Theory of Group for Physics Applications
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Lecture – 05
Lagrange's Theorem and Cayley's Theorem - I

So, we have seen what is a group the group axioms, we went through the algebraic structures and some of the algebraic rules. And the most important thing we went through was the idea of equivalence classes and, equivalence relation. And that induces what we call equivalence classes.

So, what we are going to do next is study some more generalities and, then and we also introduced the permutation group. And the claim is that permutation group really encompasses all possible groups that we can have ok.

So, what we are going to do today is actually try to see that how permutation group is the big daddy group of all the groups, but before that we just proves some more general things about a group. And to motivate let me just say that what we try to prove is what is called Lagrange's theorem.

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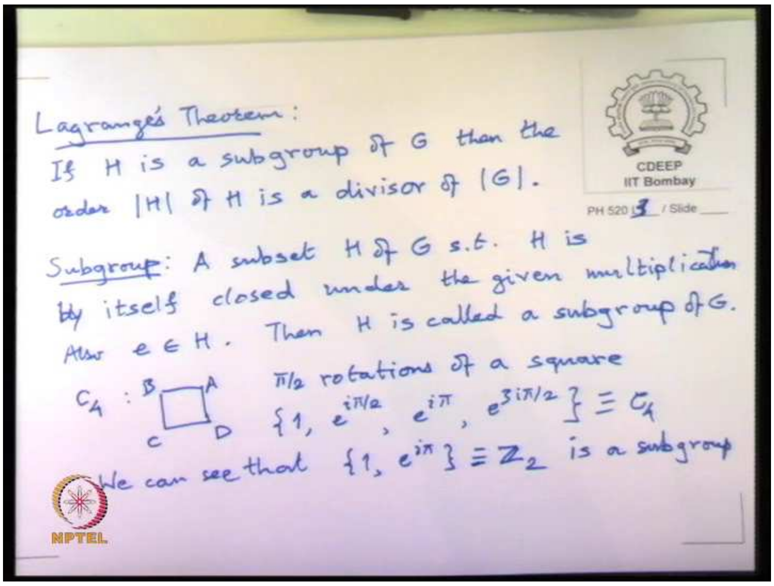

So, we have Lagrange's theorem which says that, if we have if H is a subgroup of G, then the order |H| of H is a divisor of |G|, that is the number of elements in a subgroup is such an integer that it will be it will divide G as an integer ok, divisor means it only fraction behind it will given integer number.

So, first we need to also define what is a subgroup, I am not completely sure we went through this, but the idea is rather simple, rather obvious once you state the title of what we are trying to define. A subgroup is a subset H of G such that H is itself by itself closed under the given multiplication, whatever the multiplication of the group is it went a little out right. So; obviously, also $e \in H$, then H is called a subgroup of G.

So, a simple example is let us say the we consider two groups, one was the cyclic group C_4. So, in C_4 we had basically square, which was being rotated by 90 degrees. So, here we can see that π by so, this is $\pi/2$ rotations of a square, then we can see that π rotations form a subgroup identity and π rotations. So, we have so we can write it out like this $\{1, e^{i\pi/2}, e^{i\pi}, e^{3i\pi/2}\}$.

So, this is C_4 we can see that $\{1, e^{i\pi}\}$ is a subgroup, which is actually what we had once call Z_2, I think Z_2 is just $\{1, -1\}$ subgroup. Is a sub is a group in its own right, but is now a subgroup. It is closed under multiplications and it contains identity. So, it is a subgroup ok.

So, this is the basic idea of a subgroup and one can see that in general, if I have a large group, then there will be some subgroup. There is identity and then some elements have kind of mutual relationship. So, that this is going to each other, then that becomes a subgroup.

This does not mean that they do not give some non trivial thing when multiplied with things outside the group outside that subgroup, but that subgroup kind of talks to each other and can be treated as independent. Now, we see the to see Lagrange's theorem, we introduce the concept of cosets.

