. So which is in which I transpose only 2 and 1 and leave this unchanged and then I do cyclic permutations of this. Now we have exhausted all the 6 elements of S_3.

So, if I have 3! elements in S_3, there are 6 of them and; so ideally I could write out this as one permutation then I could take this same starting and write this as the second row and so on. So these are the all the 6 possible permutations. So these are the representations of the objects not the representation of permutations. I have to write out permutation for each of them which would take too long, but you know what this means right. So these are list of permutations list of permuted order is like this. So I have 6 elements in S_3 and they are essentially going like this.

Now, we can see that this permutation that (2 1 3) is essentially one exchange right. So it is an odd permutation, so this is odd. Then if I go to (3 1 2), I can get (3 1 2) by first exchanging 1 and 2. So that 1 comes here, but 2 is here, now I exchange 2 and 3, so I will get (3 1 2), so this is an even permutation; we can check that this is also an even permutation because it will it I can exchange 3 and 1 first which will bring 1 here and 3 here and then I exchange 3 and 2. So again there are 2 permutations it is even. Suppose you made suppose as a long list which you have to exchange, you might exchange and then reverse the exchange. So you might end up doing some redundant exchanges, but those exchanges will always be even in number you will be doing up something which was not really required and then undoing it. So it will always remain like this.

So, these are all even and we of course leave the original order nothing has happened to it as even and these are all odd. So this the set of cyclic permutations, so these we sometimes called in physics sometimes these are called anti cyclic. But this is the strictly physics terminology and these are cyclic permit cyclic order. The cyclic order permutations will form a subgroup A_3, where as these do not. The other thing so we will continue next time.