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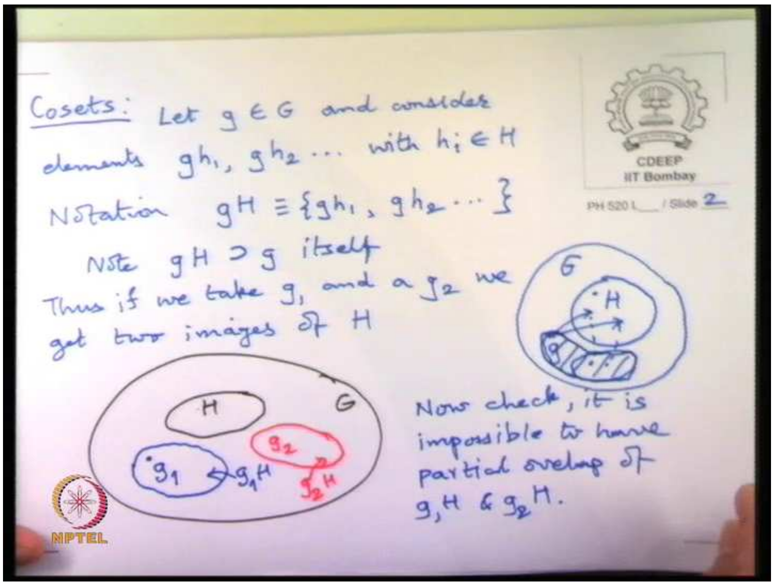

So, think of the subgroup H and consider multiplying it on the left by some element $g \in G$ and which is generic with $h_i \in H$. So, we generically write this set, the symbol $gH \equiv \{gh_1, gh_2, \ldots\ldots\}$.

Now, we see that so, let me draw a picture here, here is a subgroup H and this is the big group G, if I start if I take some element g here and start multiplying every element of this by g, then I will produce some other image ok. So, this will get mapped here, this will get mapped here, something like this, but this image has to include g, because identity is also there in H.

So, it include g as well the image of g multiplication H includes g itself. So, note $gH \supset g$ itself. Now, is consider so now, we start doing this with every element in g, take g_1, take g_2, take g_3 start multiplying everything in H by this.

We are going to after all the whole thing is finite. So, let me draw fresh picture, thus if we take g_1 and a g_2, we get 2 images of H. And let us try to draw a picture like this, where this is G and there is H and now g is here g_1 clearly this will be g_1H, we can similarly have g_2H, which is obtained by multiplying every element of H by this g_2.

Question is can g_1 and g this two sets, have some overlap these two images, of H we created can they overlap. And the answer is that if they do, if they share any element, then automatically they will be identical, there is no way of having partial overlap of it is impossible to have partial overlaps of or yeah of g_1H and g_2H, either they are disjoint or they are the same.

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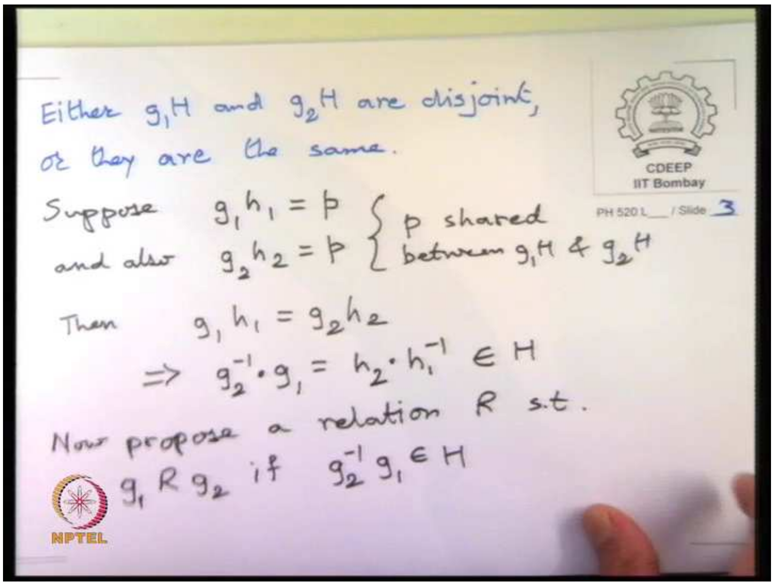

So, how do we prove that well here is we invoke our clever method of equivalence classes? So, think of the idea that they are some overlap. So, suppose they overlap. So, suppose $g_1h_1 = p$. And also $g_2h_2 = p$, it does not have to be same h, but suppose multiplying with another h_2 right.

So, if you keep this picture in mind, I multiply some element of this h and produce some element here, I multiply g_2 by something else, but somehow there is an overlap of the 2. So, that they share an element p between g_1H and g_2H. If this is so, then we then we can see that $g_1h_1 = g_2h_2 \Longrightarrow g_2^{-1} g_1 = h_2 h_1^{-1} \in H$.

So, if the two sets g_1H and g_2H share a point p, then it must be necessarily true that without any reference to this detailed elements H, $g_2^{-1} g_1 \in H$. Now, we prove that this statement the $g_2^{-1} g_1 \in H$ is an equivalence relation ok. So, now, propose a relation R such that g_1 is related to g_2. If $g_2^{-1} g_1 \in H$, now we try to see if this relation R is an equivalence relation or not.

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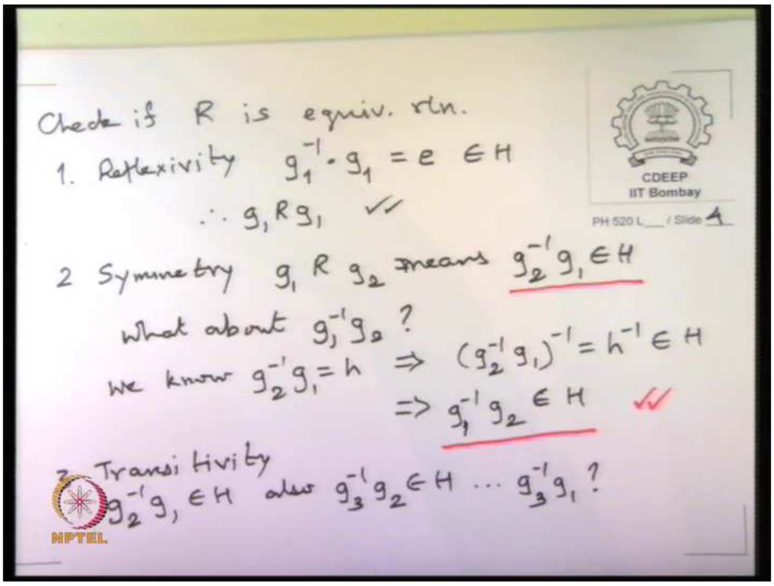

So, how do we do that check if R is equivalence relation. So, what are the things we have to check 1 is reflexivity an element has to be related to itself. So, we asked the question is $g_1^{-1} g_1 = e \in H$. So, g_1 is related to g_1 itself ok. Then number 2 we ask for symmetry, which is the question if g_1 R g_2 means $g_2^{-1} g_1 \in H$, what about $g_1^{-1} g_2$ right that would mean that the reverse, then g_2 R g_1, but we already know $g_2^{-1} g_1 = H \Longrightarrow g_2^{-1} g_1^{-1} = h^{-1} \in H \Longrightarrow g_1^{-1} g_2 \in H$. It is easy to check that that is what will happen, it will be $g_1^{-1} (g_2^{-1})^{-1}$ but $(g_2^{-1})^{-1} = g_2$ itself. So, this of course, belongs to H. So, if $g_2^{-1} g_1 \in H$, then it also means that $g_1^{-1} g_2 \in H$ that means, it is a symmetric relationship.

And finally, transitivity here we need here we are given that $g_2^{-1} g_1 \in H$ also $g_3^{-1} g_2 \in H$, what about $g_3^{-1} g_1$ that is what it means like, if g_1 and g_3 have to be related, then $g_3^{-1} g_1 \in H$, but this is easy to check because, all I have to do is multiplied the two things out.

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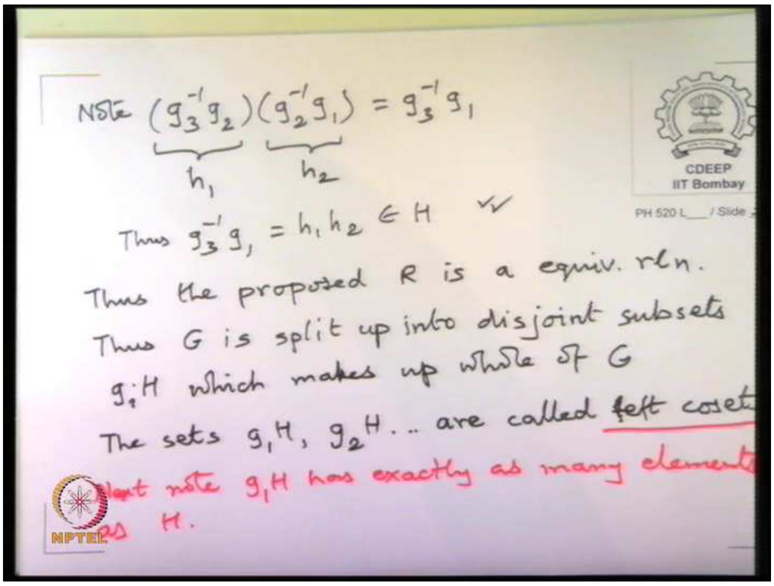

So, note that

$$(g_3^{-1}g_2)(g_2^{-1}g_1) = g_3^{-1}g_2g_2^{-1}g_1 = g_3^{-1}g_1$$

But suppose this was h_1 and this was h_2 some h_1 some h_2 it does not matter. So, it implies that g_3^{-1} g_1 is also belonging to which was the third thing to be checked. So, the requirement that g_2^{-1} $g_1 \in H$ is the relation between g_1 and g_2 which is an equivalence relation.

In other word we have proved. So, the proposed R is an equivalence relation, in other words we have proved the statement that either g_1 h and g_2 h are disjoint or they are the same. Because the equivalence relation splits the set into equivalent classes, thus G is split up into disjoint subsets g_iH, but which make up the whole G again because, that was the property of R it is and equivalence relation, every element of the set at least relates to itself that is a property of R.

So, even if we does not relate to anything it will be a point in the set. So, under equivalence relation you have a disjoint subset of subsets, whose union is the whole set. So, you have to recover the whole of g as well. So, these disjoint. So, these are all g_iH and sets g_iH are called left cosets, g_1H , g_2H etc the distinct once, or you can I mean generically you can right any of them are called cosets, left cosets I am sorry under left because it involve left multiplication.

So, if you list out like this every possible g from G of course, there will be a sum that will be same because of the equivalence relation, but you can write the whole list. But, now we see one other important thing is that any one of cosets g_1H has to have as many elements as H itself right. So, next see g_1H has exactly as many elements as H why is that true H is a subgroup. And if I start multiplying by some element g, I should generate a unique element of big G out of this multiplication. So, each of the elements that gets generated by this left multiplication has to be unique.

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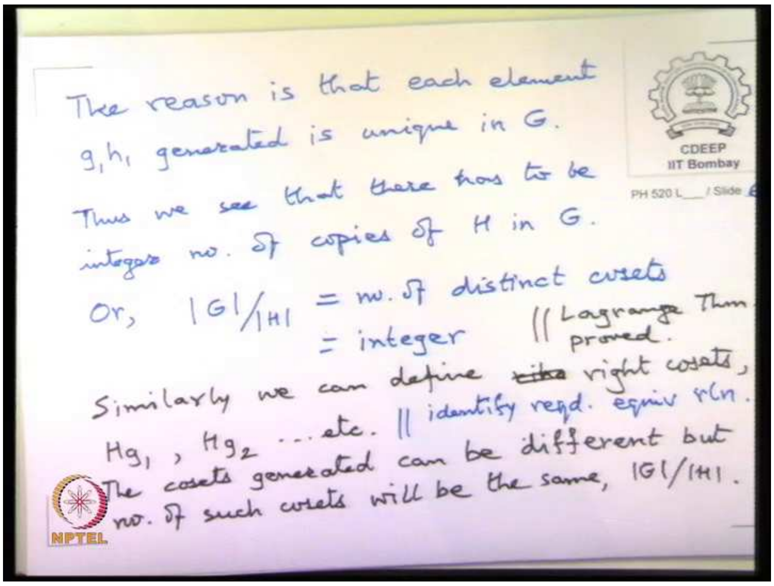

So, the left multiplication of the whole set H generates another copy of H in G with exactly as many as elements. But now you see that the union of these has to make up the whole G. So, clearly there is an integer number of copies of H in G otherwise you will not recover the whole set. So, equal to number of cosets is an integer, in fact positive number right.

So, H is a divisor of G right. So, right just to repeat what has happened is because we are equivalence classes, which have to exhaust all of G, but each equivalence classes exactly same size. So, there has to be an integer number equivalence classes into which the G is divided. So, this left coset operation is are clever way of proving the Lagrange theorem, that G slash H is an integer. Now, there is a related concept which is very easy and just a simple extension is the right coset.

So, similarly we can define right cosets. So, the this is proof Lagrange theorem proved ok. So, similarly we can define right cosets, where we take Hg_1 multiply by the from right obviously, the number of cosets generated this way, will be exactly the same as the number of cosets from left multiplication, but the two need not to be the same the two both may be different patricians of G.

The cosets generated can be different meaning we are nothing to say about it definitely in the most general case, but number of cosets will be same, $|G| / |H|$. And we can add here to do this you only need to define an equivalence relation, which is instead of $g_2^{-1} g_1$, we will say $g_2 g_1^{-1}$.

So, identify required equivalence relation right, it is clear that g_1 will be related to g_2 provided now, we will need to put $g_2 g_1^{1}$ instead of $g_2^{-1} g_1$, but that is the same thing ok. So, you can think about it and try to check directly how this works. So, we have proved what is known as Lagrange's theorem, but now this leads to one more general concept about groups, which is that there is a particular situation in which there is greater elegance of this kind of idea; which is when left cosets and right cosets turnout to be identical ok.

So, the special case is and mind your right may it all sound abstract, but later it turns out that these things actually happen in the groups of atomic I means molecules and solids and so, on. And that one can get information over the spectra from knowing other this kind of phenomenon happens or not.